Multi-particle observables from a finite Euclidean spacetime

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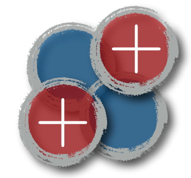
Recipe for strong force predictions

1. Lagrangian defining QCD
2. Formal / numerical machinery (lattice QCD)
3. A few experimental inputs (e.g. M_π, M_K, M_Ω)

$$\mathcal{L}_{\rm QCD} = \sum_f \overline{\Psi}_f (i \not{D} - m_f) \Psi_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$



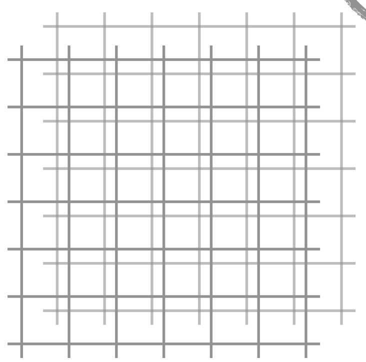
Wide range of precision pre-/post-dictions

$m_q^{\overline{\rm MS}}(\mu)$ $\alpha_{\overline{\rm MS}}(\mu)$

m_p

m_n f_π

g_A f_K



$(g-2)_\mu^{\rm HVP}$ f_D f_B

FLAG 2019 Flavour Lattice Averaging Group

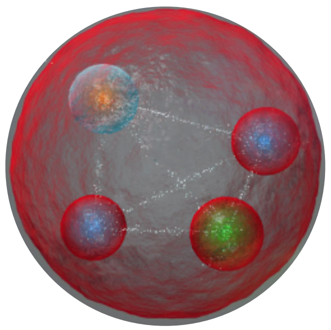
$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1182(8)$ lattice average

$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1174(16)$ PDG 18 (non-lattice)

Overwhelming evidence for QCD ✓ Tool for new-physics searches ✓

Multi-hadron observables

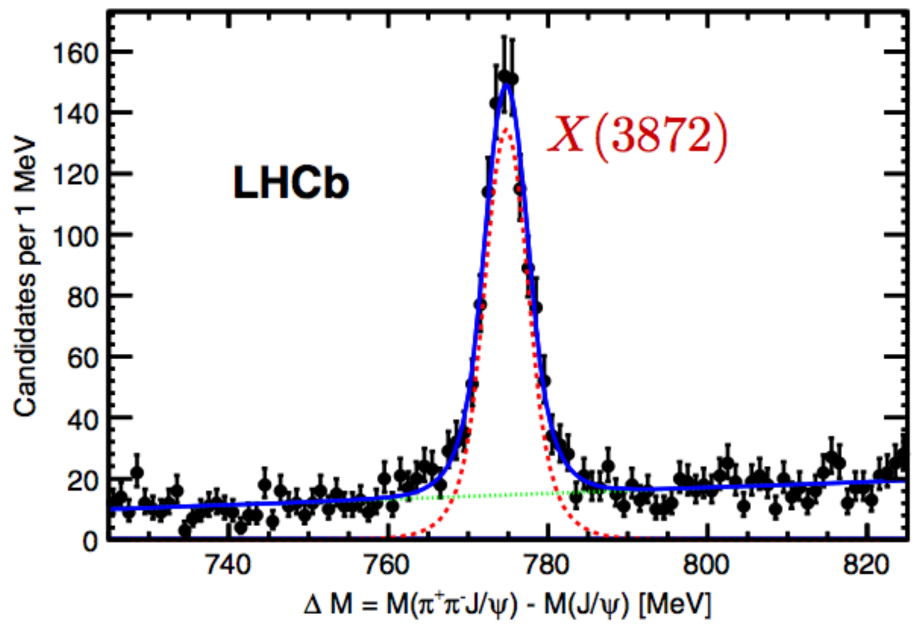
- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$$\sim |D^0\overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle \,?$$

• LHCb (PRD92, 2015) •



- Eletroweak, CP violation, resonant enhancement

CP violation in charm $\quad D \to \pi\pi, K\overline{K}$

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$$

• LHCb (PRL, 2019) •

$f_0(1710)$ could enhance ΔA_{CP}

• Soni (2017) •

Resonant B decays $\quad B \to K^* \ell\ell \to K\pi\,\ell\ell$

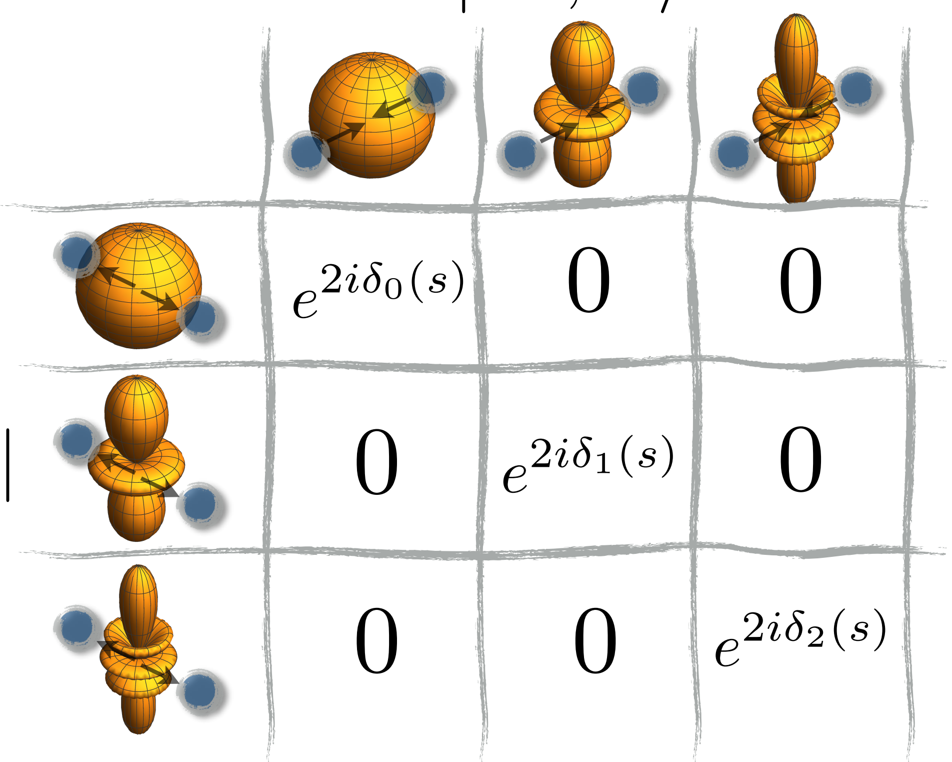
$$|X\rangle, |\rho\rangle\,, |K^*\rangle\,, |f_0\rangle \quad \notin \quad \textbf{QCD Fock space}$$

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- Overlaps of multi-hadron ***asymptotic states*** → S matrix

$$|\pi\pi, \text{in}\rangle$$

$$S(s) \equiv \langle \pi\pi, \text{out}|$$



depends on $s = E_{\text{cm}}^2$
and angular variables

diagonal in angular momentum

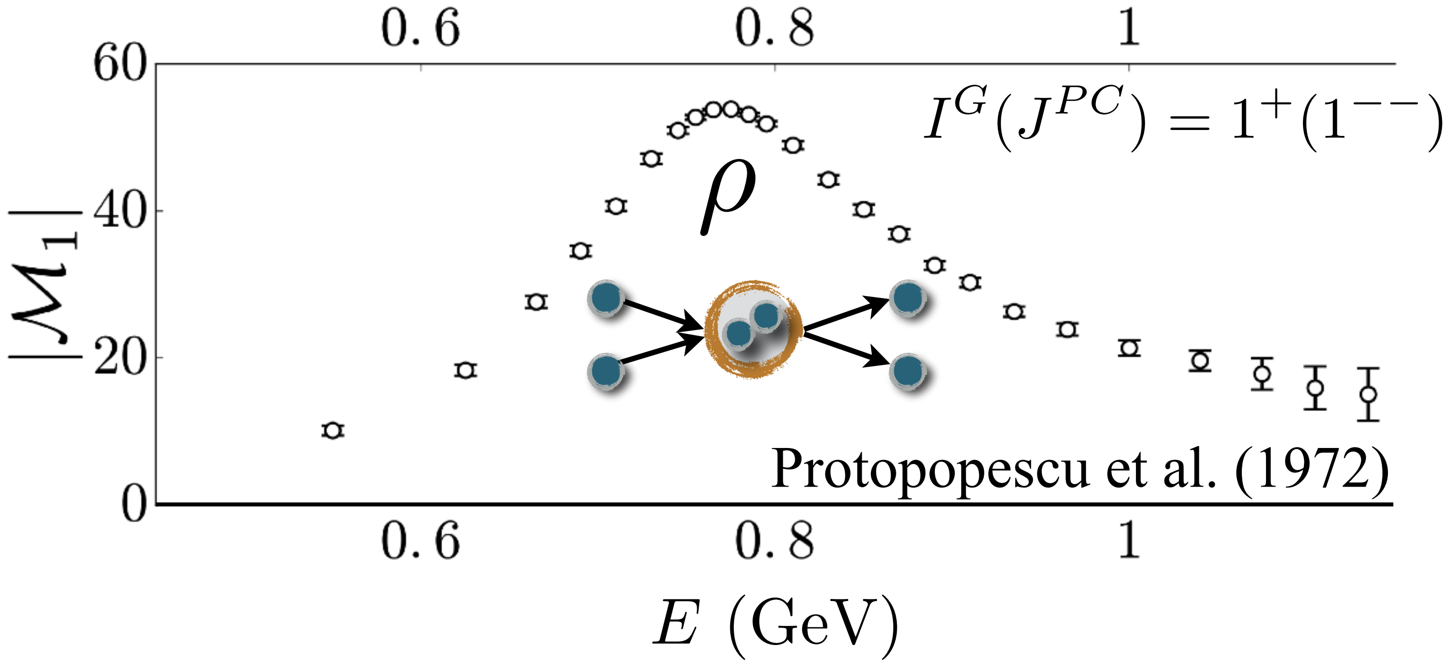
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

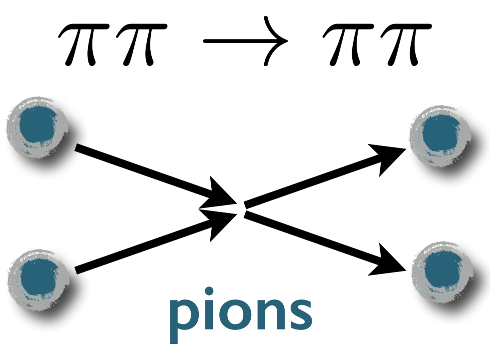
- ***An enormous space of information*** $|\pi\pi\pi\pi, \text{in}\rangle \quad |K\overline{K}, \text{in}\rangle \quad \cdots$

QCD resonances

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate





QCD resonances

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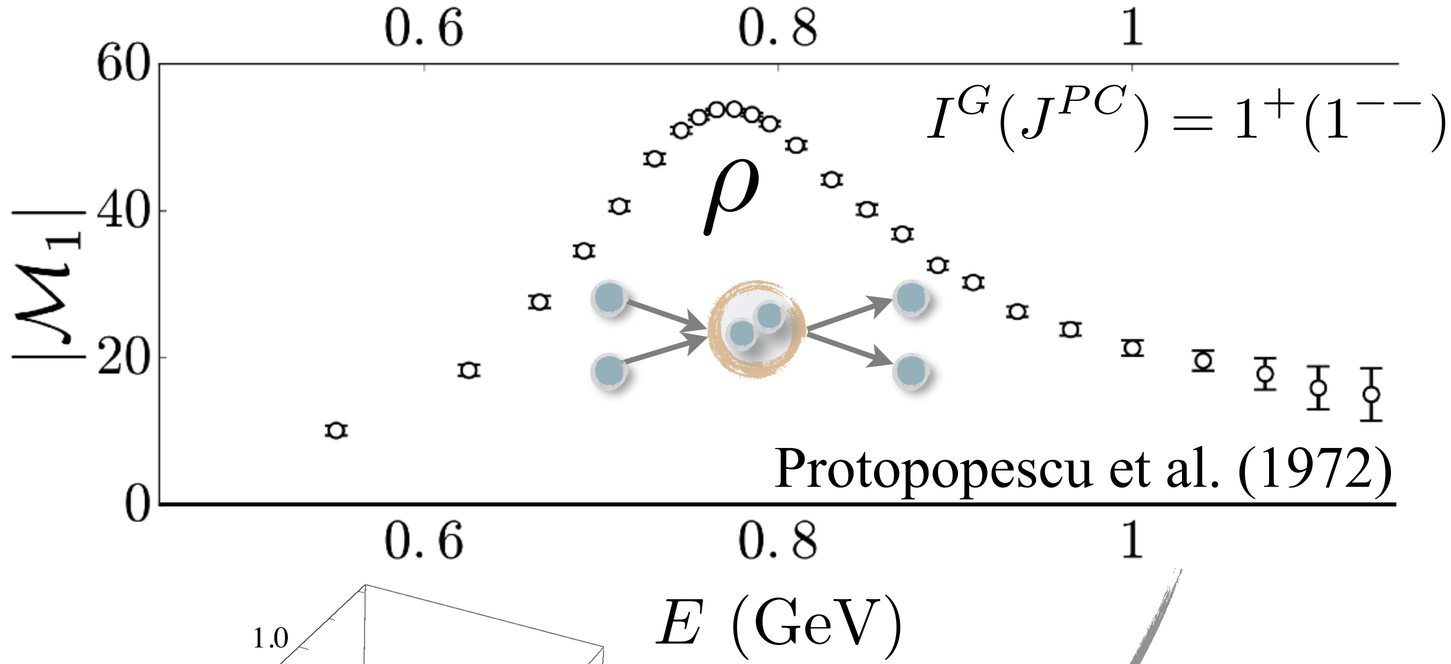
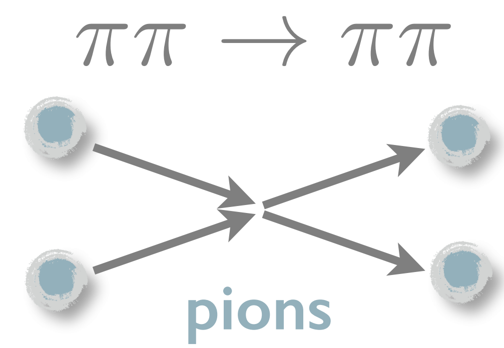
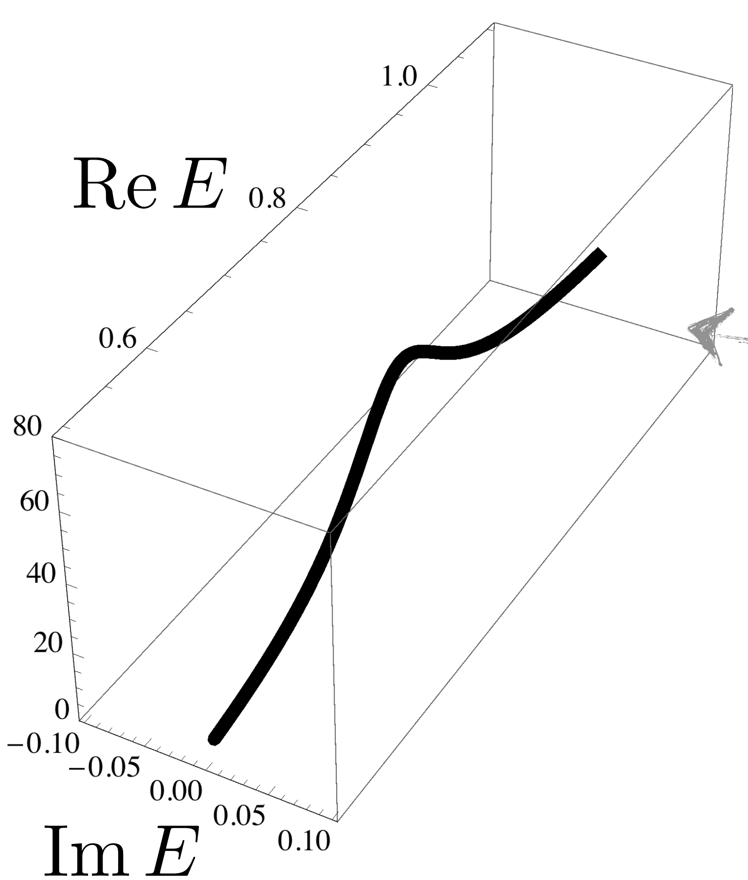




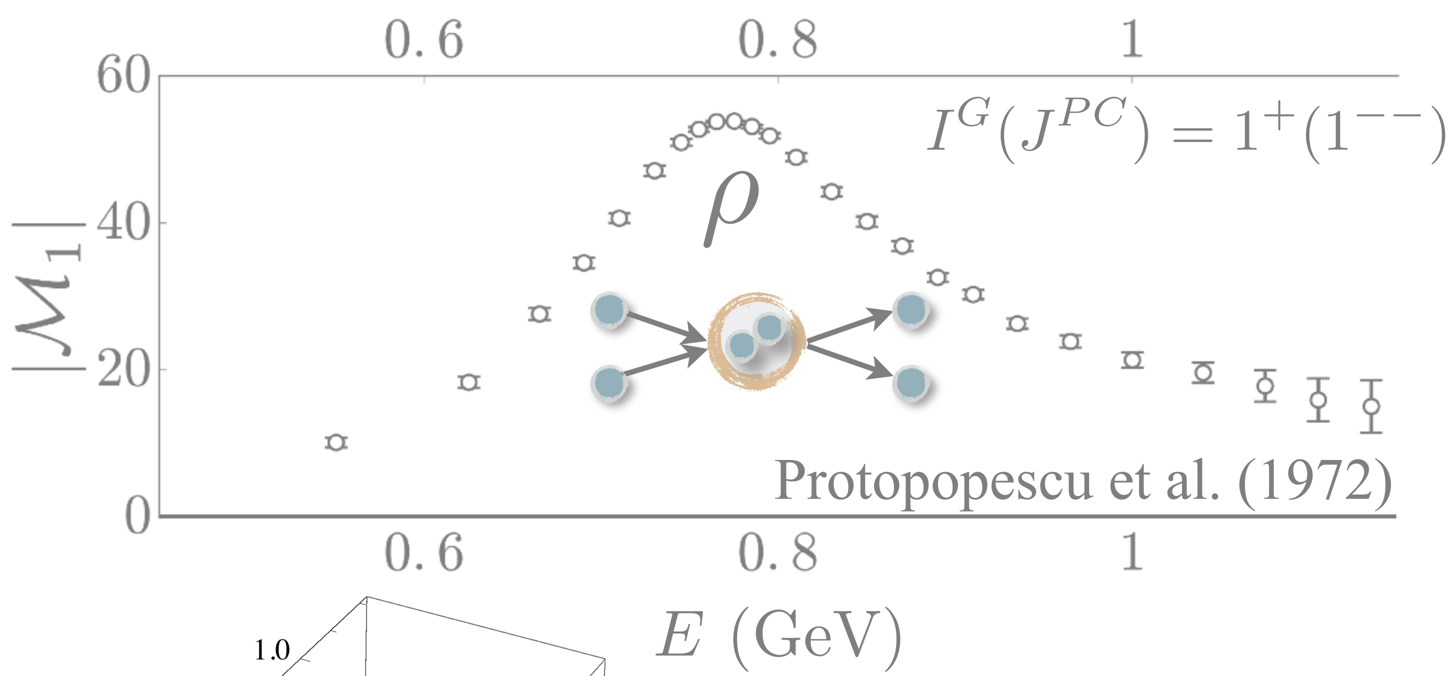


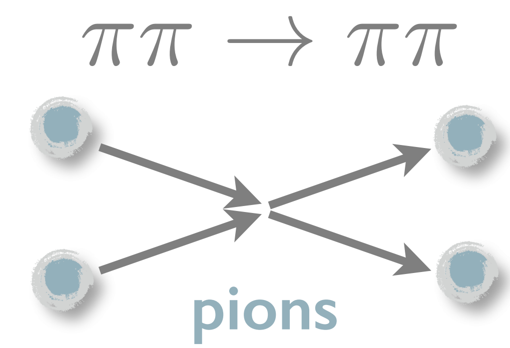
Extend to the **complex plane**

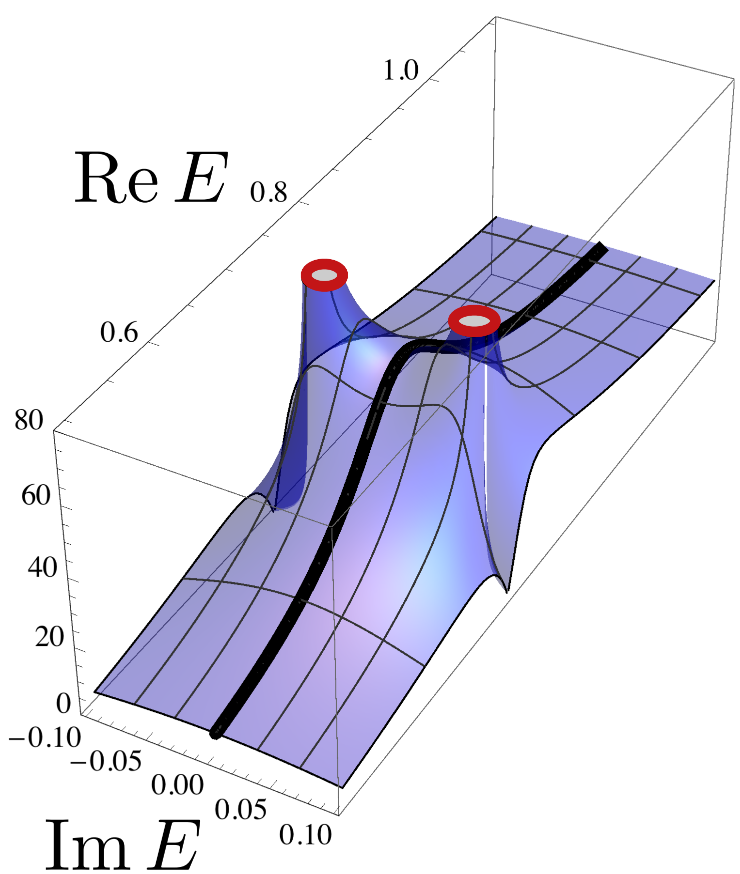
QCD resonances

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

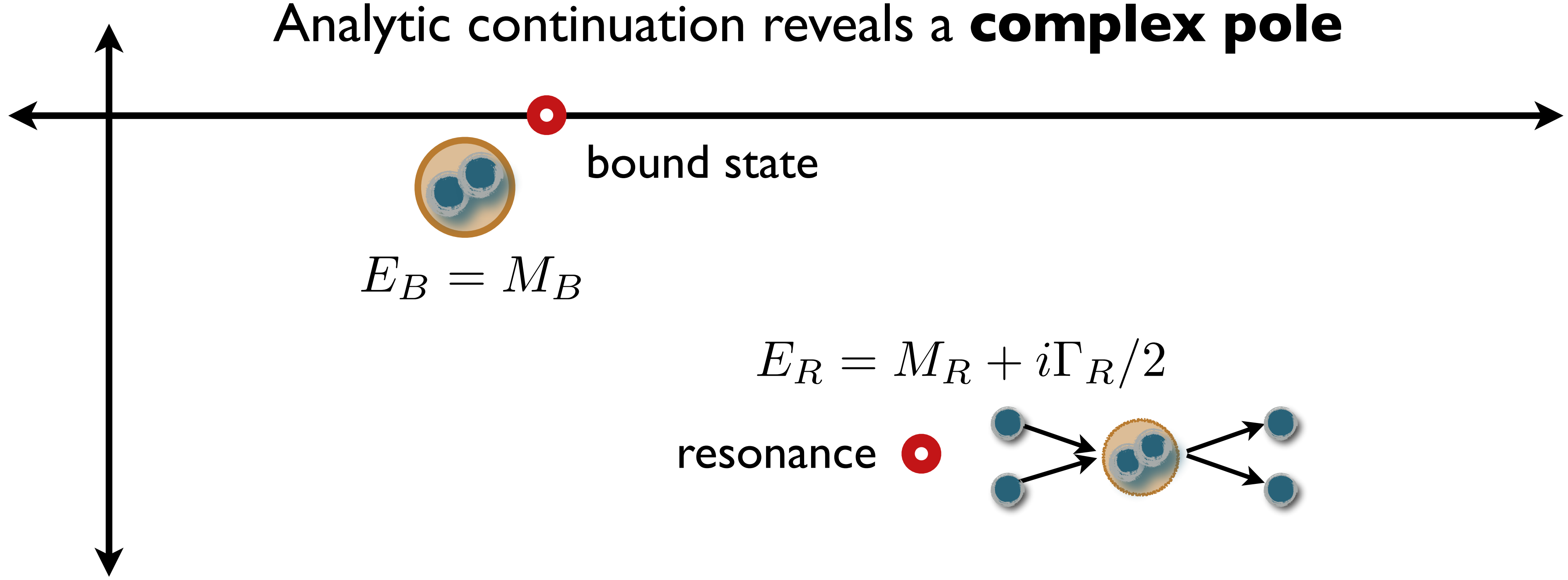
scattering rate





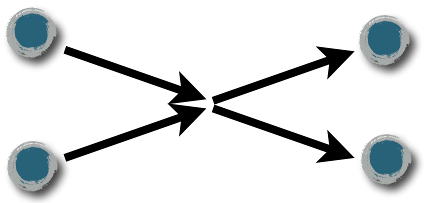


Analytic continuation reveals a **complex pole**



Analyticity

- Instead of $|\mathcal{M}(s)|^2$ → analytically continue the ***amplitude*** itself



For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?

