Non-perturbative Methods in Quantum Field Theory

Post-Quantum Quench Growth of Renyi Entropies in Perturbed Luttinger Liquids Using Truncated Spectrum Methods

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Motivation: Evolution of Entanglement

A primary aim of this work is to develop a methodology to compute time-dependent entanglement measures in *continuum* bosonic systems.

Here, apart from the very recent work of P. Emounts and I. Kukuljan arXiv:2202.11113, we have had limited methods to compute the time dependence of continuum bosonic systems:

- Generalized hydrodynamics cannot access the Renyi entropies
- Form factors of twist fields (O. Castro-Alvaredo, D. Horvath, arXiv:2103.08492) requires energy injection to be weak

Both of these approaches require integrability.

Here instead I will present a method combining truncated space methods with a generalized notion of the Renyi entropies of excited CFT states

- Able to compute the time dependence of the lower Renyi entropies
- Does not care if the system is integrable

Quantum Quenches

To induce a time-dependent behaviour, we will inject energy into our system.

Our protocol for doing so will be a quantum quench.

In a quantum quench, a parameter in the Hamiltonian is suddenly changed at time, t=0.

If the system was in the ground state, $|\psi_{initial}\rangle$,of the Hamiltonian, H(t<0), it will be in some non-trivial superposition of excited states, $|E\rangle_i, i = 1, 2, \cdots$,of the post-quench, Hamiltonian, H(t>0):

$$|\psi_{initial}\rangle = \sum_i b_i |E_i\rangle$$

and so execute non-trivial time evolution:

$$|\psi_{initial}(t)\rangle = \sum_i b_i e^{-iE_i t} |E_i\rangle$$

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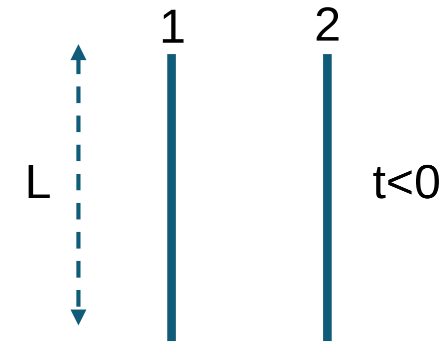
Quantum Quenches in Luttinger Liquids

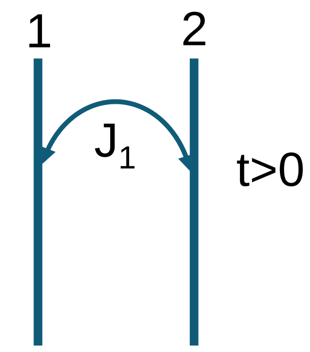
So here I am going to be interest in studying a quench involving two Luttinger liquids. In particular, I am going to imagine an interaction quench. Specifically, we initialize the system in the ground state of two uncoupled Luttinger liquids, i.e. two c=1 bosons:

$$H = H_1 + H_2 \qquad H_i = \frac{v}{2\pi}\int_0^L dx[\frac{1}{K}(\partial_x\Theta)^2 + K(\partial_x\phi)^2]$$



At t=0 we will turn a tunnel coupling between the two liquids:

$$H = H_1 + H_2 + \boxed{J_1}\int_0^L dx\cos(\phi_1(x) - \phi_2(x))$$



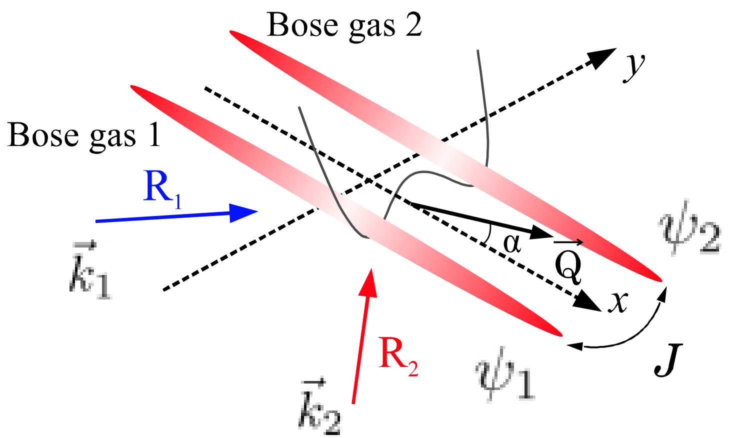
In the antisymmetric boson channel, $\phi_1 - \phi_2$, we effectively have a sine-Gordon quench where the sine-Gordon model has a value of .

$$\beta = 1/\sqrt{2K}$$

Experimental Motivation

This work has two experimental motivations:

i) Work done by the Schmiedmayer group in Vienna on coupled 1D cold atomic gases



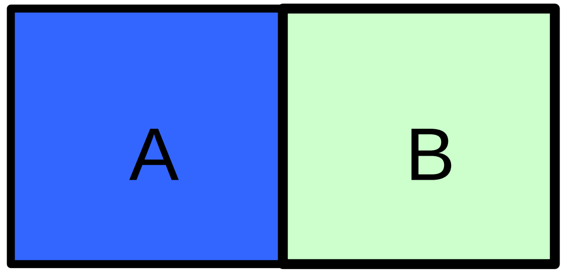
Marine Pigneur, Tarik Berrada,Marie Bonneau,Thorsten Schumm, Eugene Demler, and Jorg Schmiedmayer, Phys. Rev. Lett. 120 (2018) 173601

ii) Other systems that admit a 1D bosonic description:

- Spin chains
- 1D Hubbard like materials
- Quantum wires
- Coupled arrays of such systems

Computational Target: Renyi Entropies

Take a system and divide it into two blocks A and B



Suppose we have a pure state, $|\psi\rangle$, of the system. Performing a Schmidt decomposition in terms of subsystem A and B,

$$|\psi\rangle = \sum_{\alpha} c_\alpha |i_\alpha\rangle \otimes |j_\alpha\rangle \qquad i_\alpha \in \mathcal{H}_A,\ j_\alpha \in \mathcal{H}_B$$

we can form the reduced density matrix for subsystem A by tracing over H_B:

$$\rho_A = \mathrm{Tr}_{j_\alpha \in \mathcal{H}_B} |\psi\rangle\langle\psi| = \sum_{\alpha} c_\alpha^2 |i_\alpha\rangle\langle i_\alpha|$$

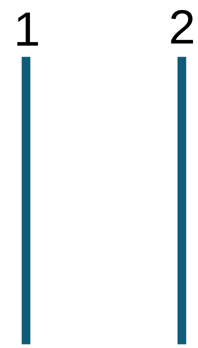
The Renyi and entanglement entropies are then

$$S_{n,Renyi} = \frac{1}{1-n} \log Tr \rho_A^n = \frac{1}{1-n} \log \sum_{\alpha} c_\alpha^{2n}$$

$$S_{entanglement} = \lim_{n \to 1} S_{n,Renyi} = -\sum_{\alpha} c_\alpha^2 \log(c_\alpha^2)$$

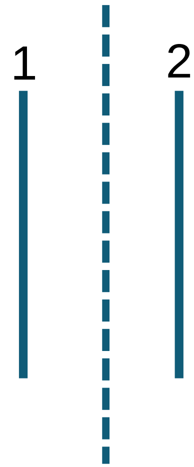
Renyi Entropies: Cutting in Two Directions

Our system appears (schematically) as:

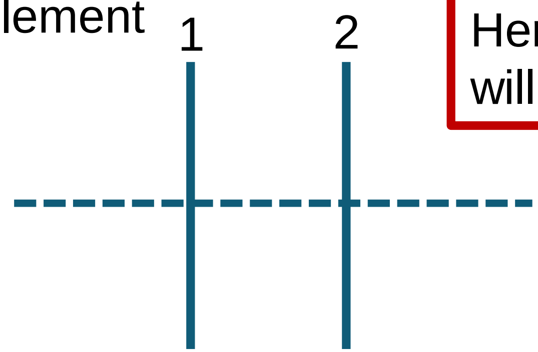


We can make two different types of cuts to compute the entanglement between subsystems:

Entanglement Cut 1



Entanglement Cut 2



Here we will focus.

Computing the entanglement along cut 1 is relatively trivial. Not so along cut 2.

Measurements of Renyi Entropies

One of the motivating reasons for focusing on the Renyi entropies is that they are potentially accessible to experiment.

There are two ways to measure the Renyi entropies:

1. Creating multiple copies of the system and measuring the swap operator of the partitions A of the copies.
    - Demler and Abanin, PRL 109 020504 (2012)
    - Cardy, PRL 106, 150404 (2011)
    - Daley et al, PRL 109 020505 (2012)
    - Islam et al, Nature 528, 77 (2015)

2. Averaging over a set of random quenches and projective measurements of the degrees of freedom in partition A.

    - Elben et al, Physical Review Letters, 120 050406 (2018)

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Q: How are we going to compute the Renyi entropies?
A: Truncated Spectrum Methods

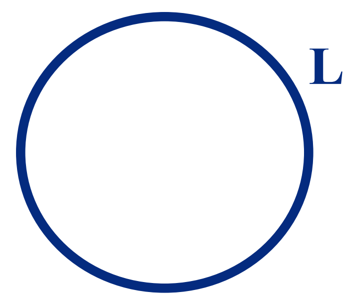
The method (Yurov and Zamolodchikov, 1990-91) can in principle study any Hamiltonian that takes the form:

$$H = \underbrace{H_{known}} + \underbrace{F_{perturbation}}$$

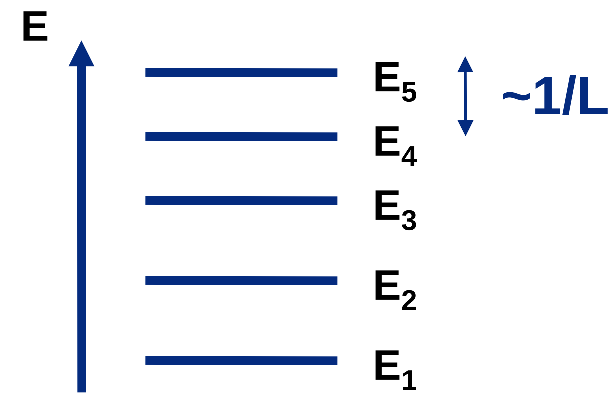
conformal/integrable theory, i.e. compact boson describing the Luttinger liquids

Luttinger liquid tunneling, cosine interaction

Consider the model on a finite sized ring of circumference, L:



Spectrum of H_{known} then becomes discrete and we can order states in terms of ascending energy (or some other measure of importance).