- The optical theorem tells us…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \mathrm{Im}\,\mathcal{M}_\ell(s)$$

where $\rho(s) = \dfrac{\sqrt{1-4m^2/s}}{32\pi}$ is the two-particle phase space

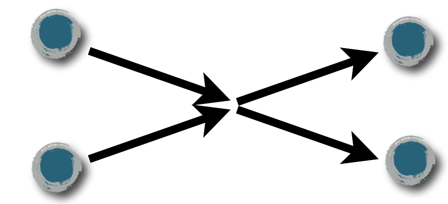
- Unique solution is…

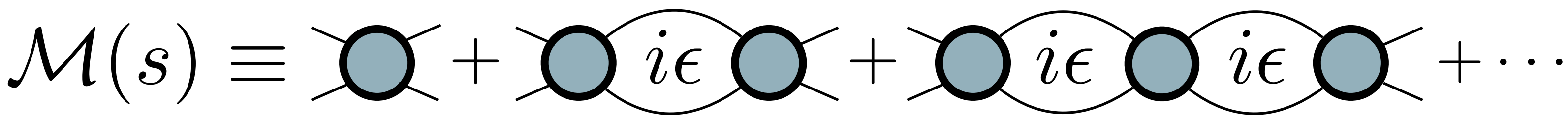
$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

K matrix (short distance) — phase-space cut (long distance)

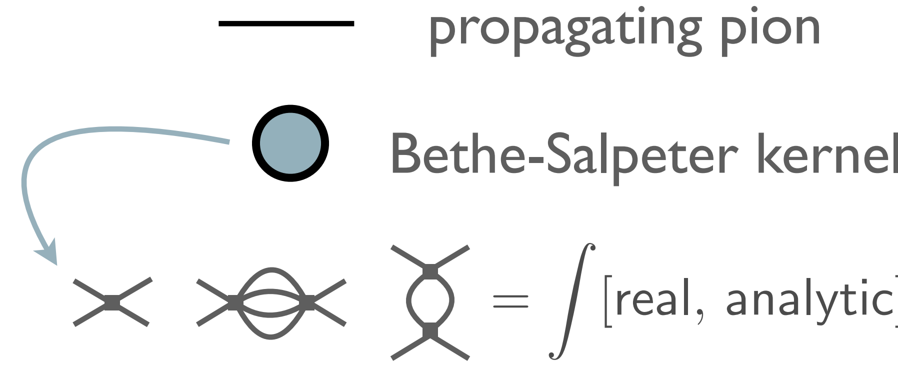
Amplitude has a branch cut ✓ K-matrix is real (useful for parametrizing) ✓

Analyticity (diagrammatic)



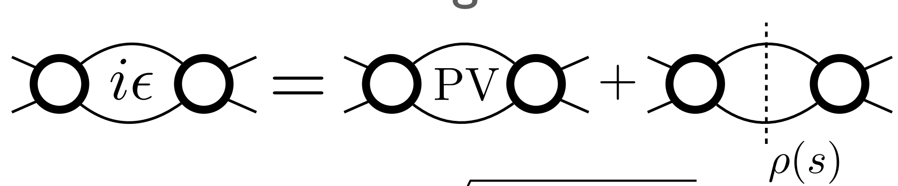


on-shell particles = singularities:
non-analytic for $(2m)^2 < s < (4m)^2$



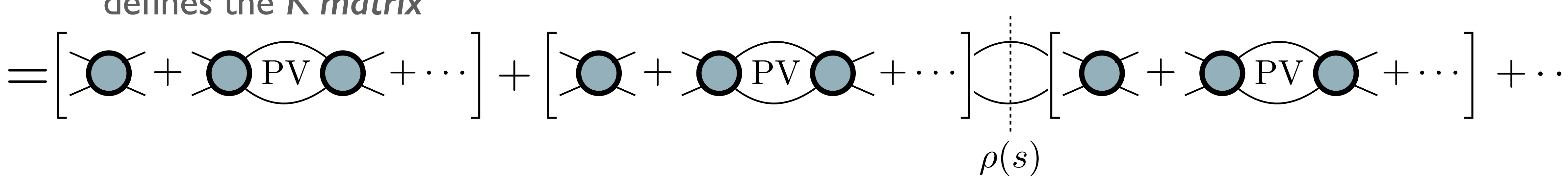
for $(2m)^2 < s < (4m)^2$

cutting rule



$\rho(s) \propto i\sqrt{s-(2m)^2}$

defines the *K* ***matrix***



$$= \mathcal{K}(s) + \mathcal{K}(s)\rho(s)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - \rho(s)}$$

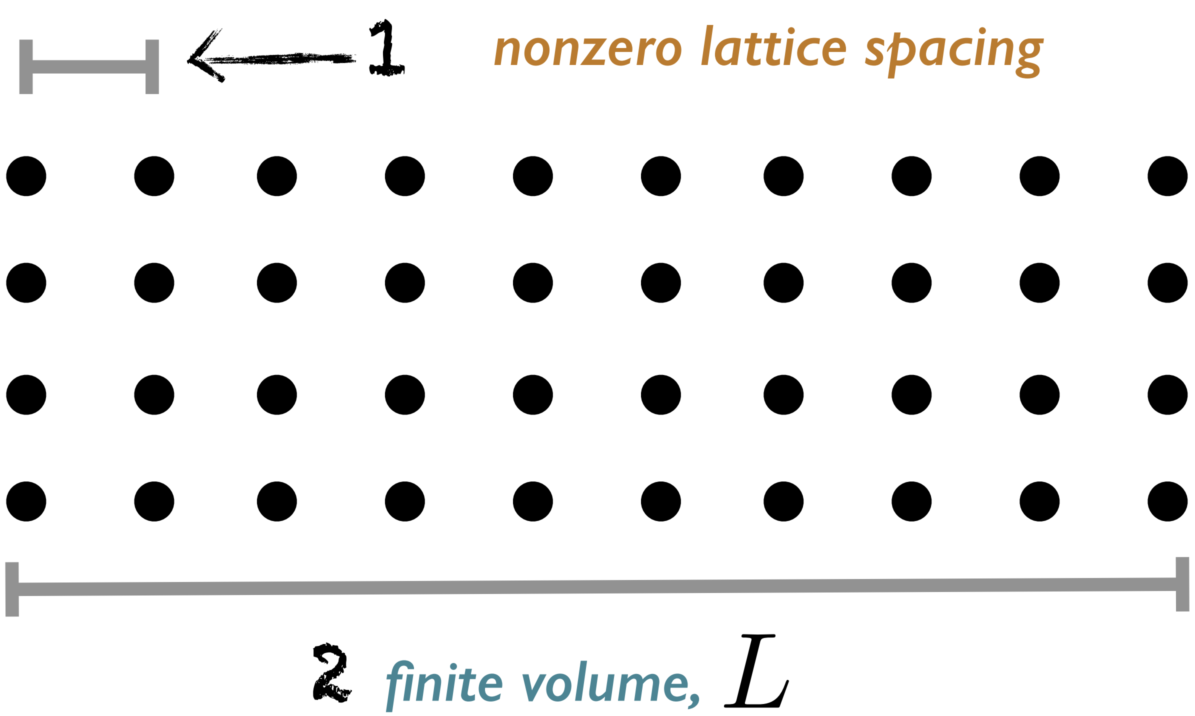
K matrix (short distance)

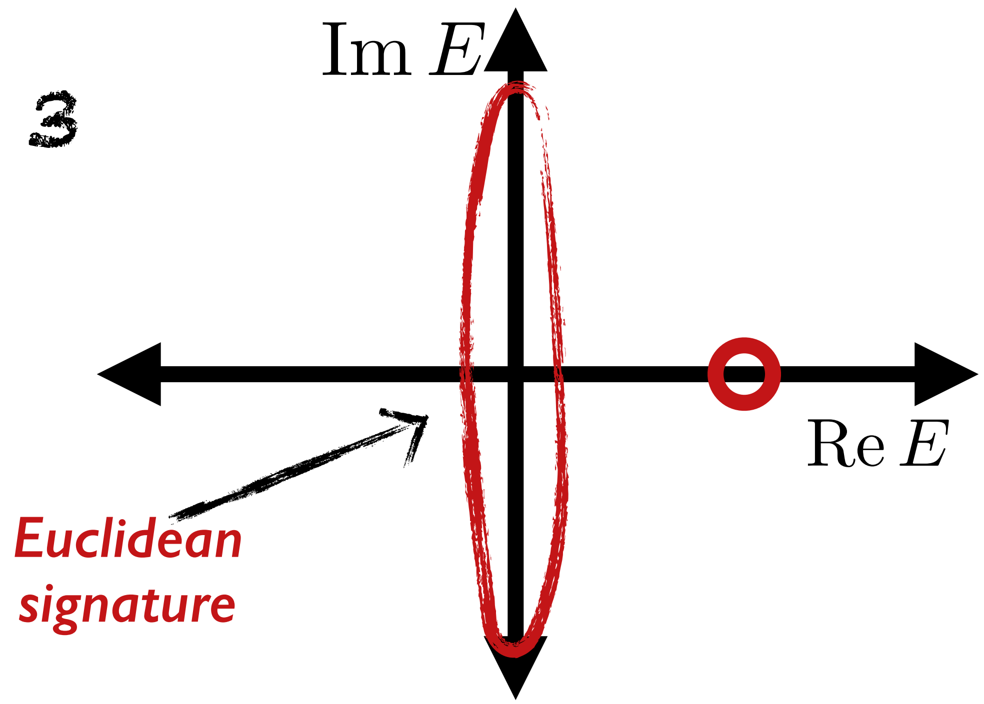
phase-space cut (long distance)

Lattice QCD

$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

To proceed we have to make *three modifications*



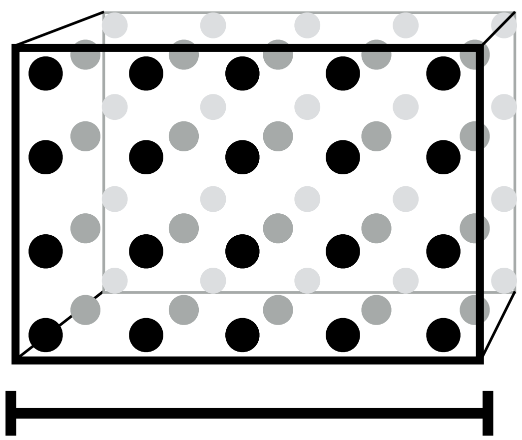


Also… $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

(but physical masses → increasingly common)

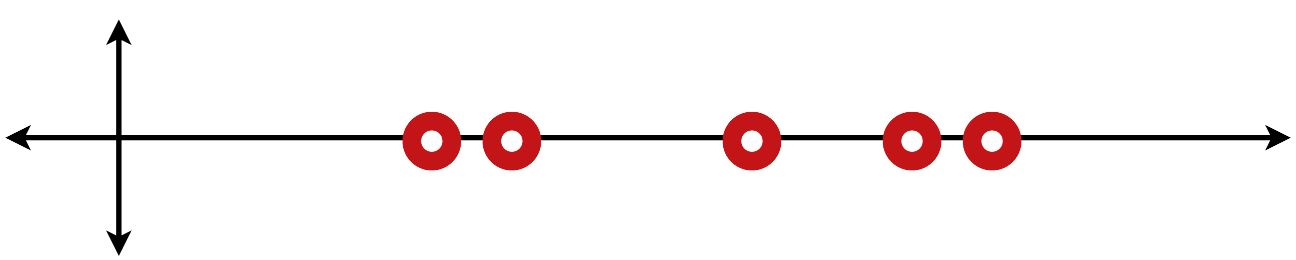
Difficulties for multi-hadron observables

- The ***Euclidean signature***…
  - ***Obscures*** real time evolution (that defines scattering)
  - ***Prevents*** normal LSZ (want $p_4^2 = -(p^2 + m^2)$, but we have only $p_4^2 > 0$)

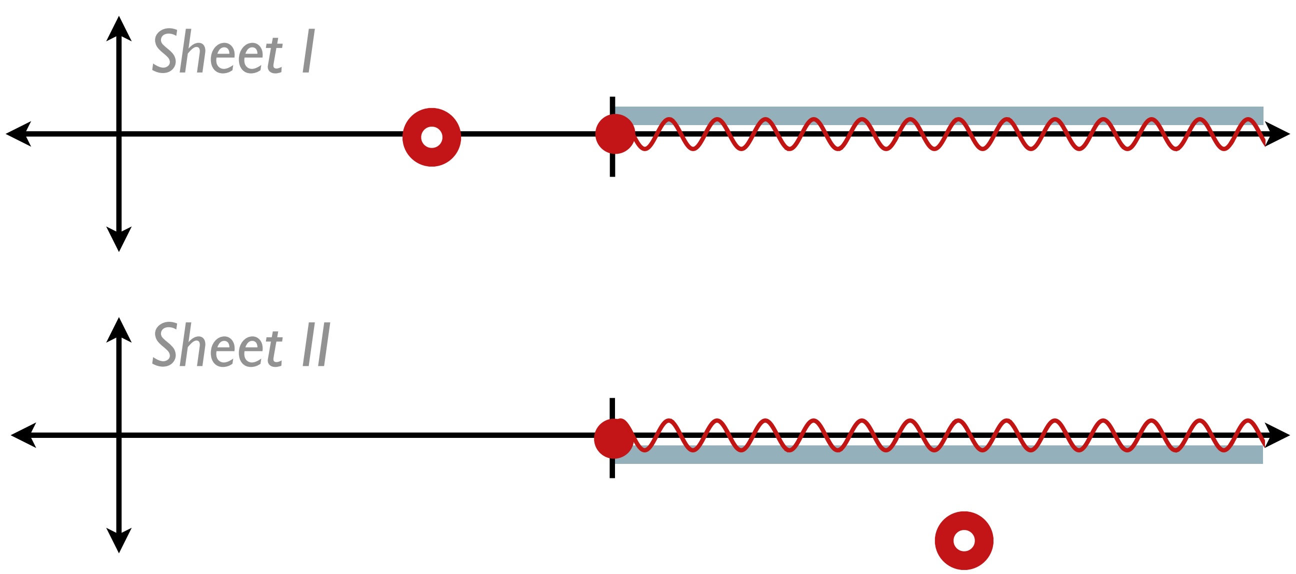


- The ***finite volume***…
  - ***Discretizes*** the spectrum
  - ***Eliminates*** the branch cuts and extra sheets
  - ***Hides*** the resonance poles

Finite-volume analytic structure



Infinite-volume analytic structure



Two strategies…

- Finite-volume as a tool
  - LQCD → Energies and matrix elements

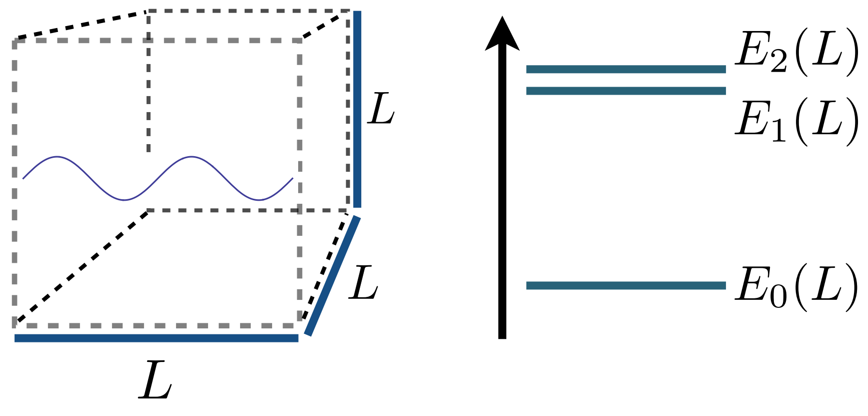
$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

  - Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***

- Spectral function method
  - An answer to… "Can't you just analytically continue?"

The finite-volume as a tool

- Finite-volume set-up

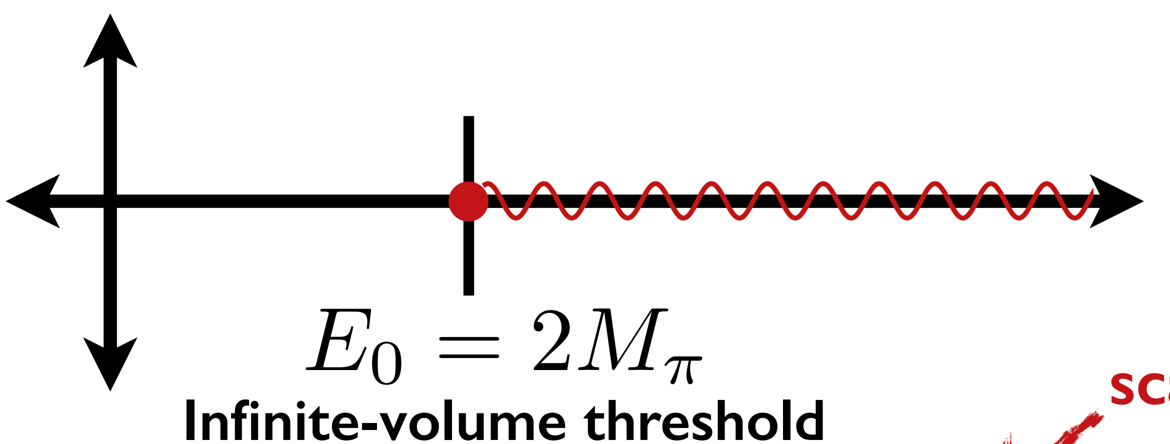


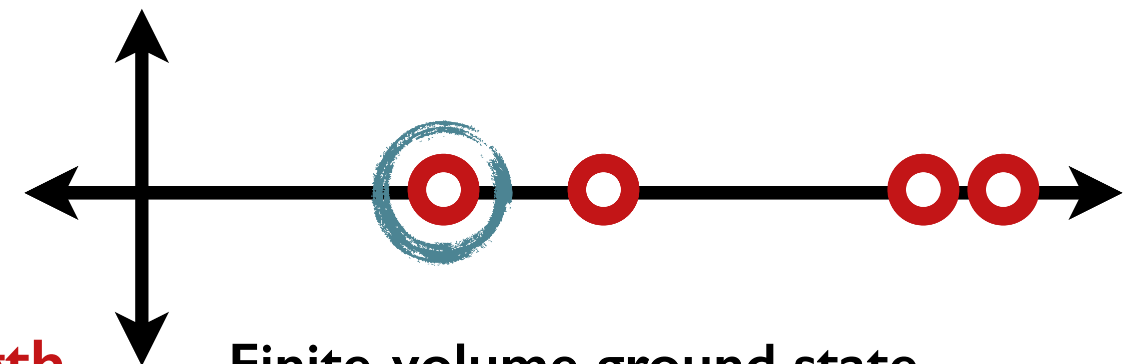
- **cubic**, spatial volume (extent *L*)
- **periodic**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \qquad \vec{n} \in \mathbb{Z}^3$$

- *L* is large enough to neglect $e^{-M_\pi L}$
- *T* and lattice also negligible

- Scattering leaves an ***imprint*** on finite-volume quantities





scattering length

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

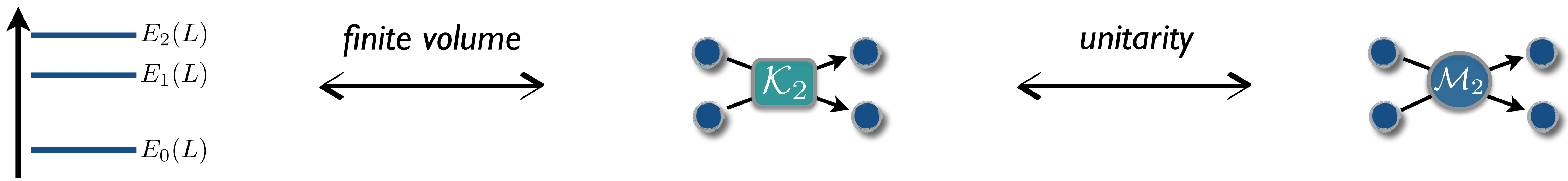
$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

General method

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

$$F(P,L) \equiv$$ Matrix of known geometric functions



Holds only for two-particle energies $s < (4m)^2$

Neglects e^{-mL}

Generalized to ***non-degenerate masses, multiple channels, spinning particles***

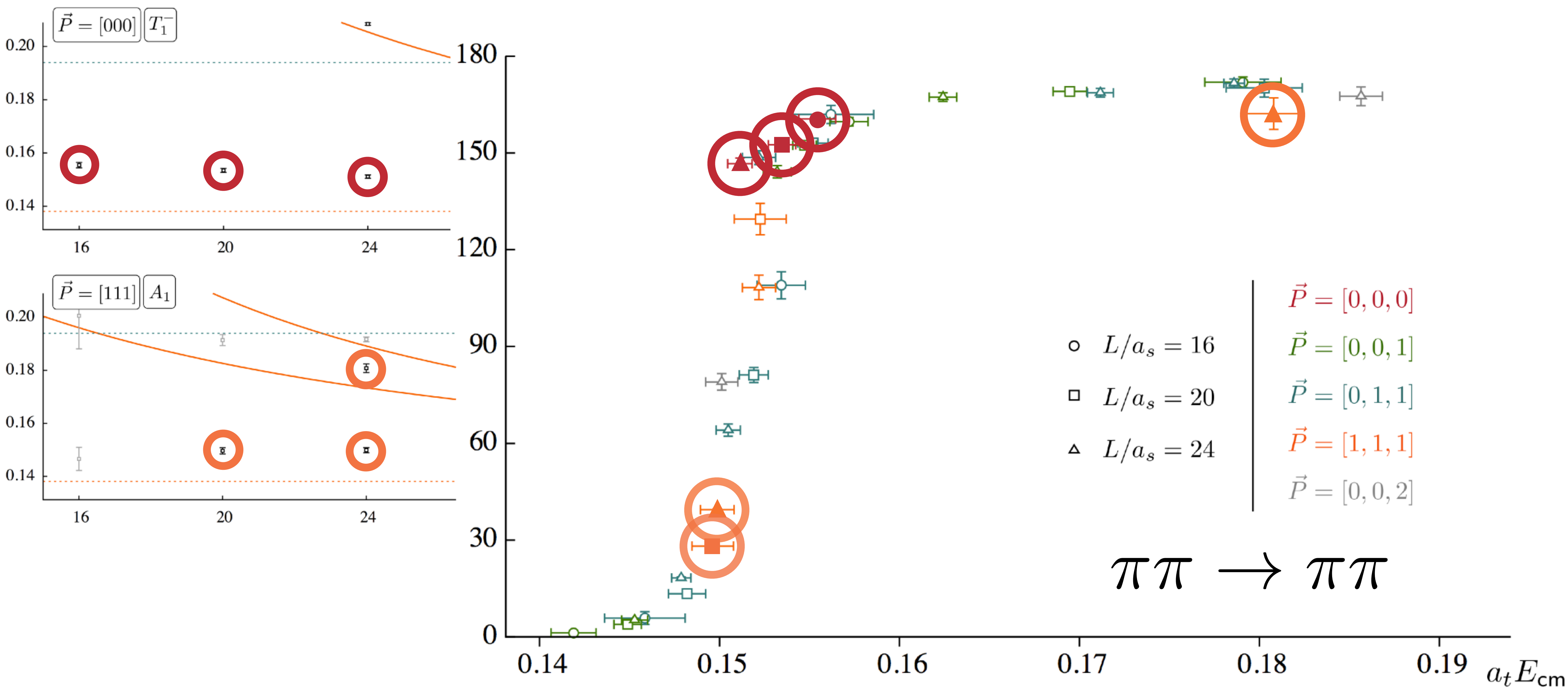
Encodes angular momentum mixing

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
Li, Liu (2013) • Briceño (2014)

Using the result

- Single-channel case *(pions in a p-wave)*

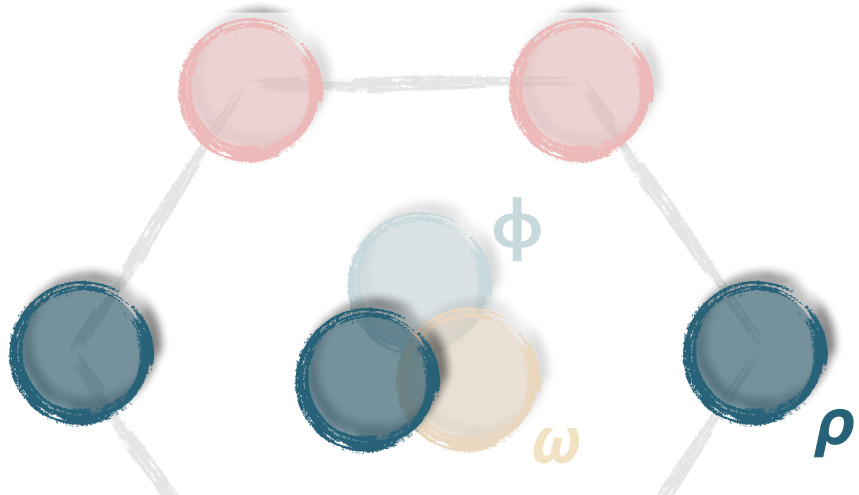
$$\mathcal{K}(s_n)^{-1} = \rho \cot\delta(s_n) = -F(E_n, \vec{P}, L)$$

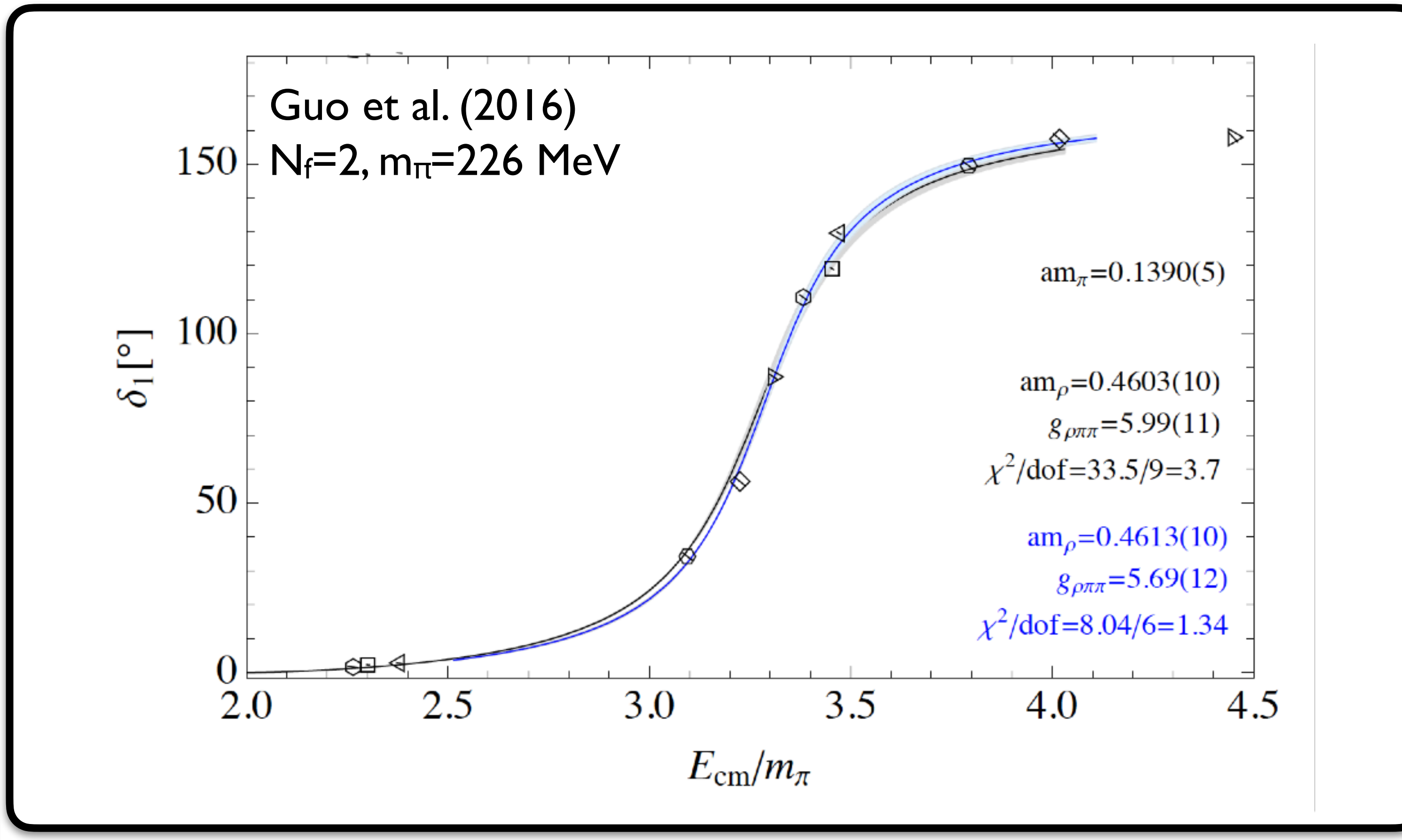


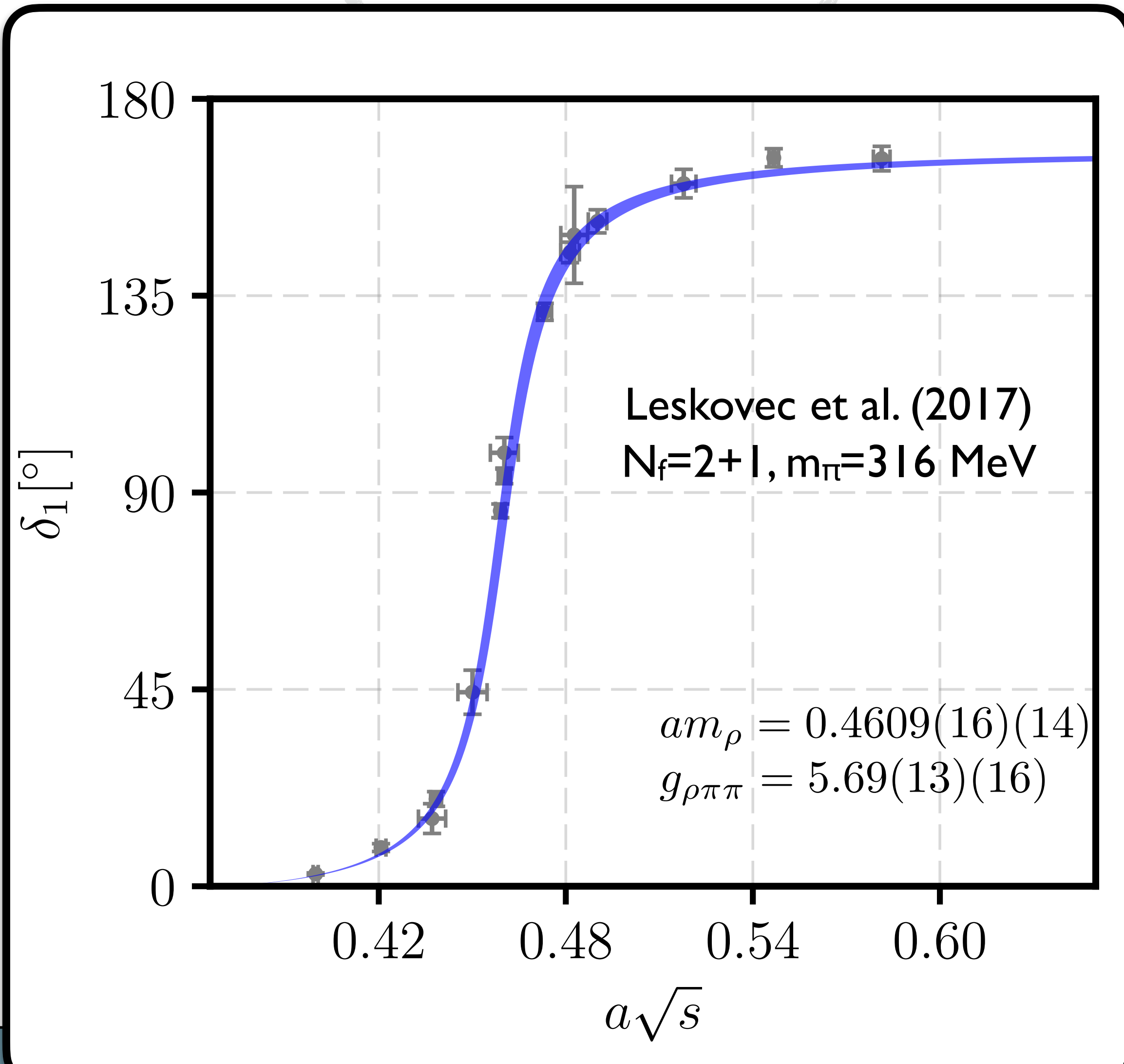
• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

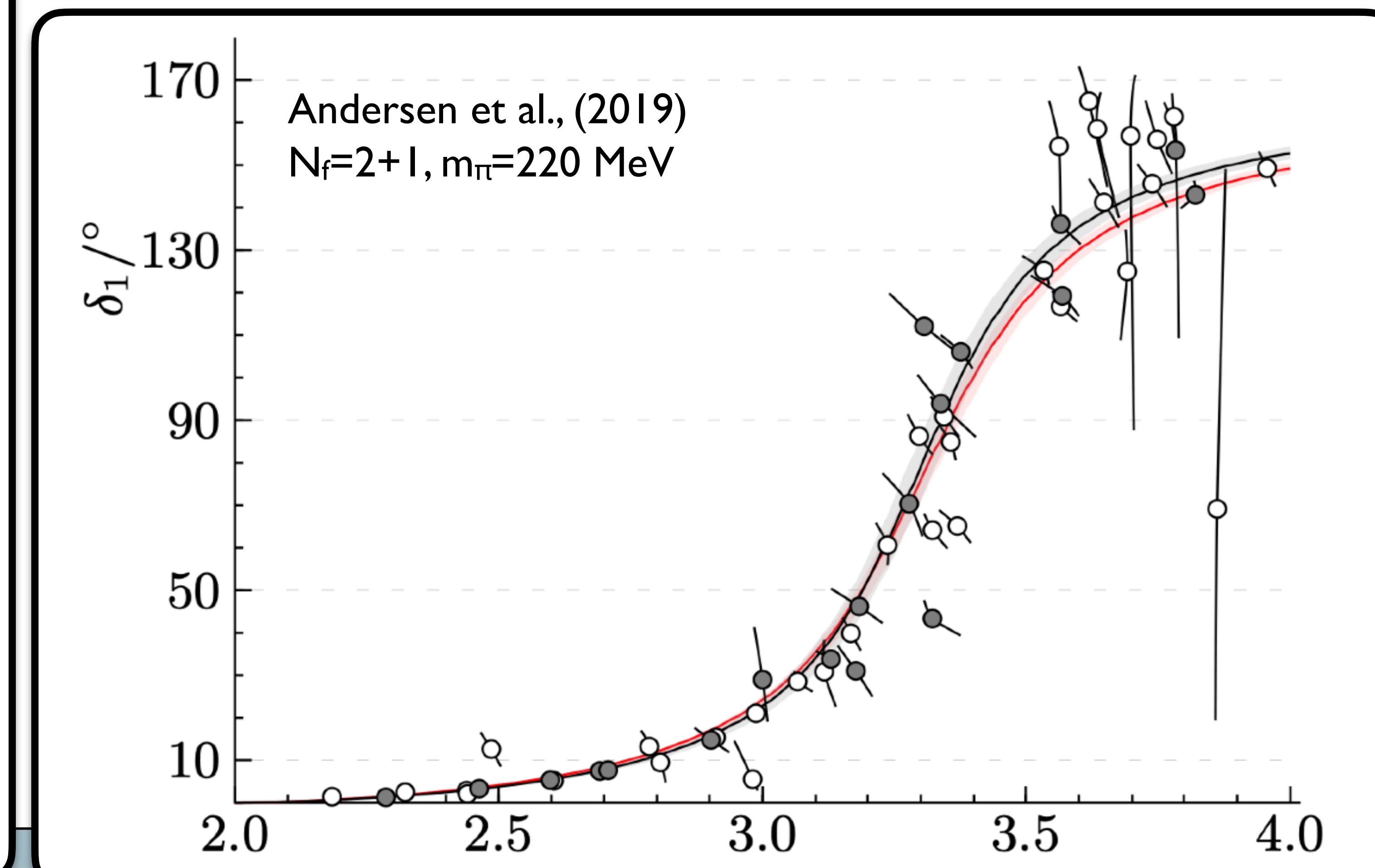
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$