What is the space of states?

The space of states is constructed from the mode expansion of the boson

$$\phi(x,t) = \underbrace{\boxed{\hat{\phi}_0}}_{\textbf{Zero mode}} + \frac{\pi}{L}\hat{\pi}_0 t + \frac{i}{2}\sum_{k\neq 0}\frac{1}{k}\left(a_k e^{i\frac{2\pi k}{L}(x-t)} - \bar{a}_{-k} e^{i\frac{2\pi k}{L}(x+t)}\right)$$

Here the zero mode is compact, i.e. takes values between 0 and 2π.

The states are then built on a set of highest weight states formed by the vertex operators:

$$e^{in\phi}|0\rangle, \quad n = 0, 1, 2, \cdots$$

A general state in our computational basis is

$$|\psi\rangle = \prod_{k<0}(a_k^{n_k}\bar{a}_k^{\bar{n}_k})e^{in\phi}|0\rangle$$

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Importance of the Compactness of the Zero Mode

We will study the system, in part, at large K or small sine-Gordon β.

At small β one might be tempted to make the substitution

$$\cos(\beta\phi) \longrightarrow \frac{\beta^2}{2}\phi^2$$

changing the problem to a quadratic, non-compact, non-interacting one. This is in general a (very) bad approximation for non-equilibrium dynamics.

The problem knows about the compact nature of the zero mode, not least through the appearance of large numbers of bound states of the solitons at small β.

With β <1, there are $2/\beta^2-1$ bound states in the low energy spectrum. The presence of these excitations will be probed in any quench. In reducing the problem to a quadratic one, this richness is lost.

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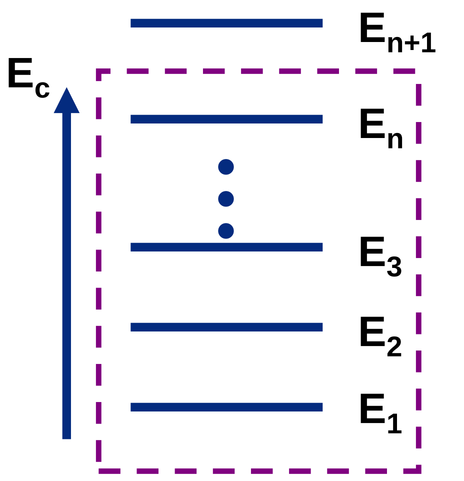
Truncated Spectrum Methods (cont.)

To understand the effects of the perturbation, we compute matrix elements of the perturbing operator.

$$F_{ij} = \langle i | F_{perturbation} | j \rangle_{\mathbf{H}_{\mathbf{Known}}}$$

Truncate Hilbert space, making it finite dimensional. This allows one to write full Hamiltonian as a finite dimensional matrix.

$$\mathbf{H} = \begin{bmatrix} E_1 & F_{12} & \cdots & F_{1n} \\ F_{21} & E_2 & & \vdots \\ \vdots & & \ddots & \\ & & E_{n-1} & F_{n-1n} \\ F_{n1} & \cdots & F_{nn-1} & E_n \end{bmatrix}$$



Diagonalize H numerically and extract spectrum and matrix elements.

Review: A. James, RMK, P. Lecheminant, N. Robinson, A. Tsvelik, Reports in Progress in Physics 81 (2018)

Truncation of Computational Basis

Our computational basis consists of states of the form:

Oscillator part

$$|\psi\rangle = \prod_{k<0} (a_k^{n_k} \bar{a}_k^{\bar{n}_k}) e^{in\phi} |0\rangle$$

Zero mode part

The (unperturbed) energy of such a state is

$$E_{zero\ mode} + E_{oscillator} = \frac{\pi n^2}{LK} + \frac{2\pi}{L}\left(\sum_{j=1}^{M} k_j + \sum_{\bar{j}=1}^{\bar{M}} k_{\bar{j}}\right)$$

In forming the computational basis, we use two cutoffs:

$$E_{zero\ mode} < E^c_{zero-mode}; \quad E_{oscillator} < E^c_{oscillator}$$

We choose

$$E^c_{zero-mode} \gg E^c_{oscillator}$$

as highly energetic zero mode states contribute to the dynamics.

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Computing the Renyi Entropy using Truncated Spectrum Methods

The initial ground state $|\psi_{initial}\rangle$ of the system is the c=1 bosonic vacuum. Using TSM, we compute the eigenstates $|E\rangle_i, i=1,2,\cdots$ of the post-quench Hamiltonian. Each $|E\rangle_i$ is expressible in terms of the TSM computational basis:

$$|E_i\rangle = \sum c_{ij}|\psi_j\rangle$$

Again this basis $\{|\psi_j\rangle\}$ has the form:

$$|\psi\rangle = \prod_{k<0}(a_k^{n_k}\bar{a}_k^{\bar{n}_k})e^{in\phi}|0\rangle$$

The time dependence of the post-quench state can then be written as

$$\begin{aligned}|\psi_{initial}(t)\rangle &= \sum_i b_i e^{-E_i t}|E_i\rangle \\ &= \sum_{ij} b_i c_{ij} e^{-E_i t}|\psi_j\rangle \\ &= \sum_j \alpha_j(t)|\psi_j\rangle\end{aligned}$$