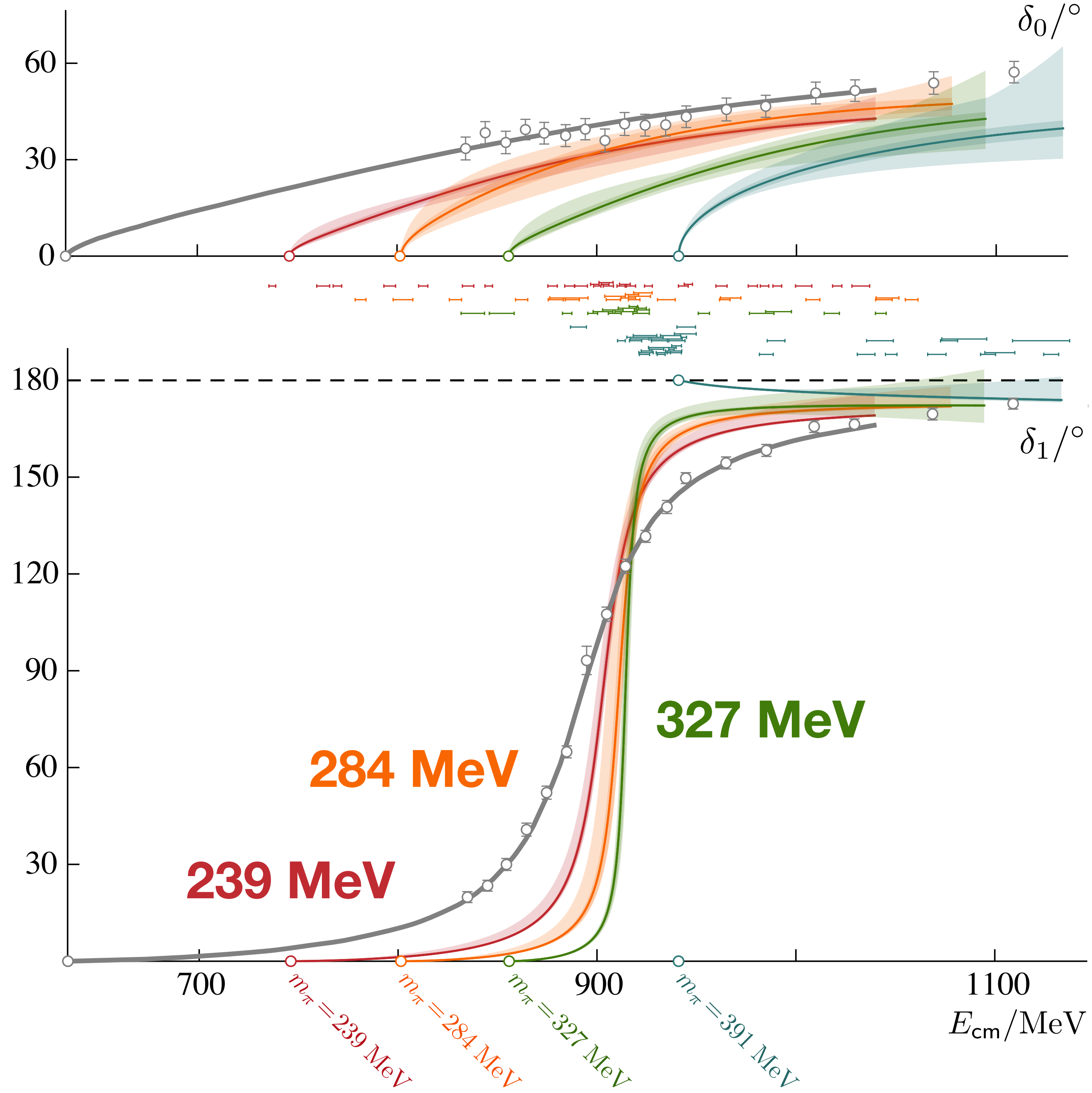


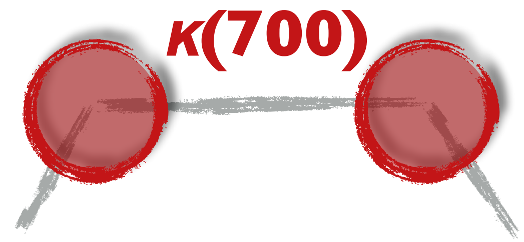




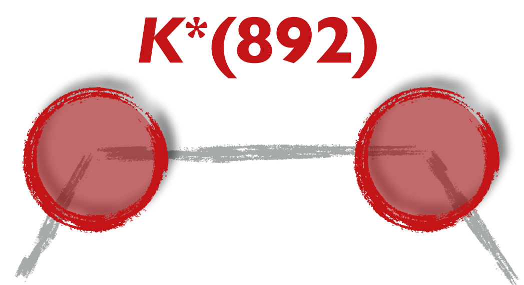


$\kappa, K^* \to K\pi$





$$I(J^P) = 1/2(0^+)$$

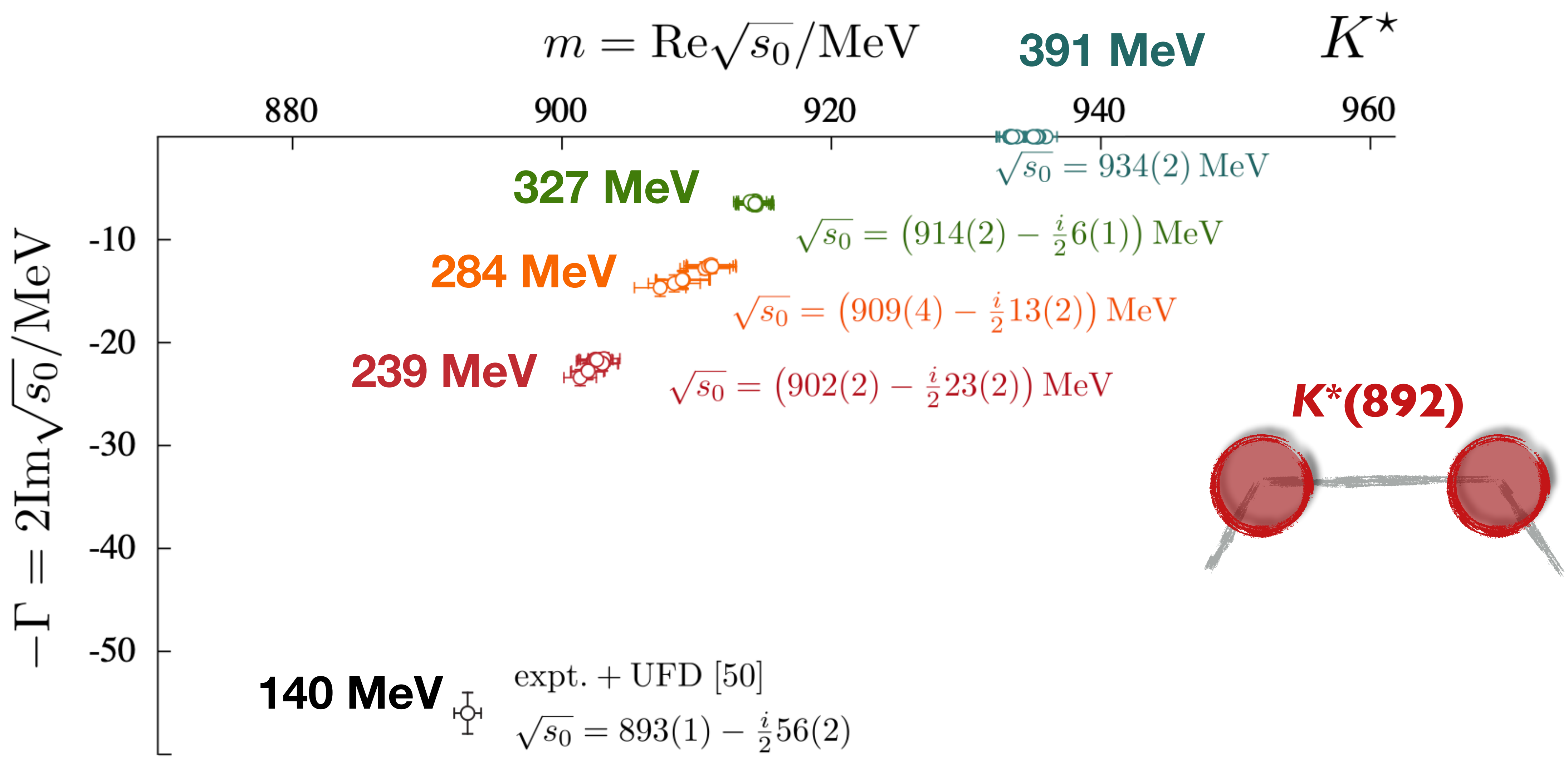


$$I(J^P) = 1/2(1^-)$$

- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002* •

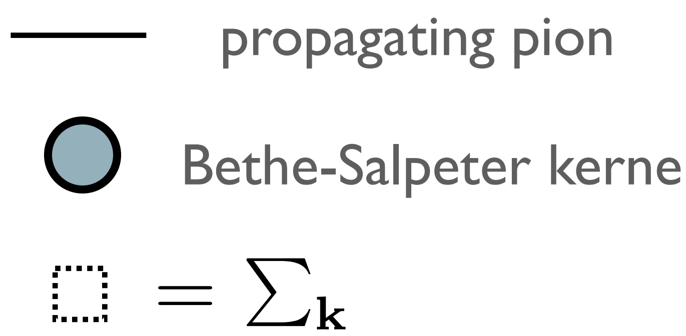
$\kappa, K^* \to K\pi$

$I(J^P) = 1/2(1^-)$

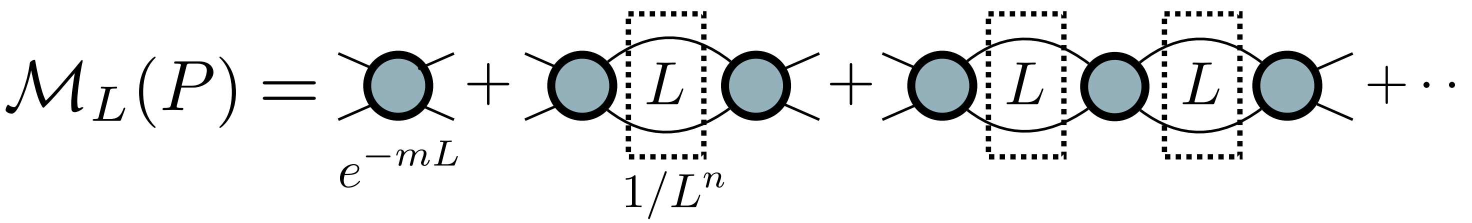


- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002*

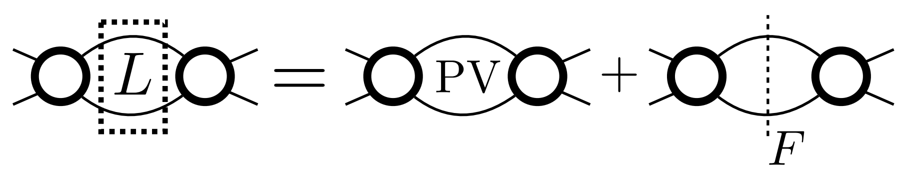
Derivation



- Consider the finite-volume correlator:

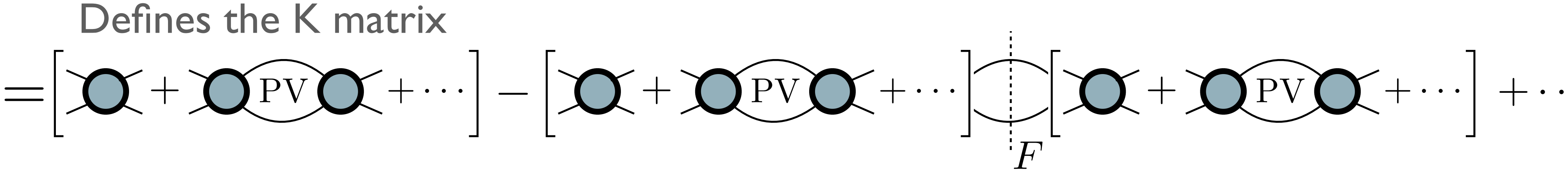


For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?

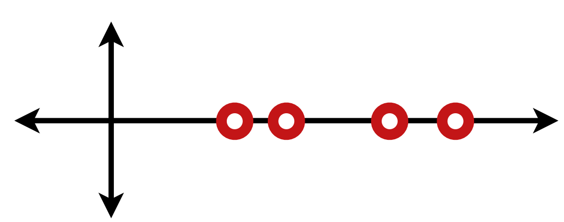


F = matrix of known geometric functions

Defines the K matrix



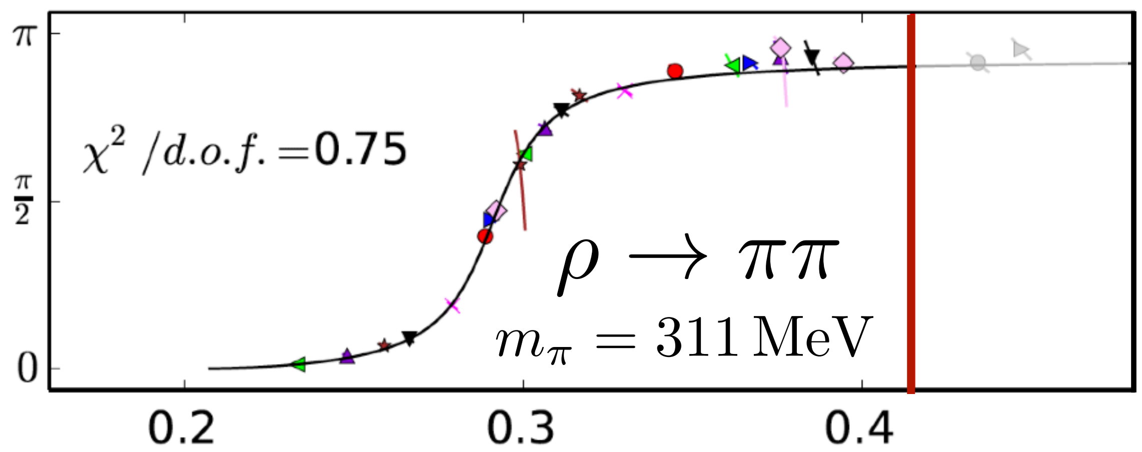
$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$



$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

- Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (*coupled channels*, 2012) •

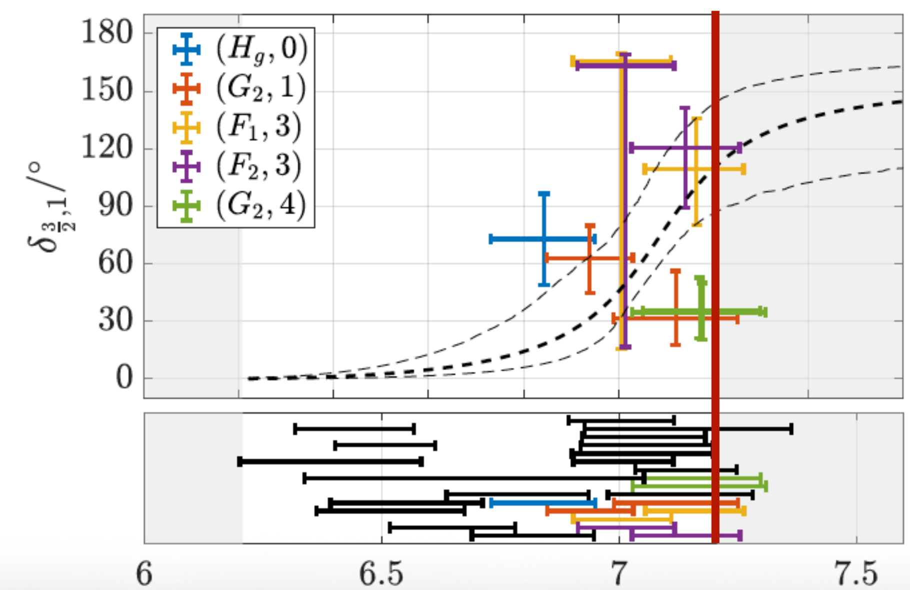
Multi-particle barrier



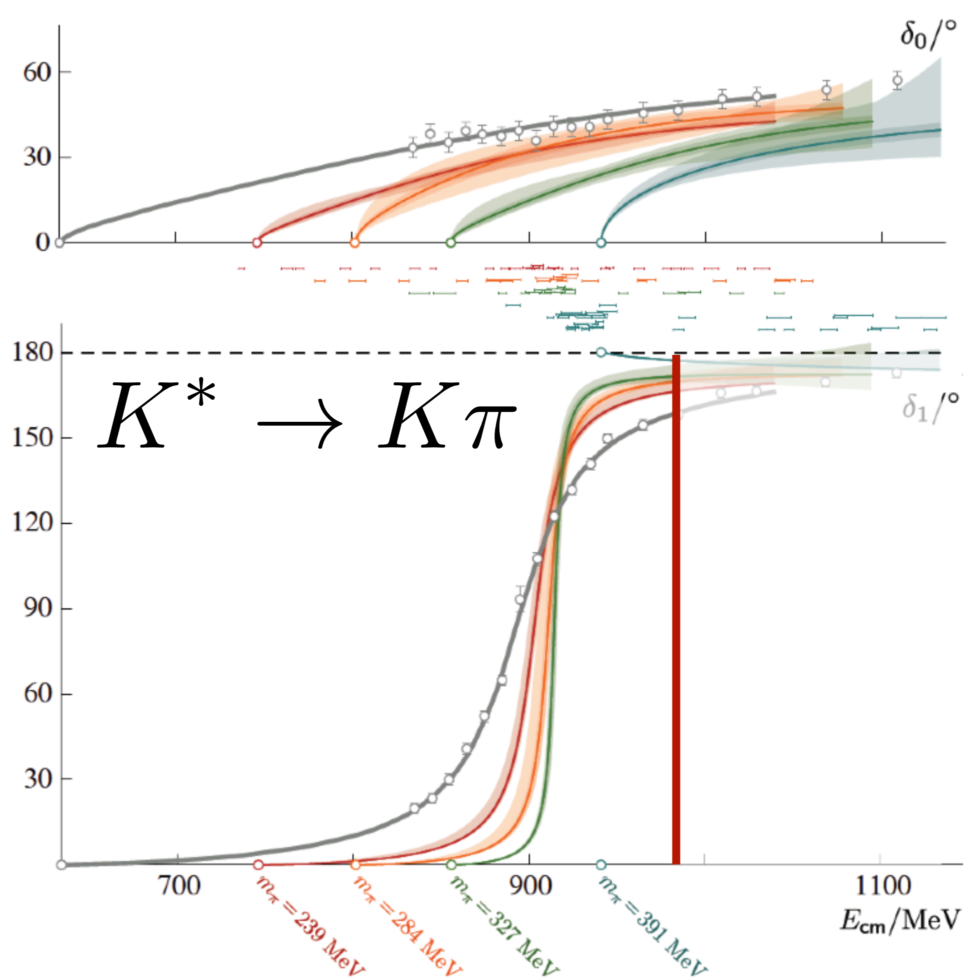
• Erben et al. (2019) •

$\Delta \to N\pi$

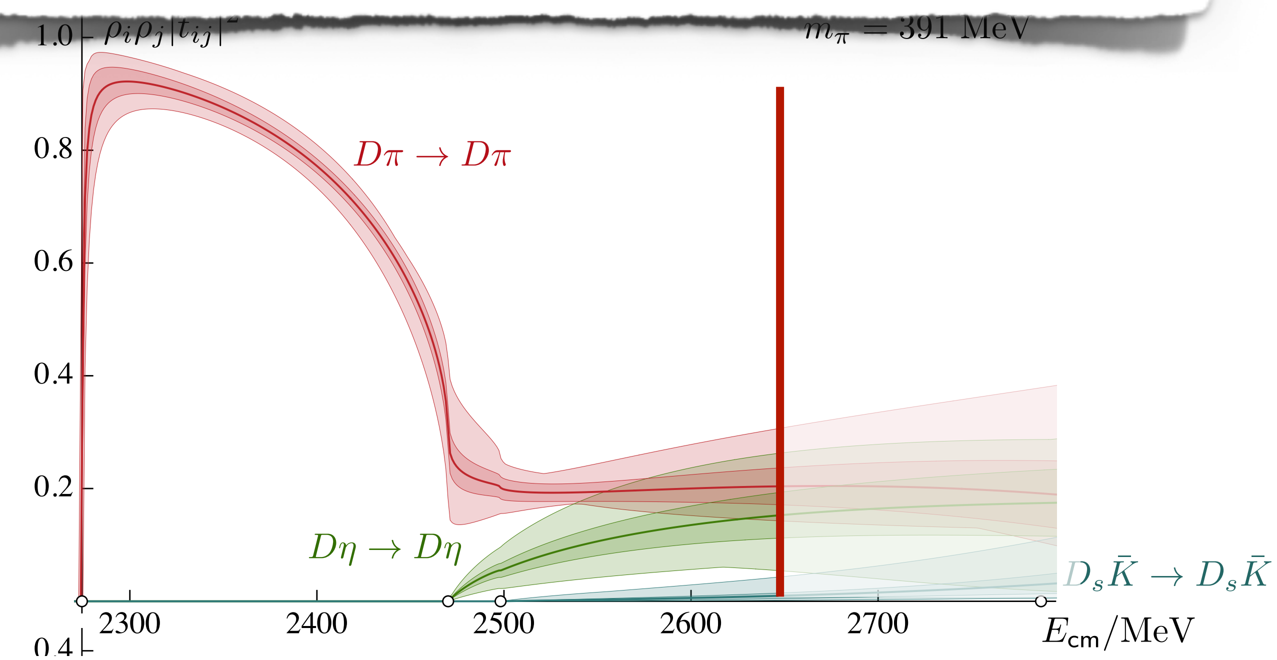
$m_\pi = 200$ MeV



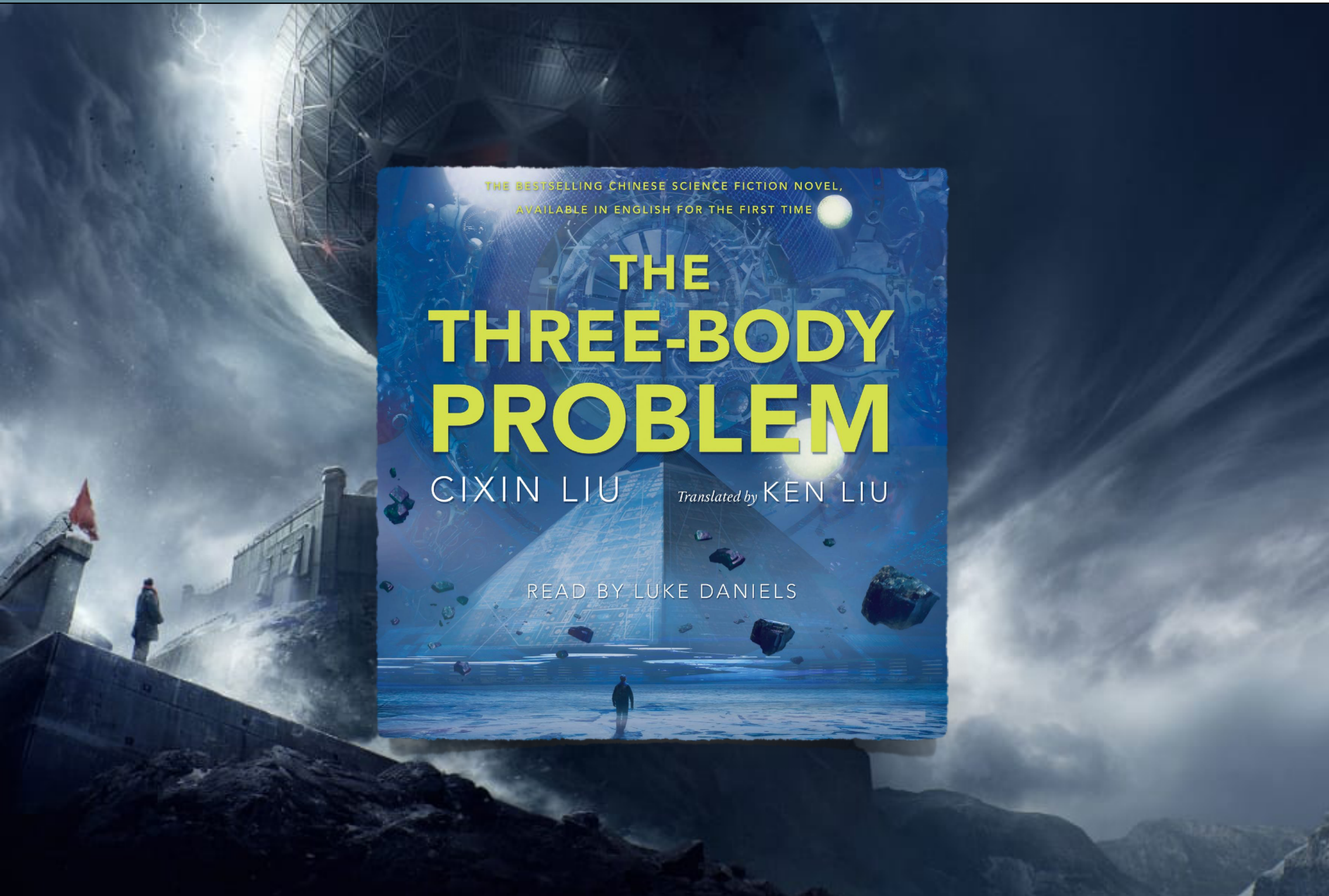
Formalism only strictly holds up to lowest 3- or 4-particle threshold



• Wilson et al. *PRL123* (2019) •



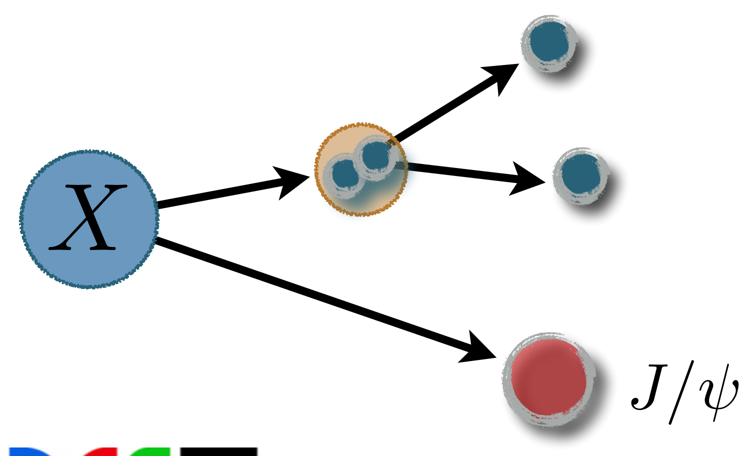
• Moir et al. *JHEP 1610* (2016) •


THE BESTSELLING CHINESE SCIENCE FICTION NOVEL,
AVAILABLE IN ENGLISH FOR THE FIRST TIME
THE THREE-BODY PROBLEM
CIXIN LIU
Translated by KEN LIU
READ BY LUKE DANIELS

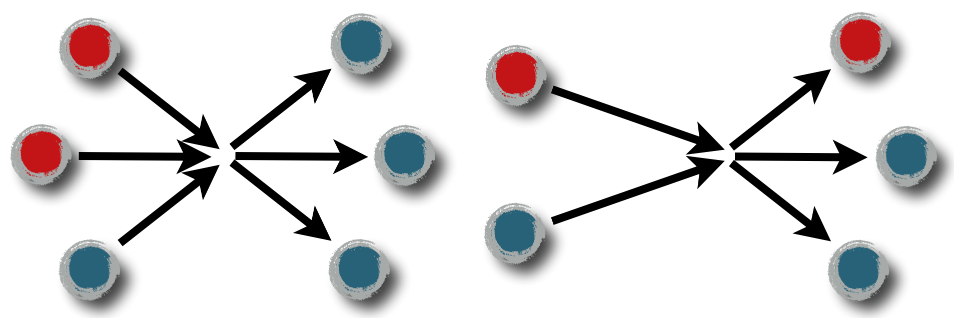
3-particle amplitudes

2-to-2 only samples J^P 0^+ 1^- $2^+ \cdots$

many interesting resonances have significant 3-body decays



Goal: ***finite-volume + unitarity formalism*** for generic two- and three-particle systems



Applications…

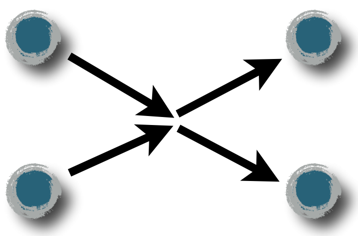
exotic resonance pole positions, couplings, quantum numbers

$\omega(782),\, a_1(1420) \to \pi\pi\pi$ $\quad X(3872) \to J/\psi\pi\pi$ $\quad X(3915)[Y(3940)] \to J/\psi\pi\pi$

form factors and transitions

and much more!… *(3-body forces, weak transitions, gluons content)*

Complication: degrees of freedom

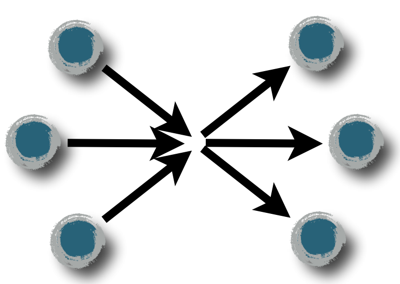


12 momentum components

-10 Poincaré generators

2 degrees of freedom

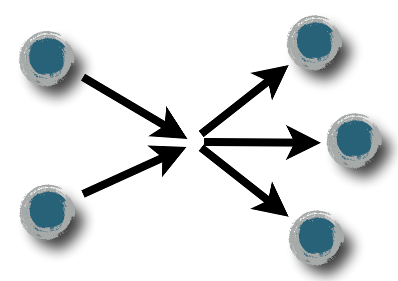
$$\vec{p}_1 + \vec{p}_2 \ \rightarrow \ \vec{p}_3 + \vec{p}_4 \quad \longrightarrow \quad \text{Mandelstam } s, t$$



18 momentum components

-10 Poincaré generators

8 degrees of freedom



15 momentum components

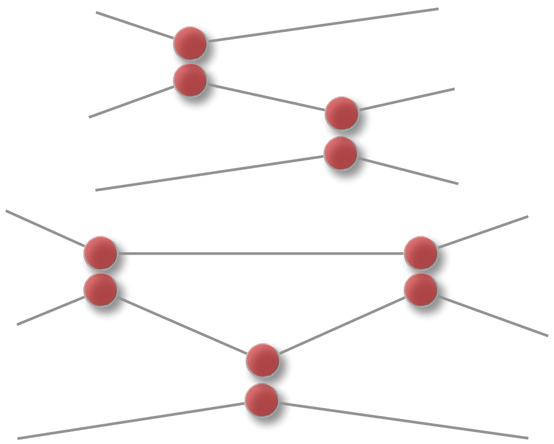
-10 Poincaré generators

5 degrees of freedom

Complication: on-shell states

- Classical pairwise scattering

for $m_1 = m_2 = m_3$ up to ***3*** binary collisions are possible



Dispersion Relations for Three-Particle Scattering Amplitudes. I*

MORTON RUBIN
Physics Department, University of Wisconsin, Madison, Wisconsin
AND
ROBERT SUGAR
Physics Department, Columbia University, New York, New York
AND
GEORGE TIKTOPOULOS
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received 31 January 1966)

$$b=\frac{(m_1+m_3)(m_2+m_3)}{m_1m_2}$$

It follows that if

$$b^{n-2}(b-1)>1\,, \qquad \text{(IV.18)}$$

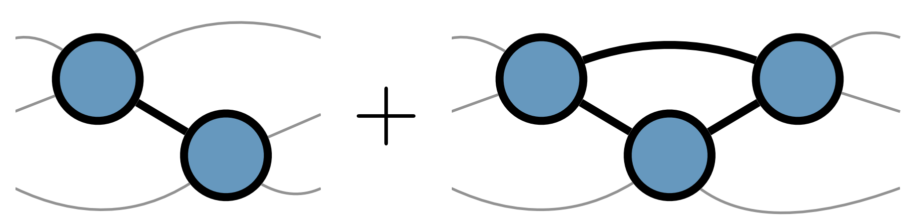
then $2n+1$ successive binary collisions are kinematically impossible.

$m_1 = m_2 = m_3 - \varepsilon$:
4 collisions possible
$\pi\pi K$

$b < 2$
5 collisions possible
$\pi K K$

- Correspond to Landau singularities

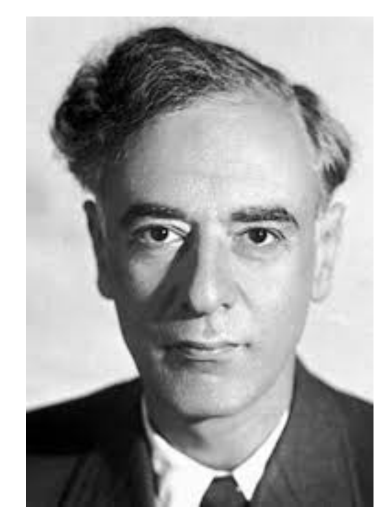
$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator $=$



$+\cdots$

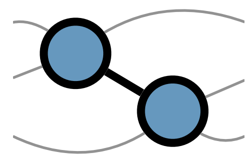
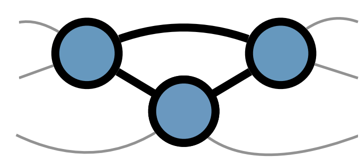
complicate analyticity & unitarity

difficult to disentangle kinematic singularities from resonance poles



Two key observations

- Intermediate $K_{\text{df},3}$ removes singularities

$\mathcal{K}_{\text{df},3} \equiv$ fully connected diagrams w/ ***PV pole prescription*** $-$  $+$  $+ \cdots$

same degrees of freedom as *M₃*

smooth real function

relation to M_3 = known

- $K_{\text{df},3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\text{df},3}(p_3, p_2, p_1; k_3, k_2, k_1) = \mathcal{K}_{\text{df},3}^{\text{iso},0} + \mathcal{K}_{\text{df},3}^{\text{iso},1}\,\Delta + \cdots \qquad \Delta = \frac{s-(3m)^2}{(3m)^2}$$

smooth real function

analogous to effective range expansion

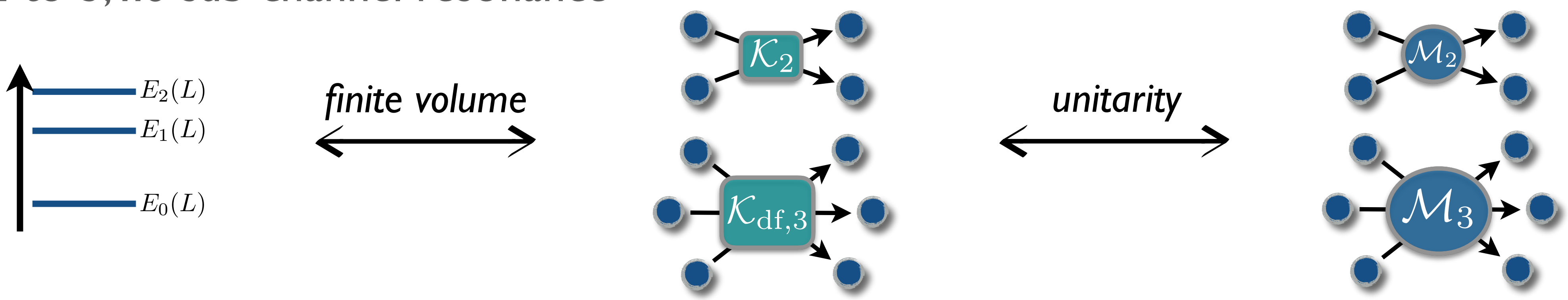
$$p\cot\delta = -\frac{1}{a} + \frac{1}{2}\,r\,p^2 + \mathcal{O}(p^4)$$

gives handle on many *degrees of freedom*
(DOFs enter order by order)

Status…

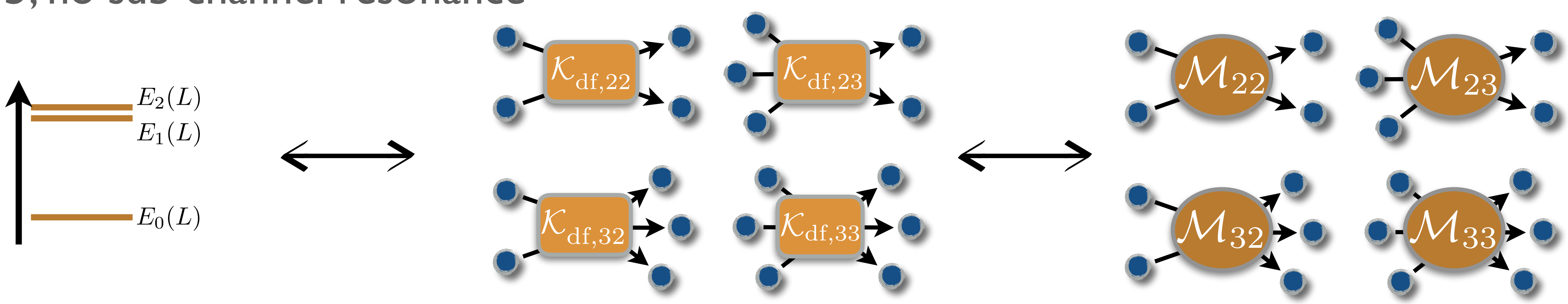
- General relation between *energies* and *two-and-three scalar scattering*

No **2-to-3,** ***no*** **sub-channel resonance**



• MTH, Sharpe (2014, 2015) •

2-to-3, no sub-channel resonance



• Briceño, MTH, Sharpe (2017) •

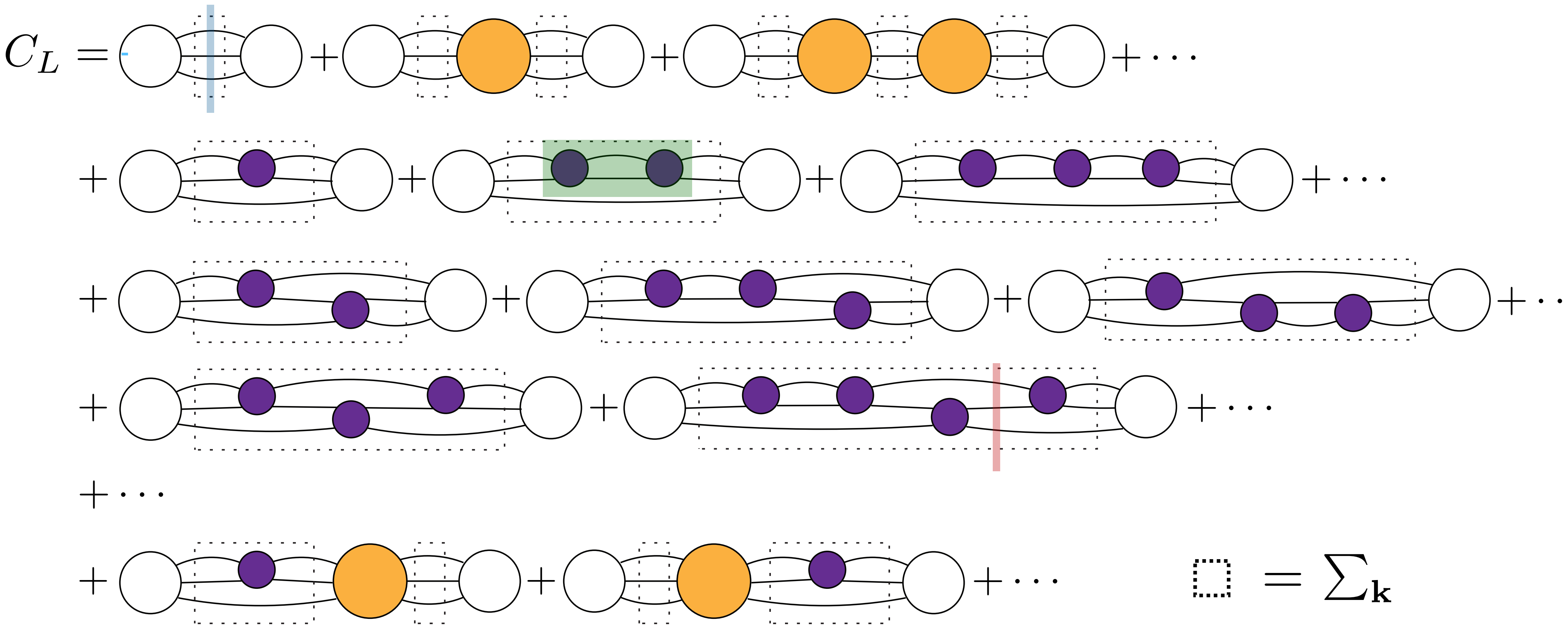
Including sub-channel resonances + *different isospins* + *non-degenerate*

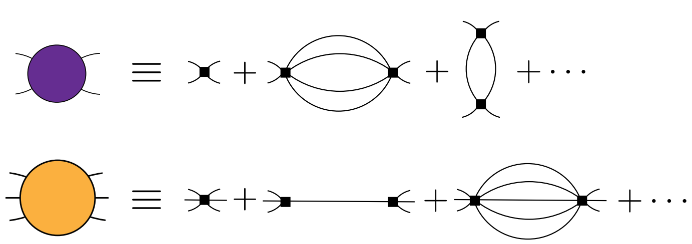
$$\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$$

• Briceño, MTH, Sharpe (2018) • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020)

3-particle derivation

- Study 3-body correlator in an ***all-orders skeleton expansion***





kernels have suppressed L dependence
lines = fully dressed hadrons

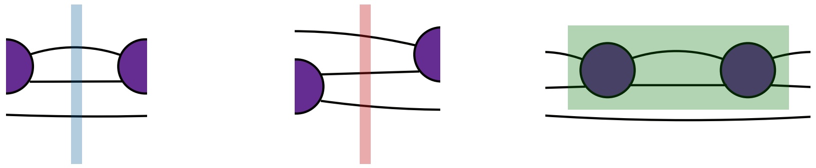
• MTH, Sharpe (2014) •

General relation

$$\det\left[\mathcal{K}_{\mathrm{df},3}^{-1}(s) + F_3(P,L|\mathcal{K}_2)\right] = 0$$

$F_3(P,L|\mathcal{K}_2) \equiv$ Matrix of functions depending on kinematics + two-particle dynamics

$$F_3 \equiv \frac{1}{3}F + F\mathcal{K}_2 \frac{1}{1-(F+G)\mathcal{K}_2} F$$



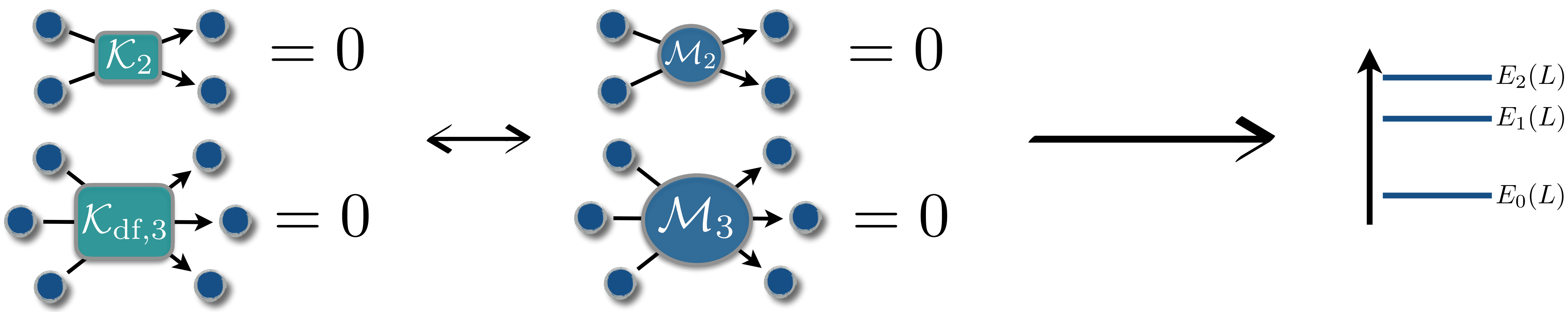
Holds only for three-particle energies

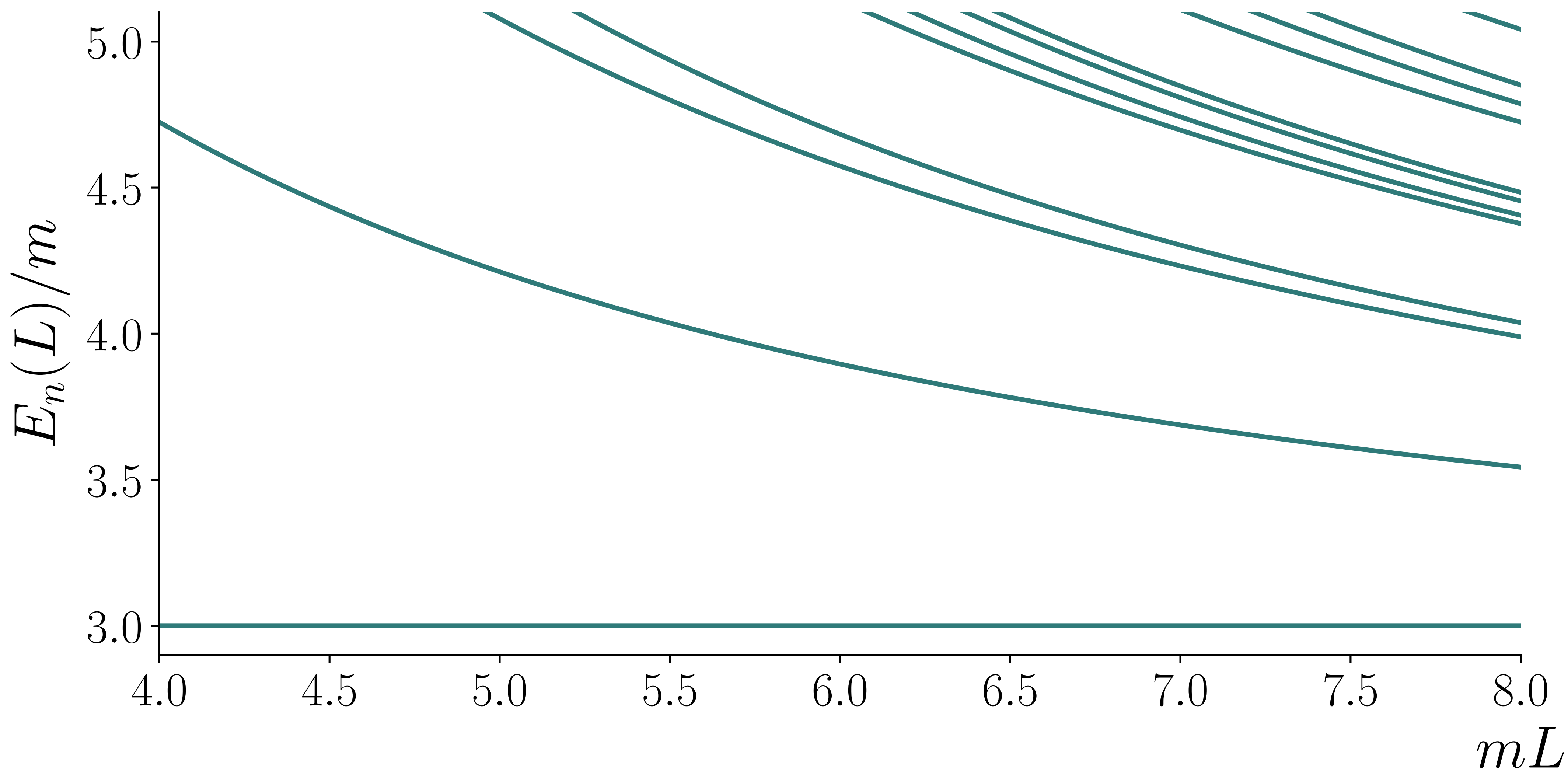
Neglects e^{-mL}

- MTH, Sharpe (2014-2016) • *See also Döring, Mai, Hammer, Pang, Rusetsky* •

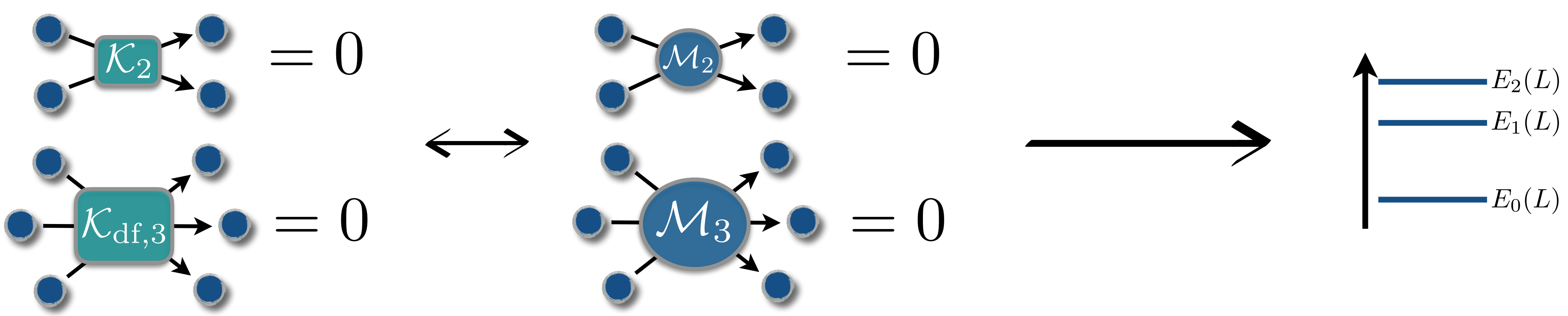
Review: *Lattice QCD and Three-particle Decays of Resonances*
MTH and Sharpe, *1901.00483*