This is not the only way to compute the time evolution – one may use Chebyshev techniques: T. Rakovszky, M. Mestyán, M. Collura, M. Kormos, and G. Takács, Nuclear Physics B 911, 805 (2016)

The time-dependence of the system's state is now expressed in terms of CFT states. The time dependent (second) Renyi entropy is then given by

$$\boxed{R_2(t) = \sum_{i,i',j,j'} \alpha_i(t)\alpha_j^*(t)\alpha_{i'}(t)\alpha_{j'}^*(t)Tr_A((Tr_B|\Psi_i\rangle\langle\Psi_j|)(Tr_B|\Psi_{i'}\rangle\langle\Psi_{j'}|))}$$

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Object We Need to Understand for Time-Dependent Renyi Entropies

We have expressed the time evolution of our state as a sum over states given by the CFT of a massless boson. To compute the time-dependent (second) Renyi entropy, we then need to be able to compute quantities of the form

$$\boxed{R_{i,i';j,j'} = Tr_A((Tr_B|\Psi_i\rangle\langle\Psi_j|)(Tr_B|\Psi_{i'}\rangle\langle\Psi_{j'}|))}$$

where each $|\psi\rangle$ is some CFT state. This is a non-standard because the states

$$|\psi_i\rangle, |\psi_j\rangle, |\psi_{i'}\rangle, |\psi_{j'}\rangle$$

are all different.

How to do this? If all of the $|\psi_i\rangle$'s are the same *and* relatively simple states, we can borrow results from the entanglement of CFT excited states:

F. C. Alcaraz, M. Ibanez Berganza, and G. Sierra, Phys. Rev. Lett. 106 201601 (2011)

While more general cases $R_{i,i;j,j}$ have been studied,

N. Lashkari, Phys. Rev. Lett. **113**, 051602 – relative entropy
S. Murciano, P. Ruggiero, P. Calabrese, J. Stat. Mech. 034001 (2019) – relative entropy
Y. Nakata, T. Takayanagi, Y. Taki, K. Tamaoka, Z. Wei, PRD **103** (2021) 026005 – pseudo-entropy
T. Palmai - Physics Letters B (2016) 439, Virasoro CFT states

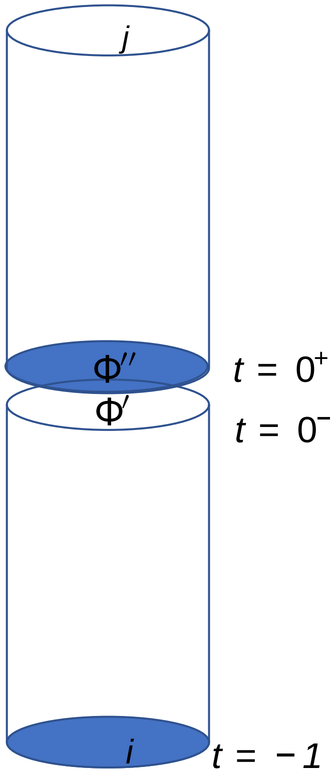
no one had studied this object in full generality for the c=1 bosonic CFT.

How to compute generalized Renyi entropies, $R_{i,i';j,j'}$

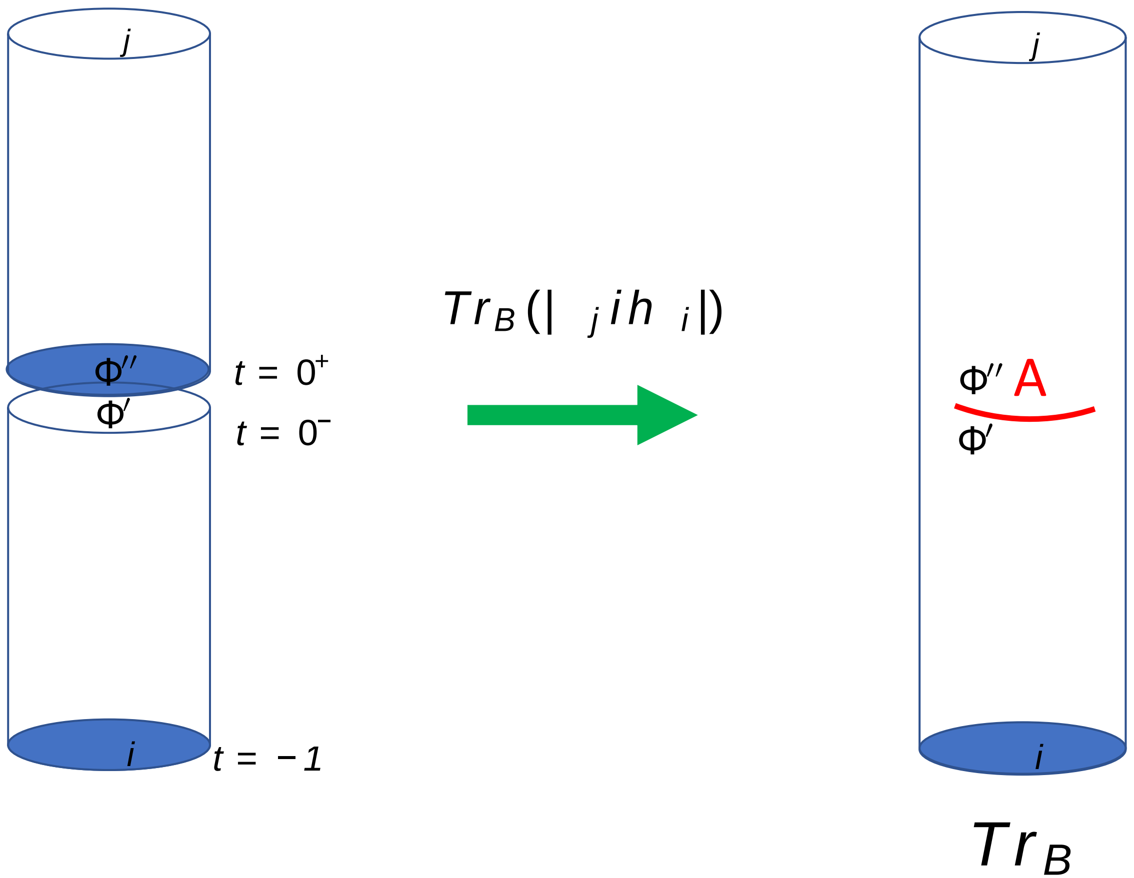
Path integral representation of $|\ _j\, i h\ \ _i|$

$$\langle\psi_j|\Phi''\rangle \propto \int_{\phi(t=0^+)=\Phi''} \mathcal{D}\phi e^{-S(\phi)}\Psi(\phi(t=+\infty))$$

$$\langle\Phi'|\psi_i\rangle \propto \int_{\phi(t=0^-)=\Phi'} \mathcal{D}\phi e^{-S(\phi)}\Psi(\phi(t=-\infty))$$



How to compute generalized Renyi entropies, $R_{i,i';j,j'}$ (cont.)



$$\langle \Phi'' | Tr_B(|\Psi_j\rangle\langle\Psi_i|)|\Phi'\rangle$$

How to Compute Mixed Renyi Entropies, $R_{i,i';j,j'}$

Playing this game then objects like $R_{i,I';j,j'}$ can be written as a path integral over a two-sheeted Riemann surface which in turn has a representation as a 4-point function in CFT

$$R_{i,i';j,j'} = Tr_A(Tr_B|\Psi_i\rangle\langle\Psi_j|)(Tr_B|\Psi_{i'}\rangle\langle\Psi_{j'}|)) \ =$$

$$C_{ij}C_{i'j'}\int_{\substack{\phi_1(x,t=0^+)=\phi_2(x,t=0^-) \\ \phi_1(x,t=0^-)=\phi_2(x,t=0^-)}\ \forall\ x\in A}\mathcal{D}\phi_1\mathcal{D}\phi_2 e^{-S(\phi_1)-S(\phi_2)}$$

$$\times\prod_{x\in A}\delta(\phi_1(x,t=0^-)-\Phi'(x))\delta(\phi_2(x,t=0^+)-\Phi''(x))$$

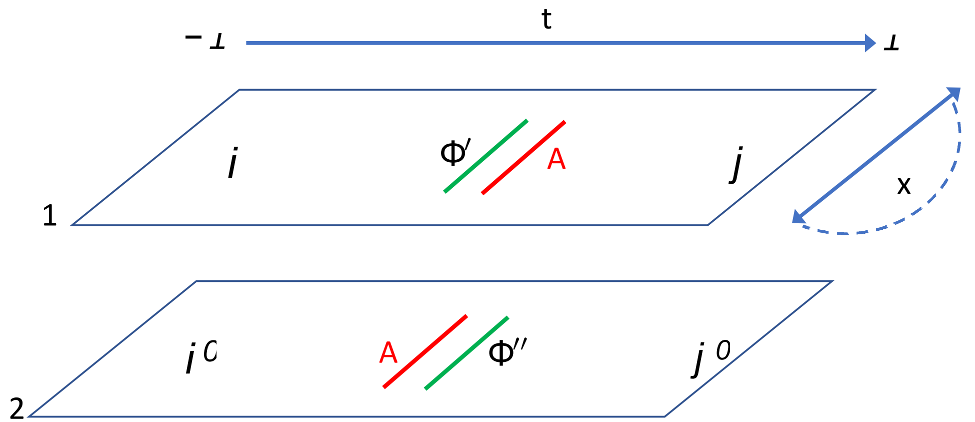
$$\times\Psi_i(t=-\infty)\Psi_j(t=\infty)\Psi_{i'}(t=-\infty)\Psi_{j'}(t=\infty)$$

$$= C_{ij}C_{i'j'}Z_2(A)\boxed{\langle\Psi_i(t=-\infty)\Psi_j(t=\infty)\Psi_{i'}(t=-\infty)\Psi_{j'}(t=\infty)\rangle}.$$

Four-point function

How to Compute Mixed Renyi Entropies, $R_{i,i';j,j'}$

Schematically the space-time one needs to evaluate the correlation function on has the form:



where the cuts A in the two planes are identified and the cuts Φ' and Φ'' are identified.

This is a peculiar spacetime, but one can map this Riemann surface onto the plane where the correlation functions can be evaluated with standard conformal field theory techniques.