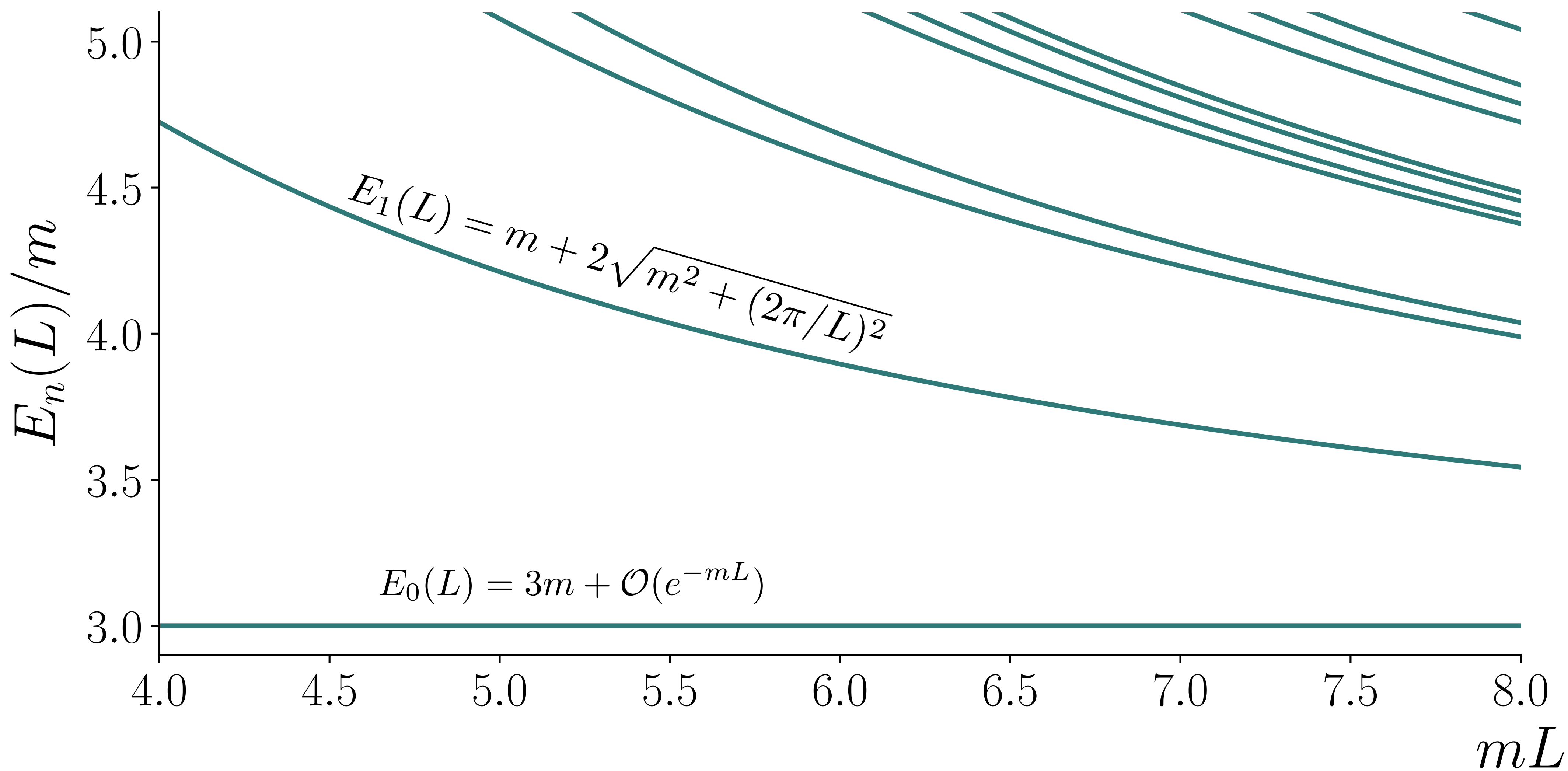
Non-interacting energies



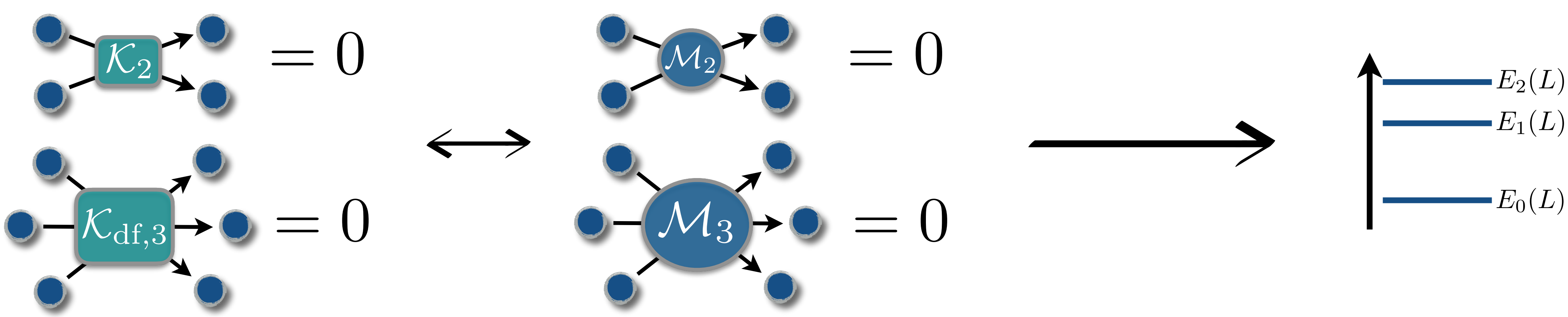


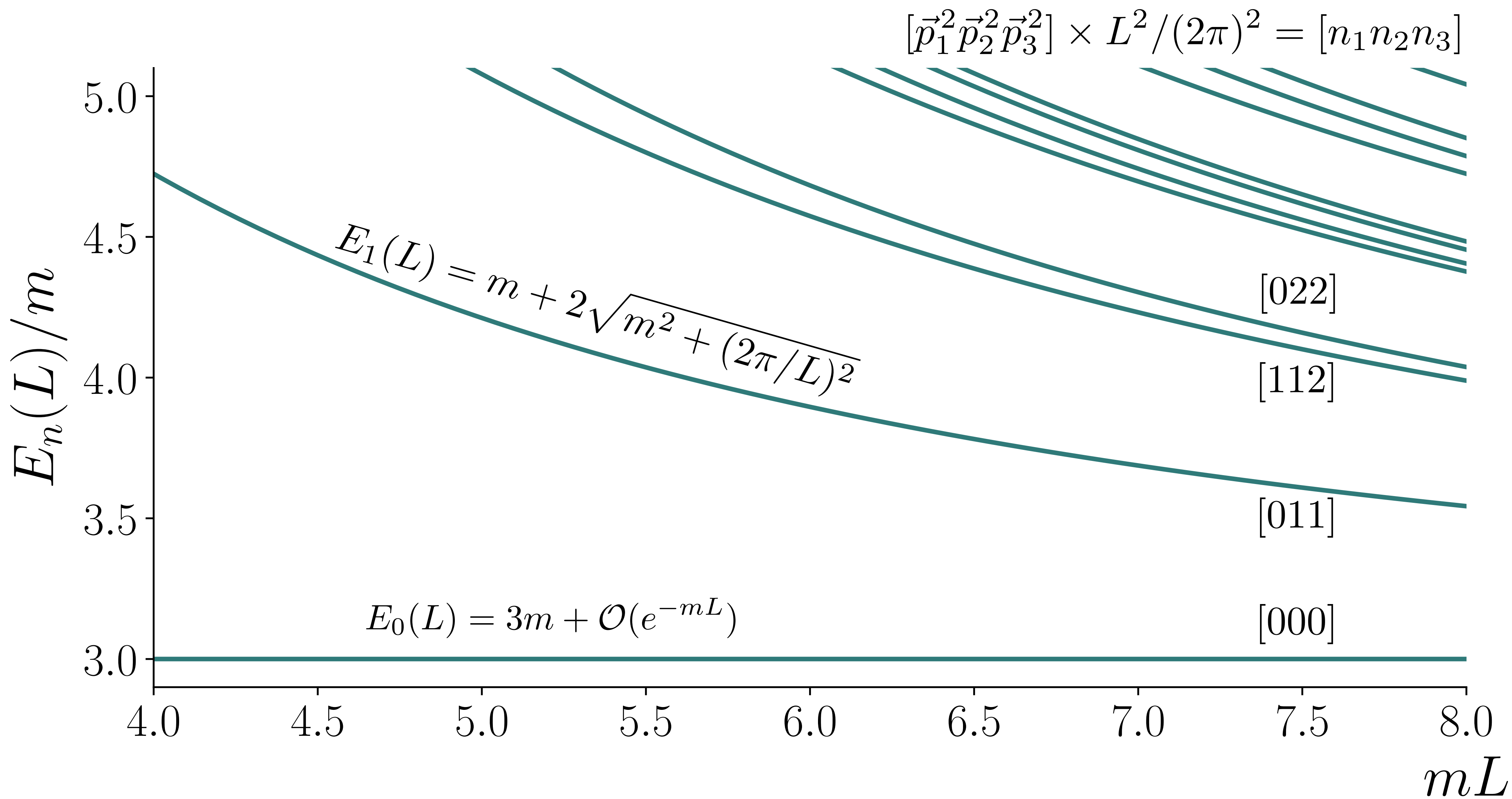
Non-interacting energies



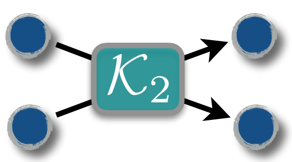


Non-interacting energies

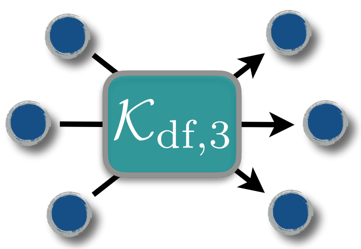




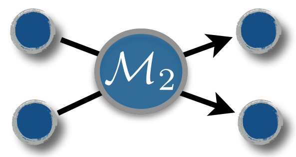
Two-particle interactions



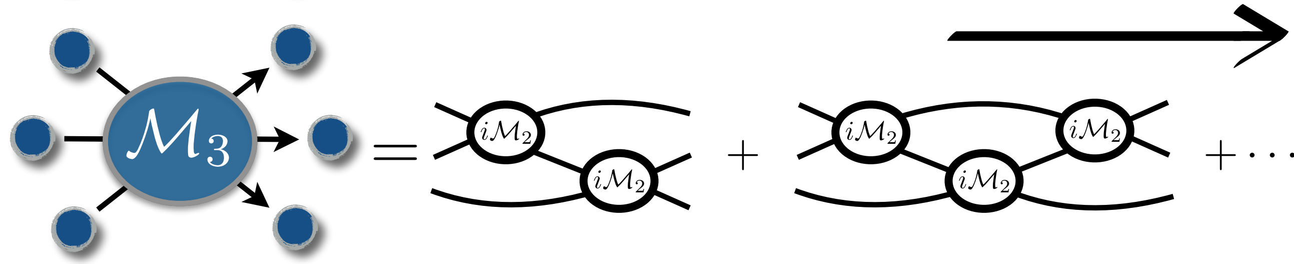
$= -16\pi\sqrt{s}\,a$

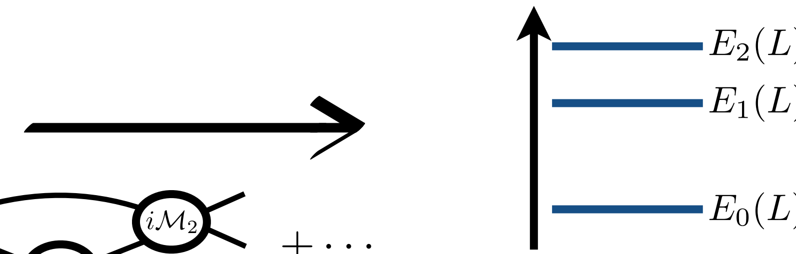


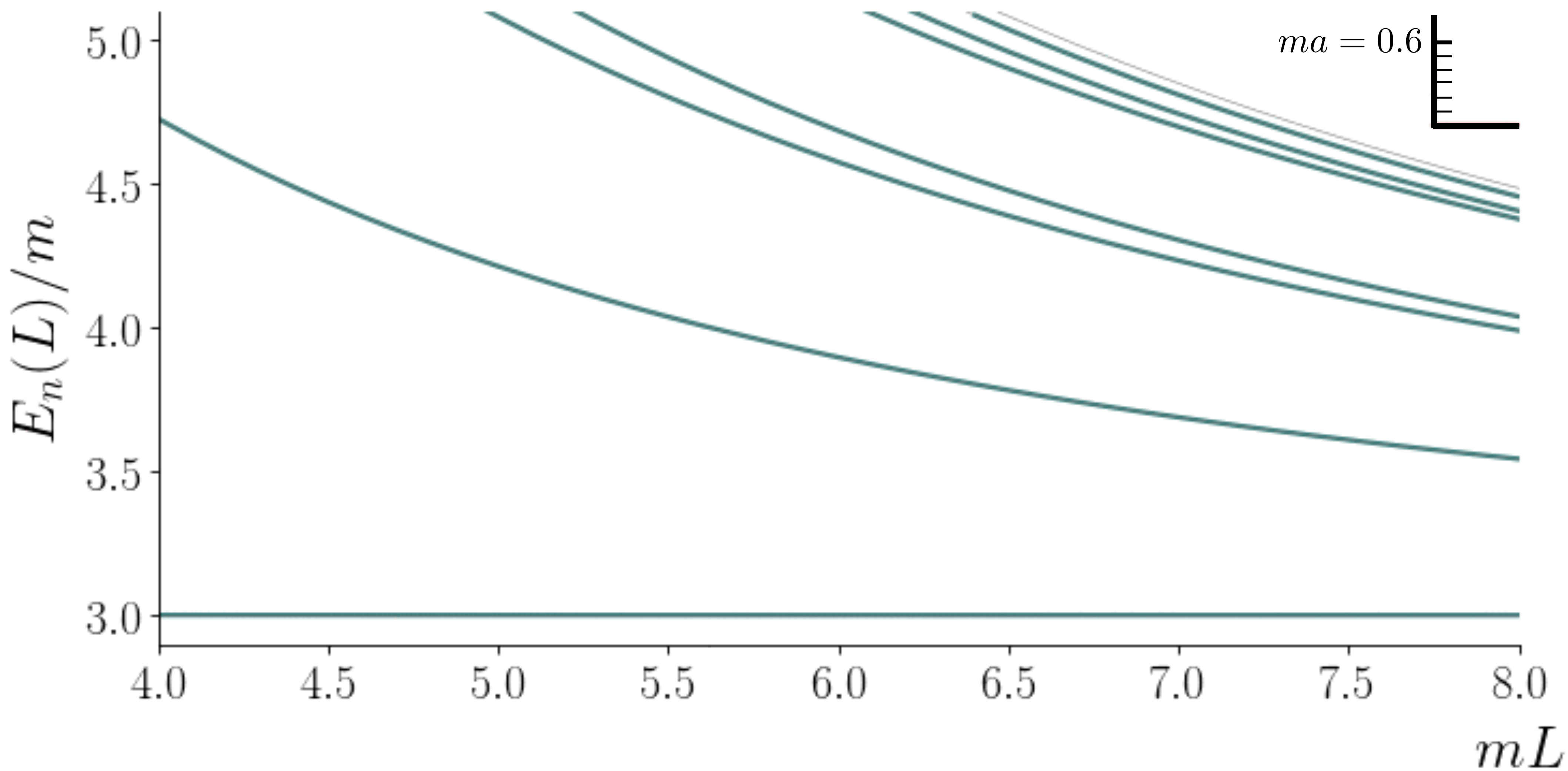
$= 0$



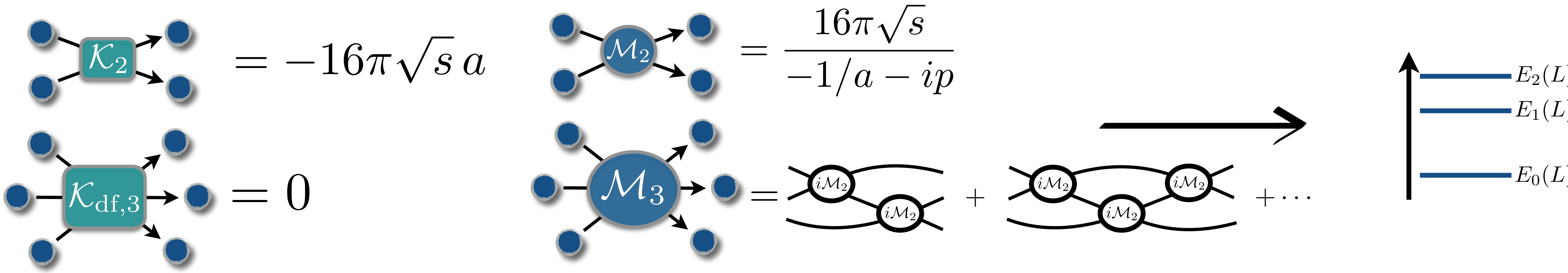
$$= \frac{16\pi\sqrt{s}}{-1/a - ip}$$

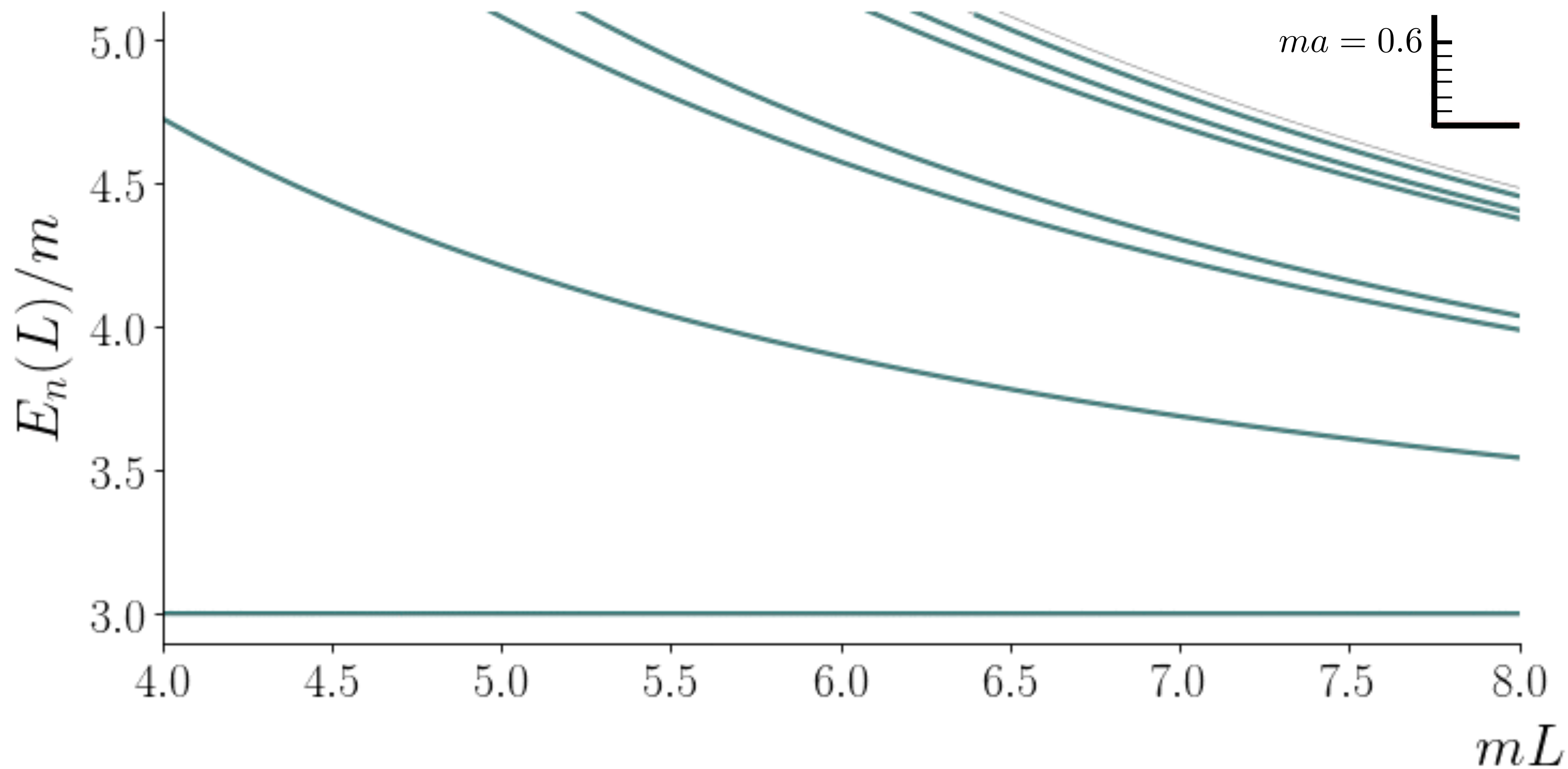






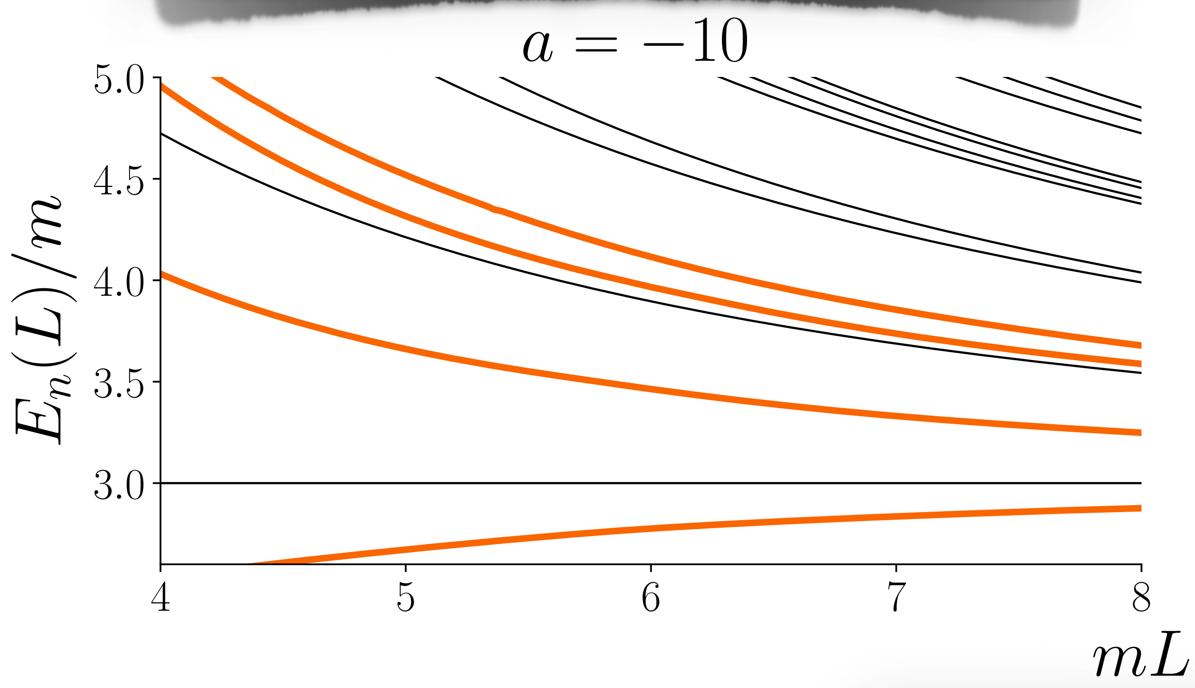
Two-particle interactions



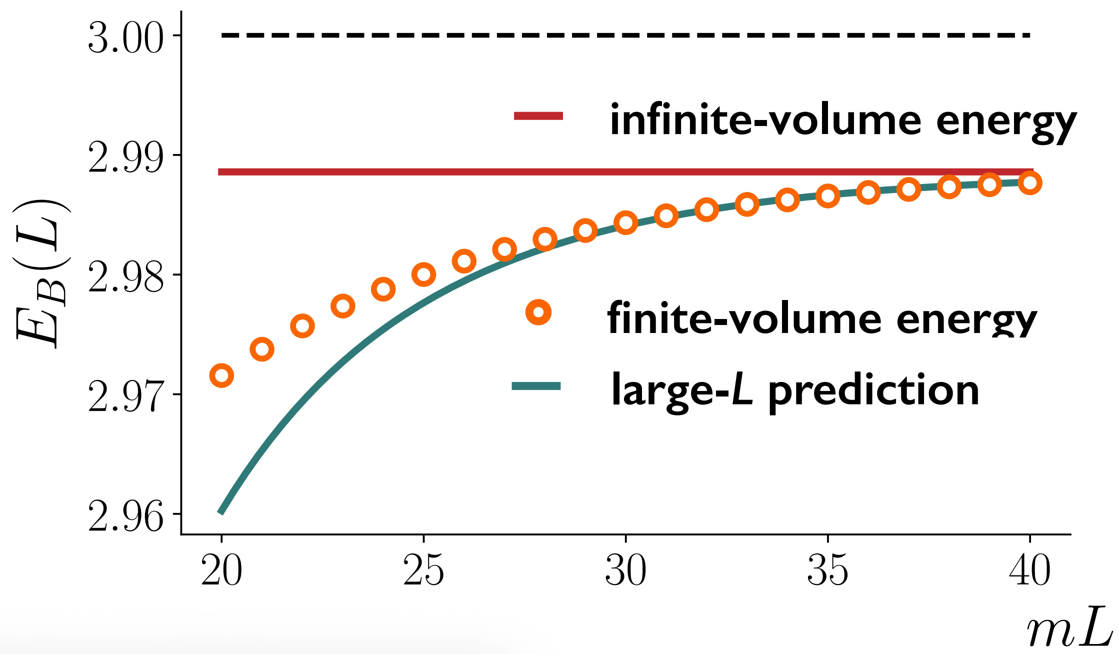


Many toy results

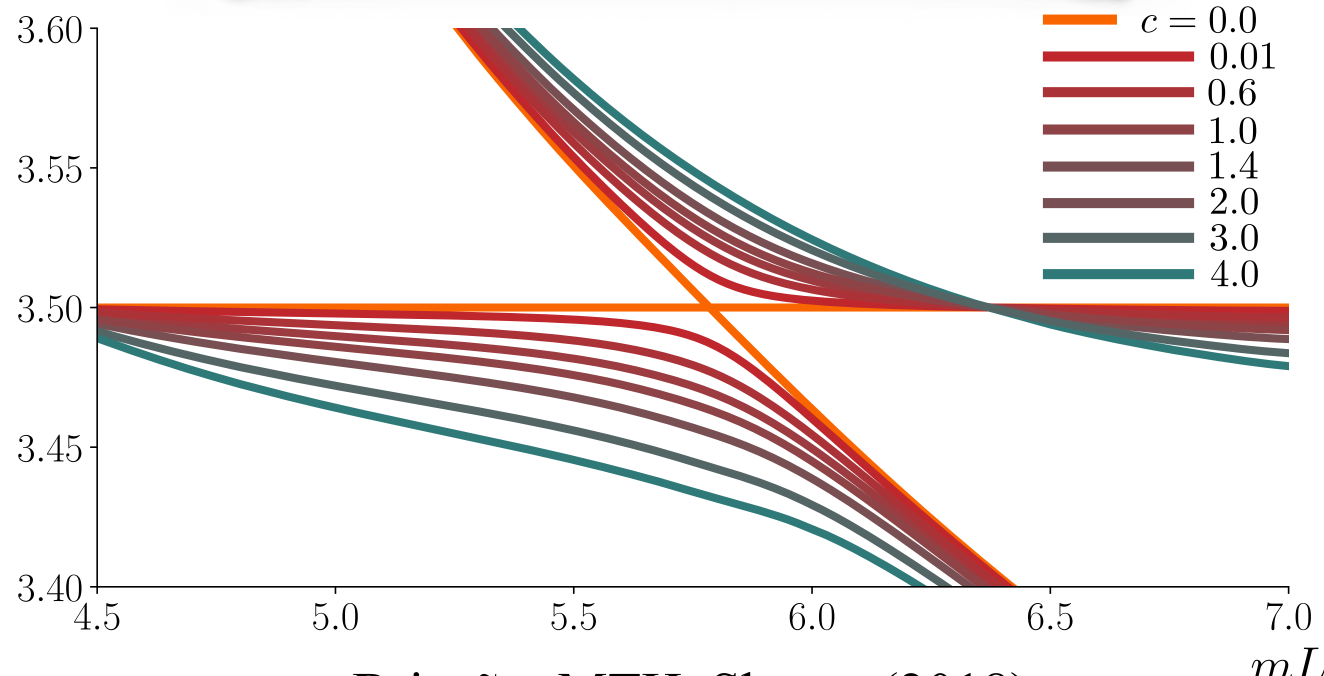
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