How to Compute Mixed Renyi Entropies, $R_{i,i';j,j'}$

One thing to note: The correlation function

$$\langle \Psi_i(t=-\infty)\Psi_j(t=\infty)\Psi_{i'}(t=-\infty)\Psi_{j'}(t=\infty)\rangle$$

is not really a 4-pt function but an n-pt function

$$C(\omega_{\alpha_1},\omega_{\alpha_2},\omega_{\alpha_3},\omega_{\alpha_4},\omega_1\cdots,\omega_N) \;=\; c_{\alpha_1}c_{\alpha_2}c_{\alpha_3}c_{\alpha_4}\langle\prod_{i=1}^{N_1}\partial_{\omega_i}\phi(\omega_i)e^{i\alpha_1\phi(\omega_{\alpha_1})}e^{i\alpha_2\phi(\omega_{\alpha_2})}\prod_{i=1}^{N_2}\partial_{\omega_{i+N_1}}\phi(\omega_{i+N_1})$$
$$\times\prod_{i=1}^{N_3}\partial_{\omega_{i+N_1+N_2}}\phi(\omega_{i+N_1+N_2})e^{i\alpha_3\phi(\omega_{\alpha_3})}e^{i\alpha_4\phi(\omega_{\alpha_4})}\prod_{i=1}^{N_4}\partial_{\omega_{i+N_1+N_2+N_3}}\phi(\omega_{i+N_1+N_2+N_3})\rangle. \qquad (53)$$

Each $|\psi_i\rangle$ leads to a product of $\partial\phi$'s in the correlation function. We can compute such correlation functions in general because the massless bosonic CFT is simple. Such correlation functions have representations in terms of Hafnians,

$$haf(A) = \frac{1}{N!2^N}\sum_{\sigma\in S_{2N}} A_{\sigma(2j-1),\sigma(2j)}$$

A is a 2Nx2N matrix

the symmetric cousins of the anti-symmetric Pfaffians. If, however, we had to compute this quantity for a general CFT, we would be stuck.

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Formula for Renyi Entropies, $R_{i,i';j,j'}$

For Renyi entropies involving states with no vertex operators, we have

$$R_{k_1,\cdots,k_N} = Tr_A(Tr_B(\prod_{i=1}^{N_1} a_{-k_i}|0\rangle\langle 0|\prod_{i=1}^{N_2} a_{k_{N_1+i}})Tr_B(\prod_{i=1}^{N_3} a_{-k_{N_1+N_2+i}}|0\rangle\langle 0|\prod_{i=1}^{N_4} a_{k_{N_1+N_2+N_3+i}}))$$

We get

$$\frac{R_{k_1,\cdots,k_N}}{R_{\mathbb{1},\mathbb{1};\mathbb{1},\mathbb{1}}} = (-1)^{N_1+N_3} e^{i\frac{2\pi}{R}v(P_1+P_3-P_2-P_4)}$$

$$\times \sum_{\substack{\sigma\in S_N \\ \sigma_{2i}<\sigma_{2i+1} \\ \sigma_1<\sigma_3<\cdots<\sigma_{2n-1}}} \prod_{i=1}^{N/2} W(k_{\sigma_{2i-1}}, k_{\sigma_{2i}}, y_{\sigma_{2i-1}}, y_{\sigma_{2i}})$$

Hafnian

$$W(k_i,k_j,y_i,y_j) = \begin{cases} \frac{1}{\Gamma(k_i)} \sum_{l=0}^{k_i-1} \binom{k_i-1}{l} \frac{\Gamma(k_i-l+1)}{\Gamma(k_i+k_j-l+1)} & \\ \qquad \times(\partial_z^l f^{k_i})(z=y_i,y_i)(\partial_z^{k_i+k_j-l} f^{k_j})(z=y_j,y_j), & \text{if } y_i = y_j; \\ \frac{1}{\Gamma(k_i)\Gamma(k_j)}\partial_{z_i}^{k_i-1}\partial_{z_j}^{k_j-1}\left(\frac{(f(z_i,y_i))^{k_i}(f(z_j,y_j))^{k_j}}{(z_i-z_j)^2}\right)\Bigg|_{\substack{z_i=y_i \\ z_j=y_j}}, & \text{if } y_i \neq y_j. \end{cases}$$

$$f(z_1,y_1) = \frac{z_1^2-(y_1^*)^2}{z_1+y_1}$$

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Computational Details: Which Renyi Entropies, $R_{i,i';j,j'}$, to Compute

In our truncated spectrum computations, our computational basis has sizes ranging from ~1×10^3 to 4×10^4 .

Thus, we need (nominally) to compute between 10^{12} to 10^{18} Renyi entropies, $R_{i,i';j,j'}$, in order to compute $R_2(t)$. Clearly, we can't do this. Instead, we extrapolate.