Model of a 3-particle resonance



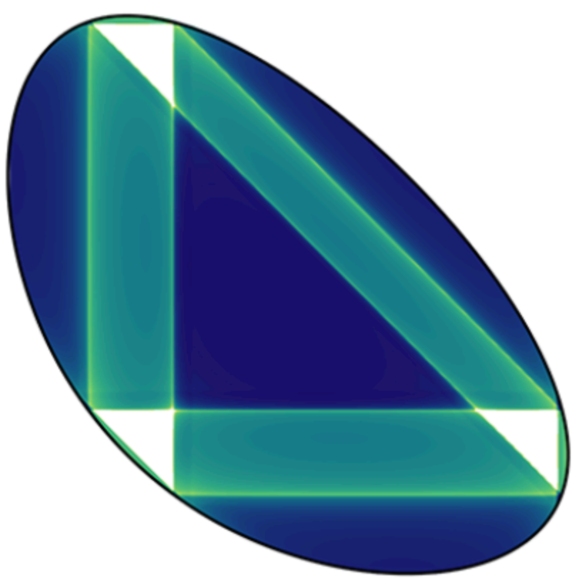
• Briceño, MTH, Sharpe (2018) •

Energy-Dependent $\pi^+\pi^+\pi^+$ Scattering Amplitude from QCD

Maxwell T. Hansen,[1,2,*] Raul A. Briceño,[3,4,†] Robert G. Edwards,[3,‡]
Christopher E. Thomas,[5,§] and David J. Wilson[5,‖]

(for the Hadron Spectrum Collaboration)



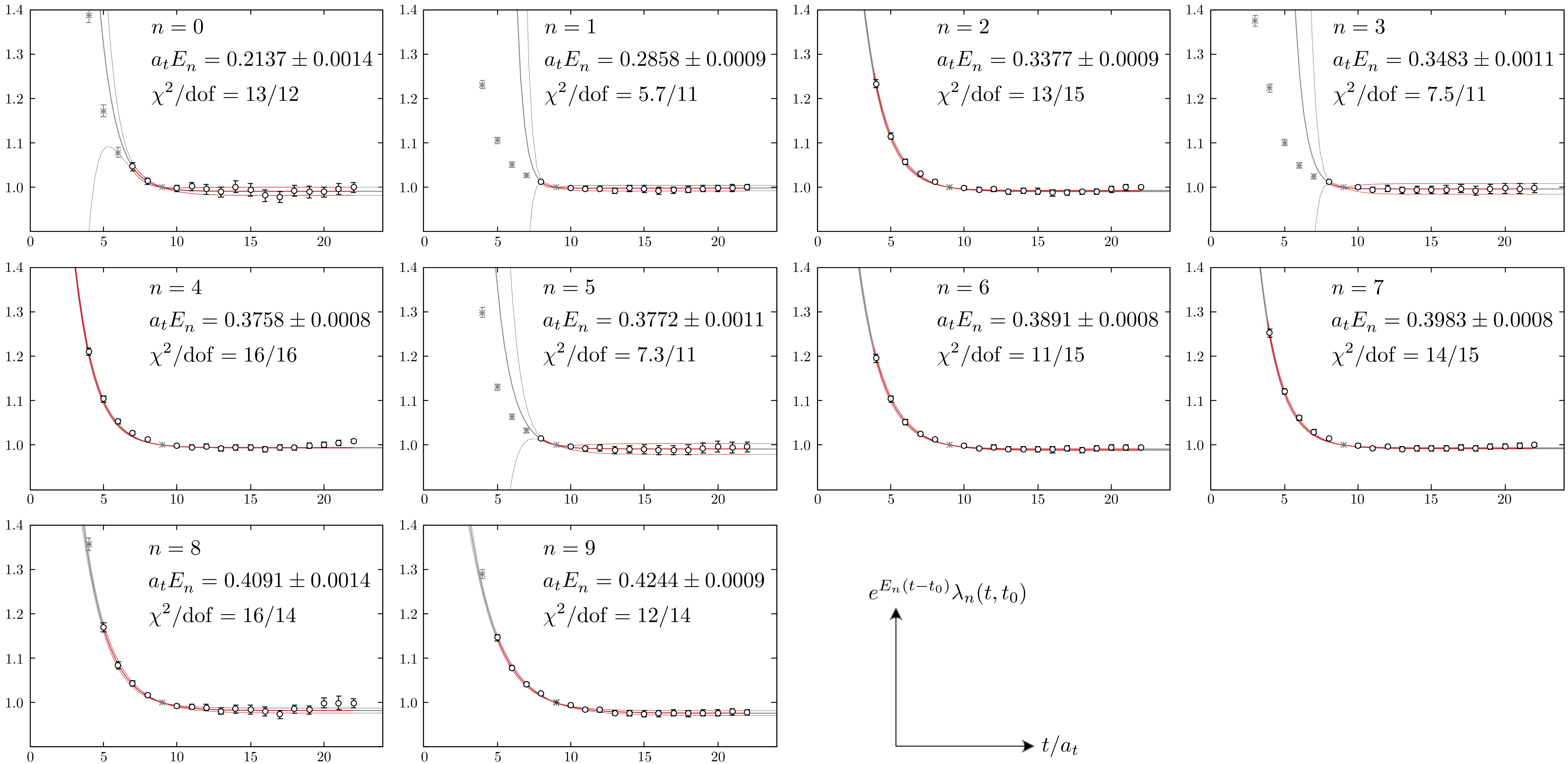
EDITORS' SUGGESTION

Energy-Dependent $\pi^+\pi^+\pi^+$ Scattering Amplitude from QCD

A three-hadron scattering amplitude is computed using lattice QCD for the first time.

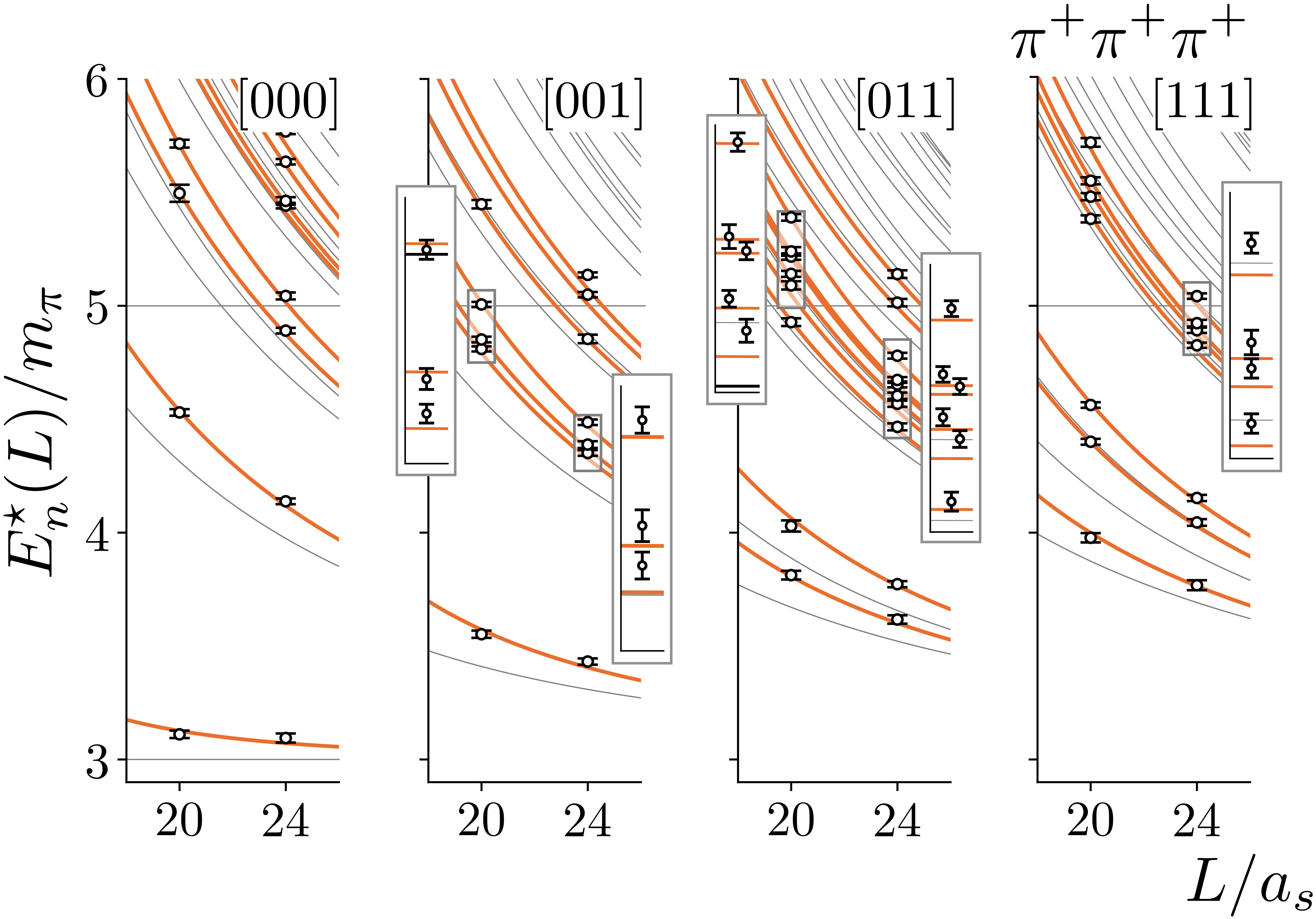
Maxwell T. Hansen *et al.*
Phys. Rev. Lett. **126**, 012001 (2021)

$$I = 3\,(\pi^+\pi^+\pi^+),\ \boldsymbol{P} = [000],\ \Lambda = A_1^-,\ L/a_\mathsf{s} = 24$$

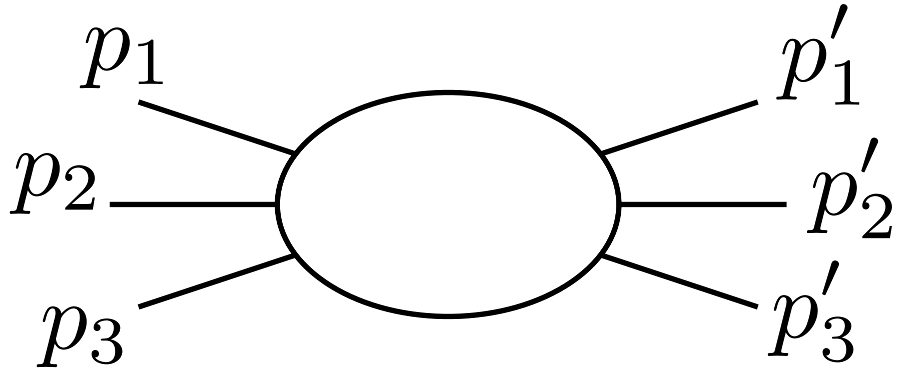


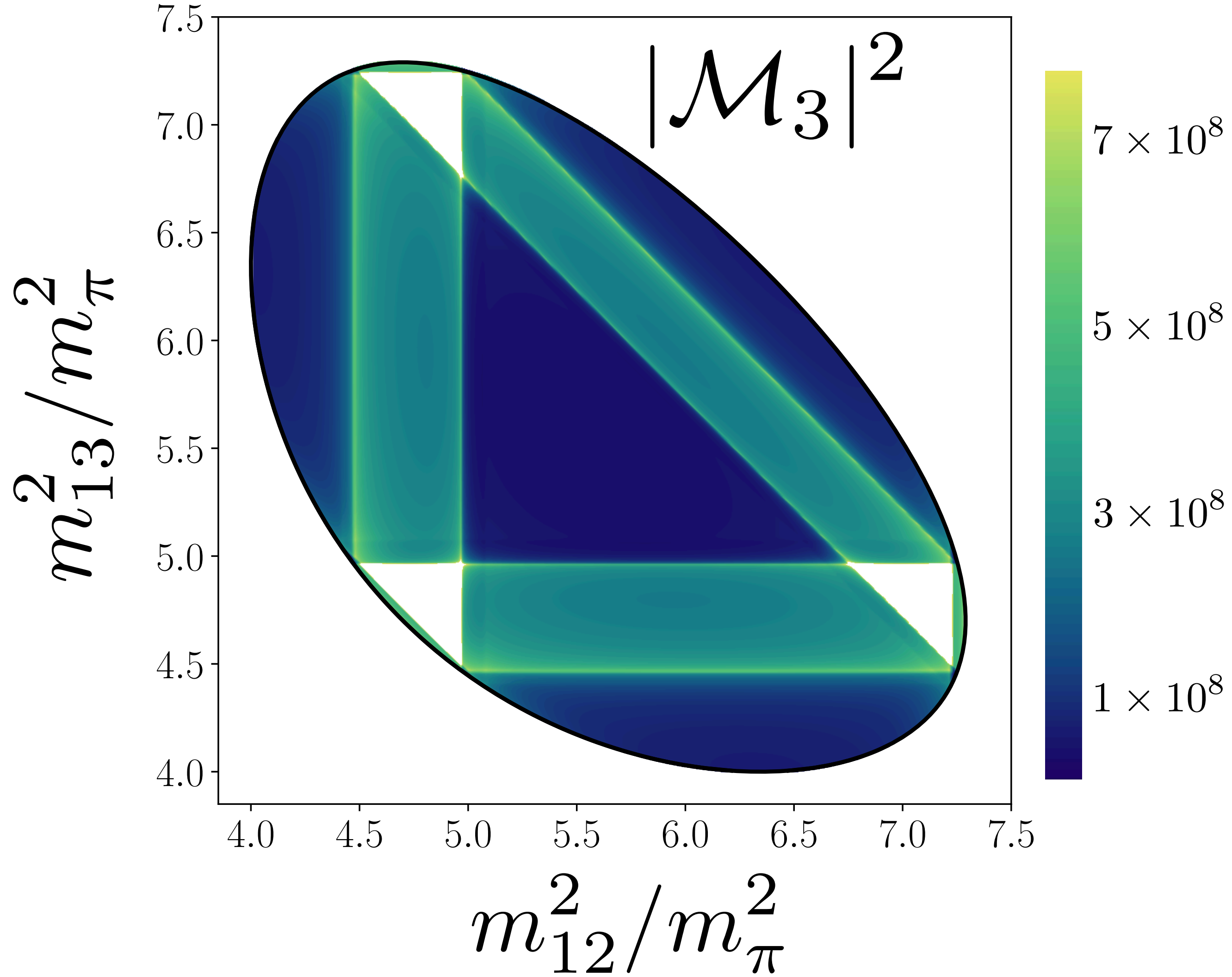
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

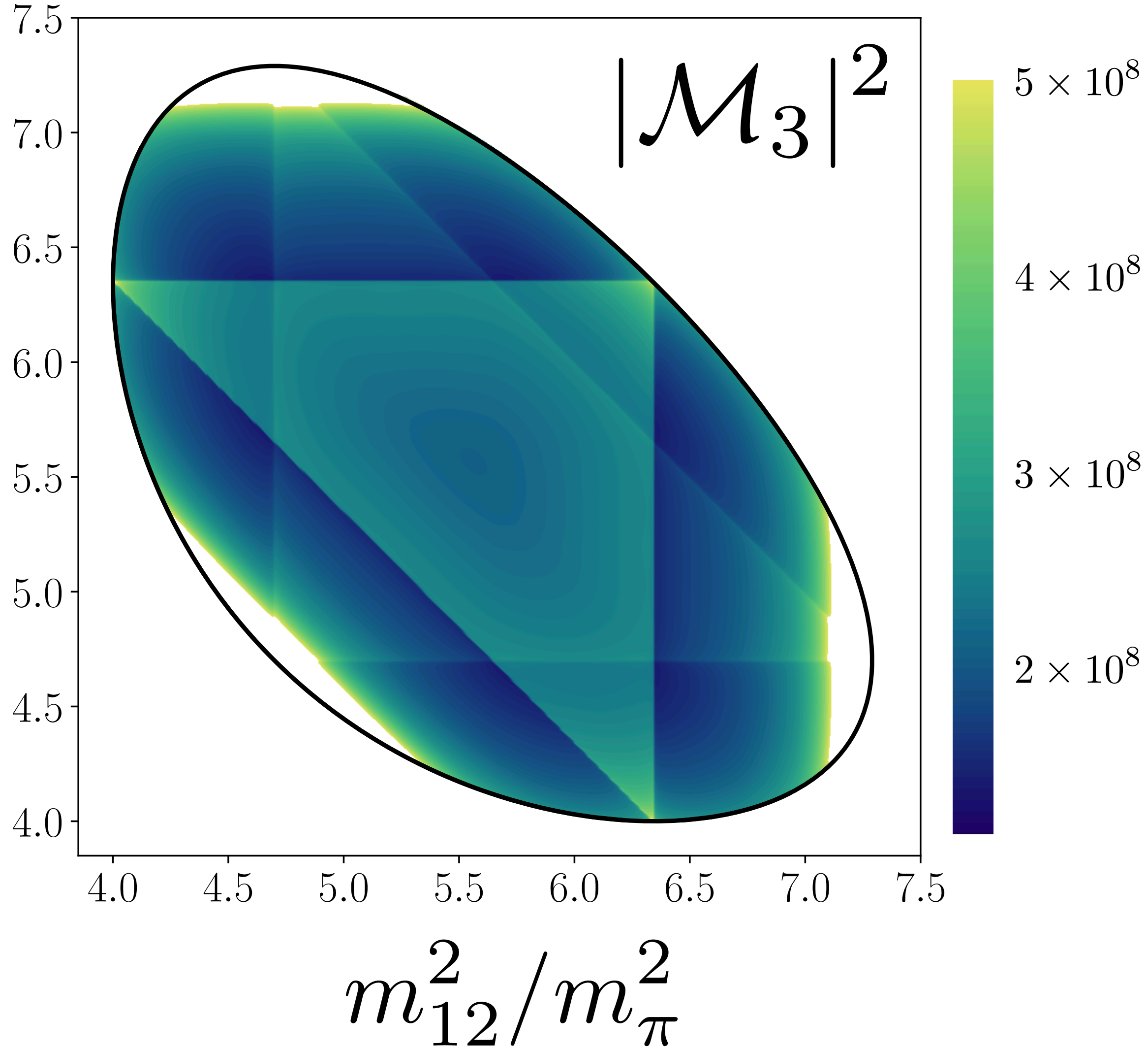
$\pi^+\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001







MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

Two strategies…

- ☑ Finite-volume as a tool
  - LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

  - Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***

- ☐ Spectral function method
  - An answer to… "Can't you just analytically continue?"
  - Expected to be most competitive in multi-particle regime

Reconstruction methods

$$\underbrace{G(\tau)}_{\text{have}} = \int d\omega\, e^{-\omega\tau}\, \underbrace{\rho(\omega)}_{\text{want}}$$

- ***Linear, model-independent reconstruction*** (e.g. Backus-Gilbert-like, Chebyshev)

$$\sum_\tau \mathcal{K}(\bar{\omega},\tau) G(\tau) = \sum_\tau \mathcal{K}(\bar{\omega},\tau) \int d\omega\, e^{-\omega\tau}\, \rho(\omega)$$

$$= \int d\omega \left[\sum_\tau \mathcal{K}(\bar{\omega},\tau)\, e^{-\omega\tau} \right] \rho(\omega)$$

$$= \int d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho(\omega)$$

δ is exactly known

- Maximum Entropy Method (MEM) ← **Not discussed here…**
- Direct fits
- Neural networks

See multiple ECT and CERN workshops, work by Aarts, Allton, Amato, Brandt, Burnier, Del Debbio, Francis, Giudice, Hands, Harris, Hashimoto, Jäger, Karpie, Liu, Meyer, Monahan, Orginos, Robaina, Rothkopf, Ryan, …*