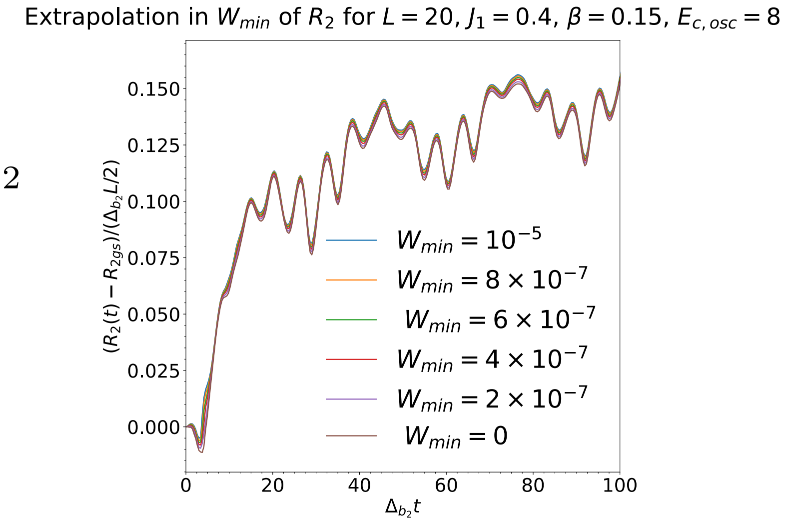
Our expression for the Renyi entropy $R_2(t)$ is

$$R_2(t) = \sum_{i,i',j,j'} \alpha_i(t)\alpha_j^*(t)\alpha_{i'}(t)\alpha_{j'}^*(t) Tr_A((Tr_B|\Psi_i\rangle\langle\Psi_j|)(Tr_B|\Psi_{i'}\rangle\langle\Psi_{j'}|))$$

In this 4-sum, we only keep terms satisfying,

$$W < |\bar{\alpha}_i\bar{\alpha}_j\bar{\alpha}_{i'}\bar{\alpha}_{j'}|, \quad \bar{\alpha}_i \equiv \left[\frac{1}{T}\int_0^T dt|\alpha_i(t)|^2\right]^{1/2}$$

We do this for several (small) values of W and then extrapolate W to 0.



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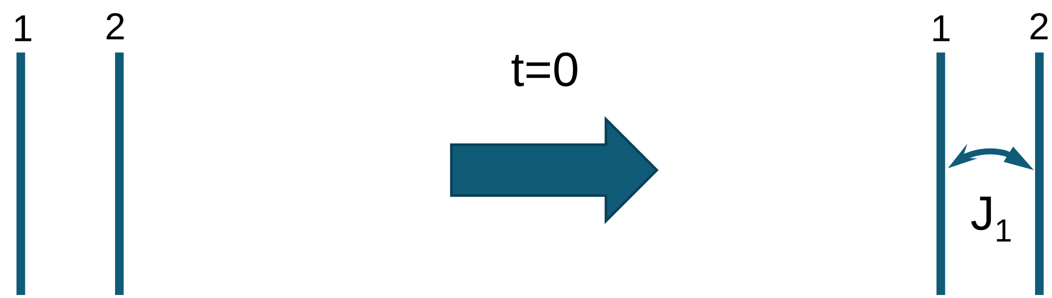
Results

Let me remind you of the quench in which we are interested:



$$H = H_1 + H_2$$

$$H_i = \frac{v}{2\pi}\int_0^R dx[\frac{1}{K}(\partial_x\Theta_i)^2 + K(\partial_x\phi_i)^2]$$

Uncoupled Luttinger liquids, t<0

$$H = H_1 + H_2 + J_1\int_0^R dx\cos(\phi_1(x) - \phi_2(x))$$

Coupled ($J_1 \neq 0$) Luttinger liquids, t>0

We will compare the growth of the 2nd Renyi entropy after the quench with the growth of the order parameter:

$$\langle\cos(\phi_1(0) - \phi_2(0))\rangle(t)$$

At short times

Using unitary perturbation theory at **short times and (the experimentally relevant) large K (small** :

Renyi entropy:

$$R_2(t) = (4(\log 2)^2 - \frac{447}{256})\boxed{\frac{4}{K^2}}L^2 J_1^2(\frac{2\pi}{L})^{4/K} t^2$$

$1/K^2$ prefactor: physics is probing field theoretic degrees of freedom

L: system size

Cosine order parameter:

$$\langle cos(\phi)\rangle(t) = -\frac{2\pi}{K}\left(\frac{2\pi}{L}\right)^{-2+2\beta^2} J_1\left(\frac{2\pi t}{L}\right)^2$$

$1/K$ prefactor: physics is quantum mechanical, that of the compact zero mode

Upshot: At short times, the Renyi entropy probes at leading order the quantum field theoretic nature of the quench while the order parameter evolution is purely quantum mechanical.

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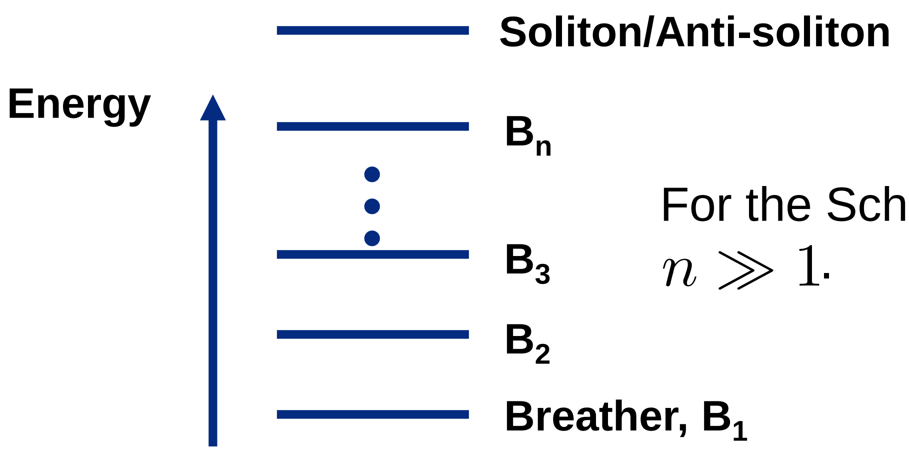
At longer times

At longer times the physics is going to be expressible in terms of the IR degrees of freedom for the post quench (sine-Gordon) Hamiltonian.

For the Schmiedmayer experiments, the Luttinger parameter K is large. In sine-Gordon language the equivalent coupling

$$\beta = \sqrt{1/2K}$$

is small. The spectrum thus consists of many breathers. The soliton and anti-soliton are very heavy and won't figure significantly in the quench dynamics.



For the Schmiedmayer experiment, $n \gg 1$.

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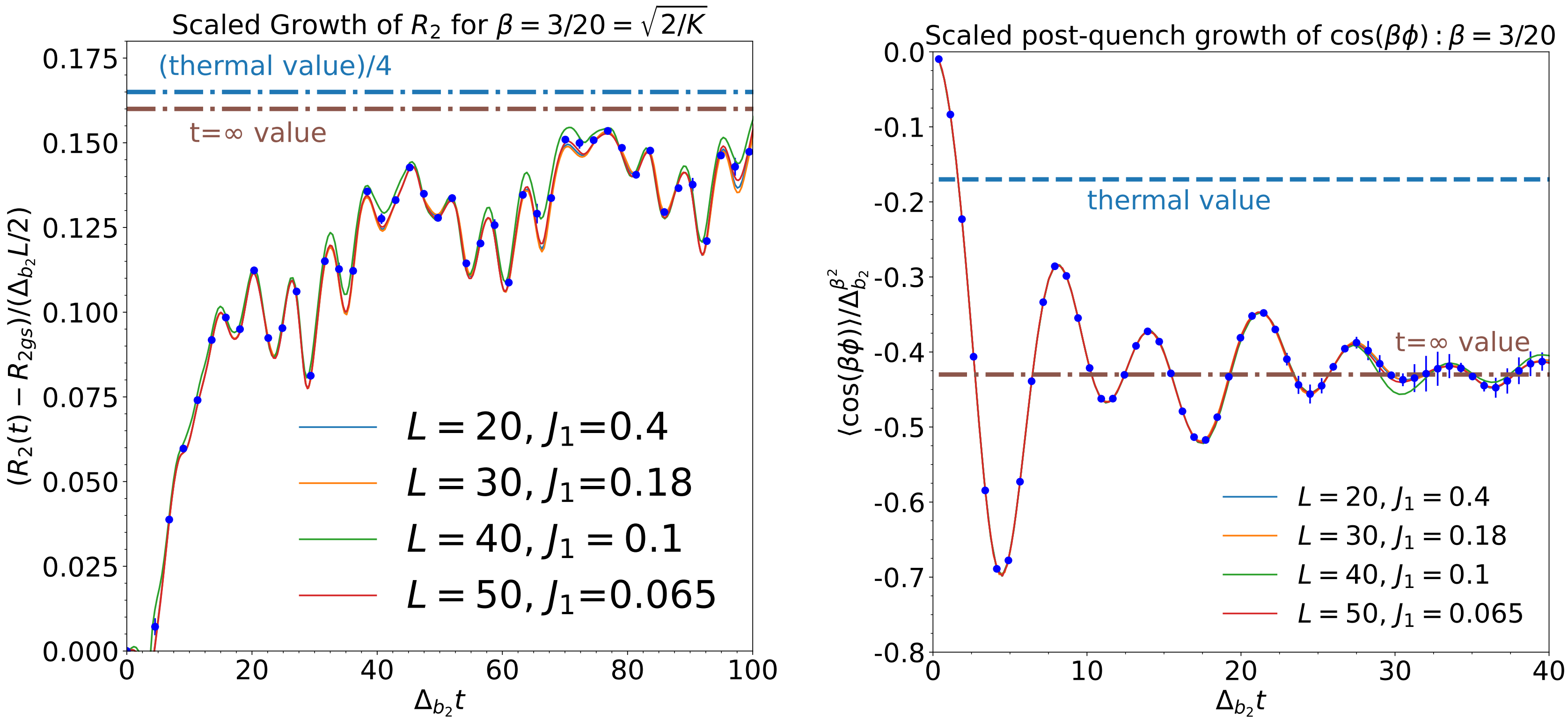
Over longer times, scaling collapse



We see scaling collapse at different system sizes (except at the longest times) provided $L\Delta_{b2}$~20 is kept constant. Here Δ_{b2} is the gap of the second breather.

The frequencies of the observed oscillations can be directly connected to the breather masses.

Conclusions

- We have developed the necessary tools to explore the evolution of the Renyi entropies in quantum quenches in 1D bosonic systems.

- This amounts to deriving expressions for what we term generalized Renyi entropies and combining this with data coming from a truncated spectrum method analysis of the quantum quench.

- Specifically, we have explored the case of joining two Luttinger liquids via a tunneling coupling, a quench relevant to experiments conducted by the Schmiedmayer group.

- Here we have been able to show that the short time evolution of the Renyi entropy depends on all degrees of freedom of the problem (zero mode and oscillators) while the order parameter evolution depends solely on the zero mode. Thus, in some sense the Renyi entropy is a more sensitive probe of the field theoretic nature of the problem.