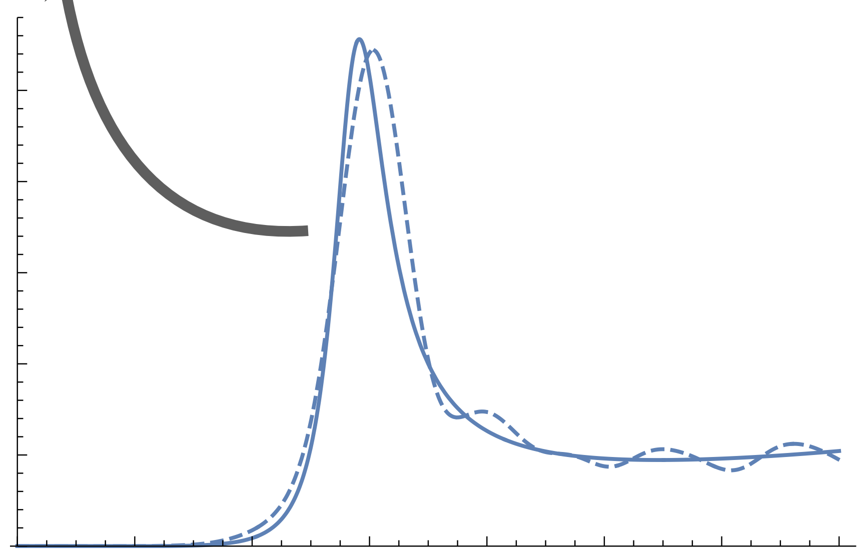
- Key idea here… we aim only to construct

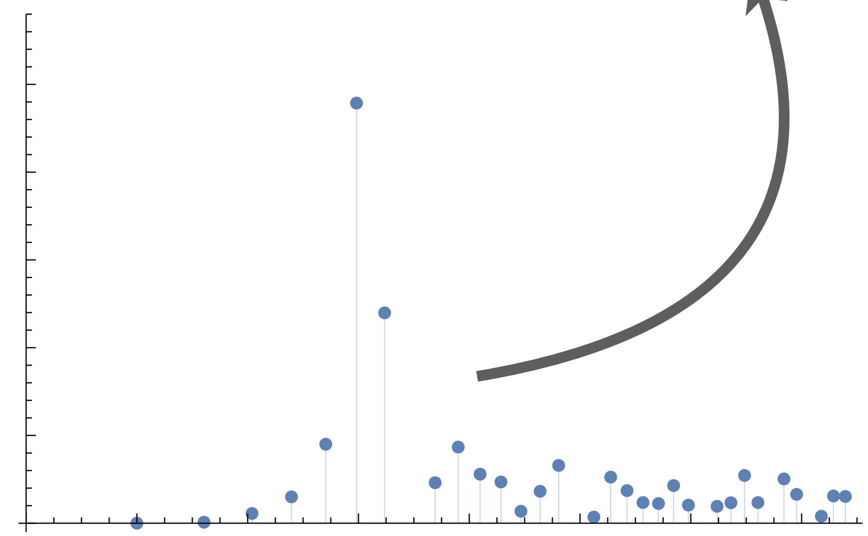
$$\widehat{\rho}(\overline{\omega}) = \int_0^\infty d\omega\, \widehat{\delta}(\overline{\omega},\omega)\, \rho(\omega)$$

Role of the finite volume

$$\widehat{\rho}_L(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho_L(\omega)$$

$$G_L(\tau) = \int d\omega\, e^{-\omega\tau}\, \rho_L(\omega)$$





- Any reconstructed spectral function that ≠ forest of deltas…

contains implicit smearing (or else $L \to \infty$)

We require…

$$1/L \ll \Delta \ll \mu_{\text{physical}}$$

smearing function covers many delta peaks

smearing does not overly distort observable

MTH, Meyer, Robaina (2017)

Linear reconstruction

$$\widehat{\rho}^{[\mathcal{K}]}(\overline{\omega}) = \sum_{\tau} \mathcal{K}(\overline{\omega}, \tau)\, G(\tau) = \int_0^{\infty} d\omega\, \widehat{\delta}^{[\mathcal{K}]}(\overline{\omega}, \omega)\, \rho(\omega)$$

$$\widehat{\delta}^{[\mathcal{K}]}(\overline{\omega}, \omega) = \sum_{\tau} \mathcal{K}(\overline{\omega}, \tau)\, e^{-\omega\tau}$$

- Given $\delta^{\text{target}}(\bar{\omega}, \omega)$, best $K(\bar{\omega}, \tau)$ = whatever minimizes combination of

$$\Delta(\mathcal{K}\,|\,\bar{\omega}, \omega) = \left|\delta^{\text{target}}(\bar{\omega}, \omega) - \sum_{\tau} \mathcal{K}(\bar{\omega}, \tau) e^{-\omega\tau}\right|$$

+

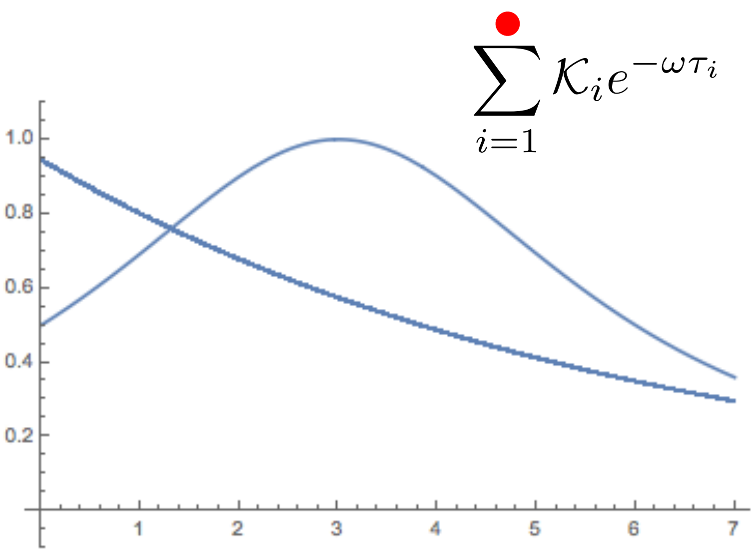
statistical uncertainty on $\widehat{\rho}(\omega)$

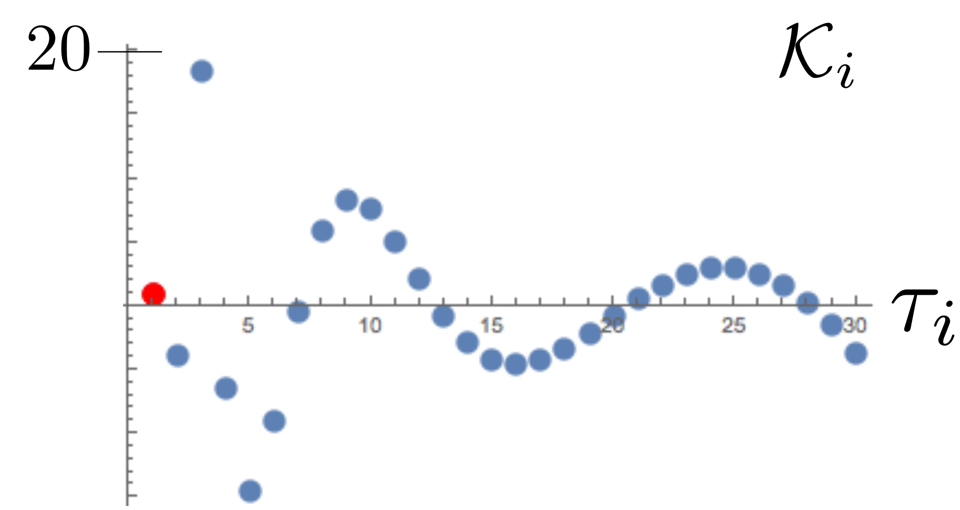
- $\Delta(K\,|\,\bar{\omega}, \omega)$ is known, but the difference we really want is unknown…

$$\left|\widehat{\rho}^{\text{target}}(\overline{\omega}) - \rho^{[\mathcal{K}]}(\overline{\omega})\right|$$

Extended Backus Gilbert example

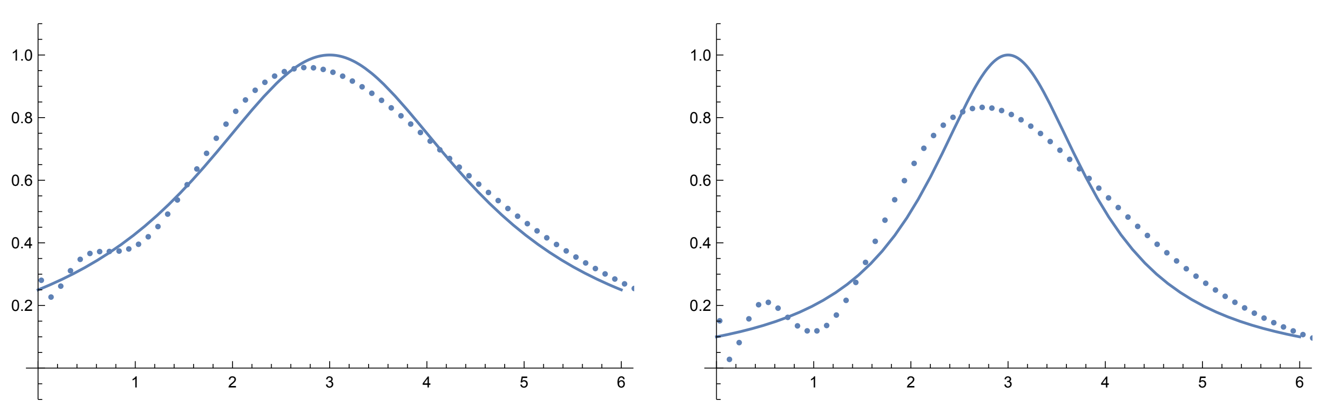
- e.g. target a Breit-Wigner



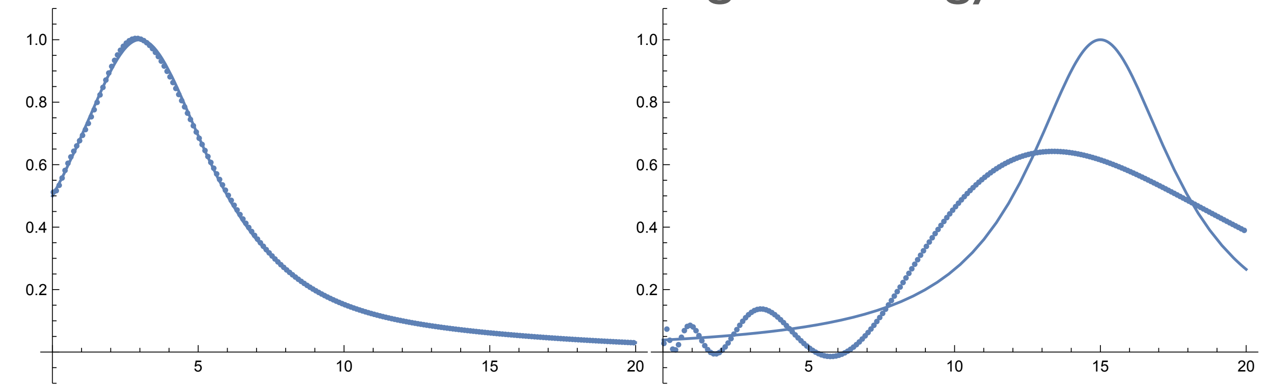


- At a fixed precision ("fixed K_i behavior"), method will fail if one tries to…

make the resolution function too narrow



move the resolution too high in energy



Hansen, Lupo, Tantalo (2019) method

Monte-Carlo test

- Full lattice calculation in two-dimensional O(3) non-linear sigma model
  - An integrable theory → compare to the *known analytic result*

$$\rho^{(2)}(E) = \frac{3\pi^3}{8\theta^2} \frac{\theta^2 + \pi^2}{\theta^2 + 4\pi^2} \tanh^3 \frac{\theta}{2} \Big|_{\theta = 2\cosh^{-1} \frac{E}{2m}}$$

A. B. Zamolodchikov and A. B. Zamolodchikov, (1978)

- Demonstrating the modified Backus-Gilbert (HLT) method for the "R-ratio"

$$C(t) \equiv \int d\boldsymbol{x} \, \langle \Omega | \, \hat{j}_1^a(0, \boldsymbol{x}) \, e^{-\hat{H} t} \, \hat{j}_1^a(0) \, | \Omega \rangle = \int_0^\infty d\omega \, e^{-\omega t} \, \rho(\omega)$$

- Data + theory driven analysis of finite-L and -T effects and discretization

Bulava, MTH, Hansen, Patella, Tantalo (2021), arXiv:*2111.12774*

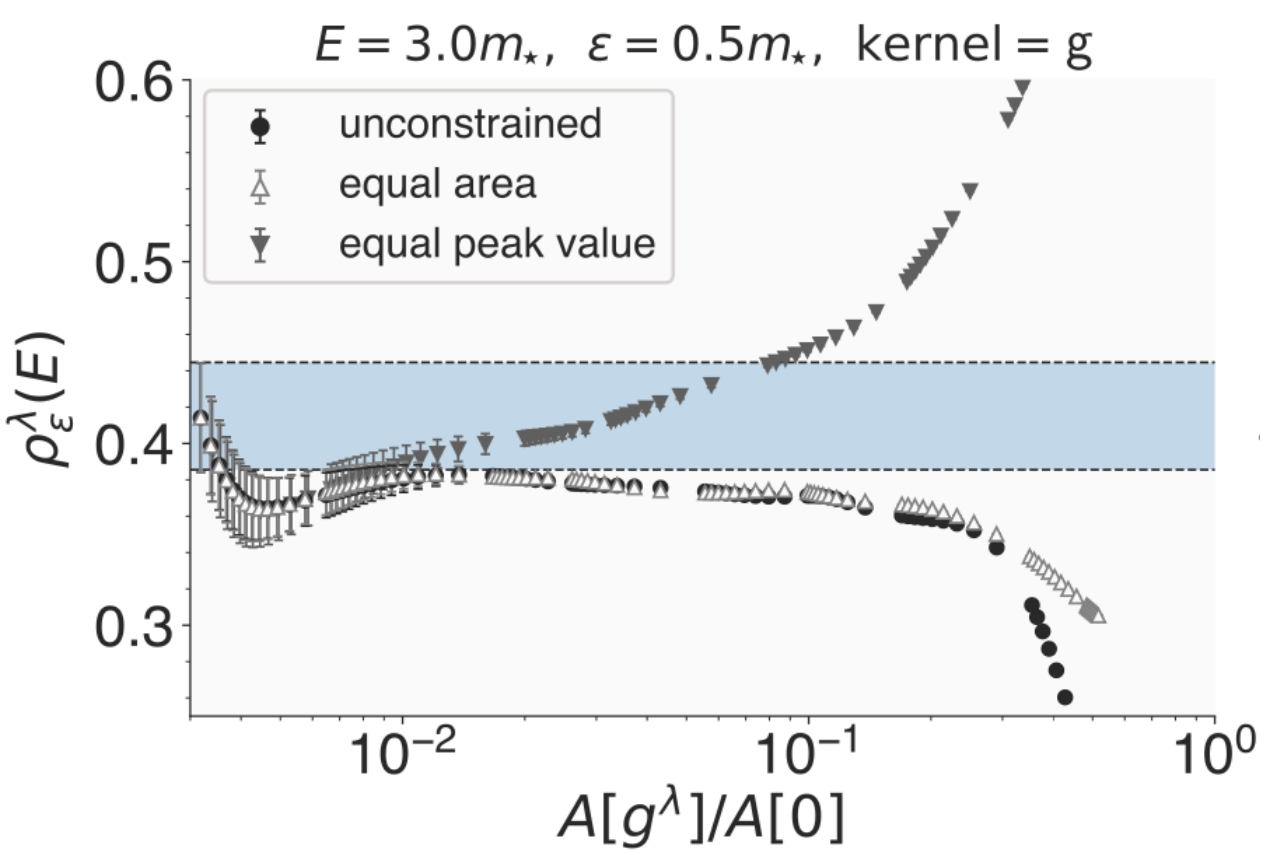
Backus-Gilbert-like algorithm (HLT)

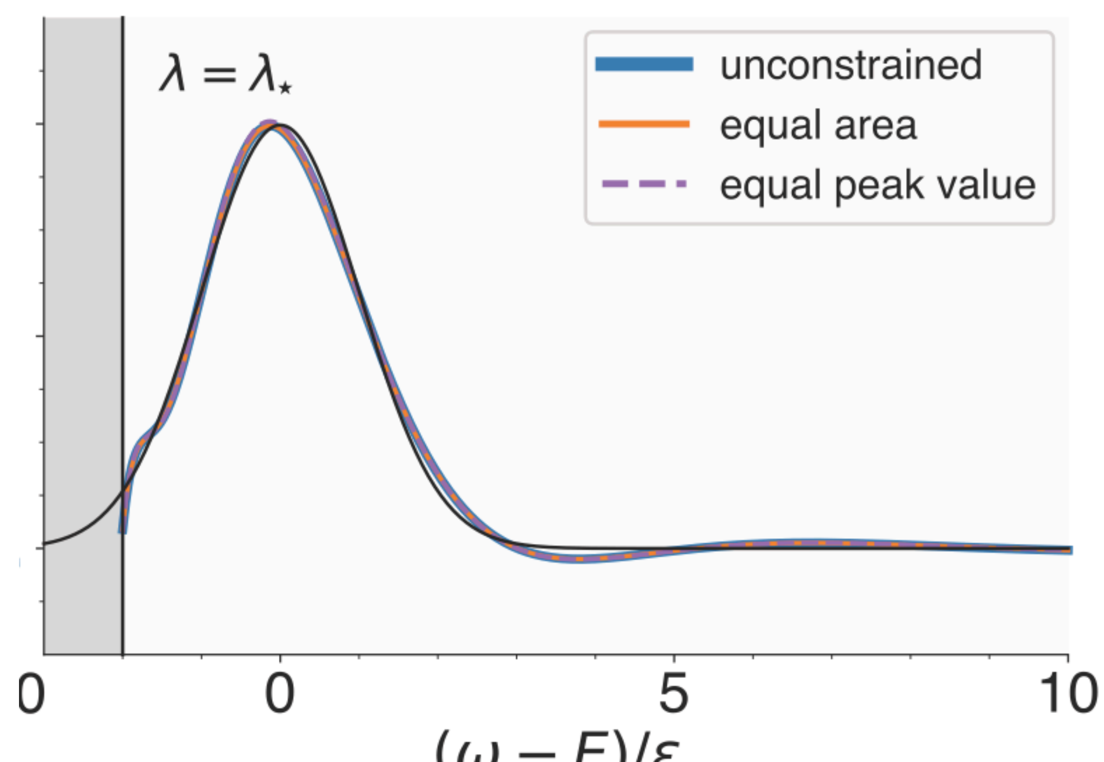
- Construct different smearings of $\rho(\omega)$

$$\rho_\epsilon^\lambda(E) = \int_0^\infty d\omega\, \delta_\epsilon^\lambda(E,\omega)\, \rho(\omega)$$

$$\delta_\epsilon^{\rm g}(x) = \frac{1}{\sqrt{2\pi}\epsilon}\exp\left[-\frac{x^2}{2\epsilon^2}\right], \qquad \delta_\epsilon^{\rm c0}(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2},$$

$$\delta_\epsilon^{\rm c1}(x) = \frac{2}{\pi}\frac{\epsilon^3}{(x^2+\epsilon^2)^2}, \qquad \delta_\epsilon^{\rm c2}(x) = \frac{8}{3\pi}\frac{\epsilon^5}{(x^2+\epsilon^2)^3}.$$





Bulava, MTH, Hansen, Patella, Tantalo (2021), arXiv:*2111.12774*

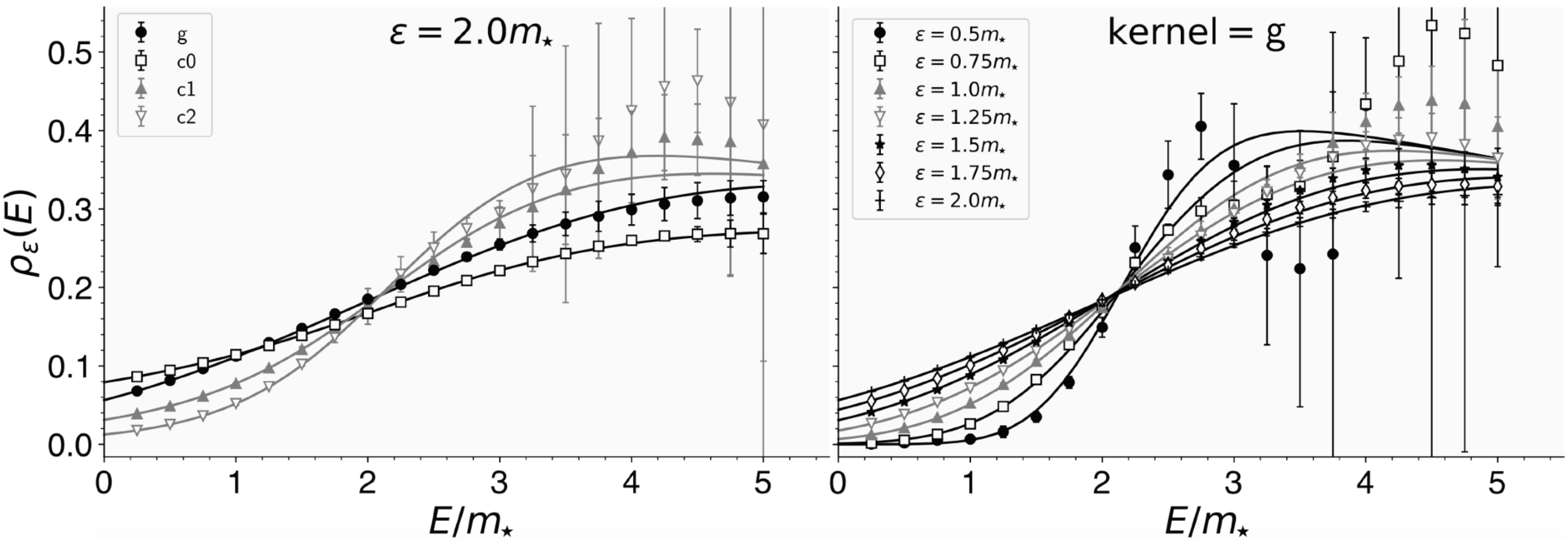
Smeared spectral function vs analytic result

- ❑ Construct different smearings of $\rho(\omega)$

$$\rho_\epsilon^\lambda(E) = \int_0^\infty d\omega\, \delta_\epsilon^\lambda(E,\omega)\,\rho(\omega)$$

$$\delta_\epsilon^{\rm g}(x) = \frac{1}{\sqrt{2\pi}\epsilon}\exp\left[-\frac{x^2}{2\epsilon^2}\right], \qquad \delta_\epsilon^{\rm c0}(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2},$$

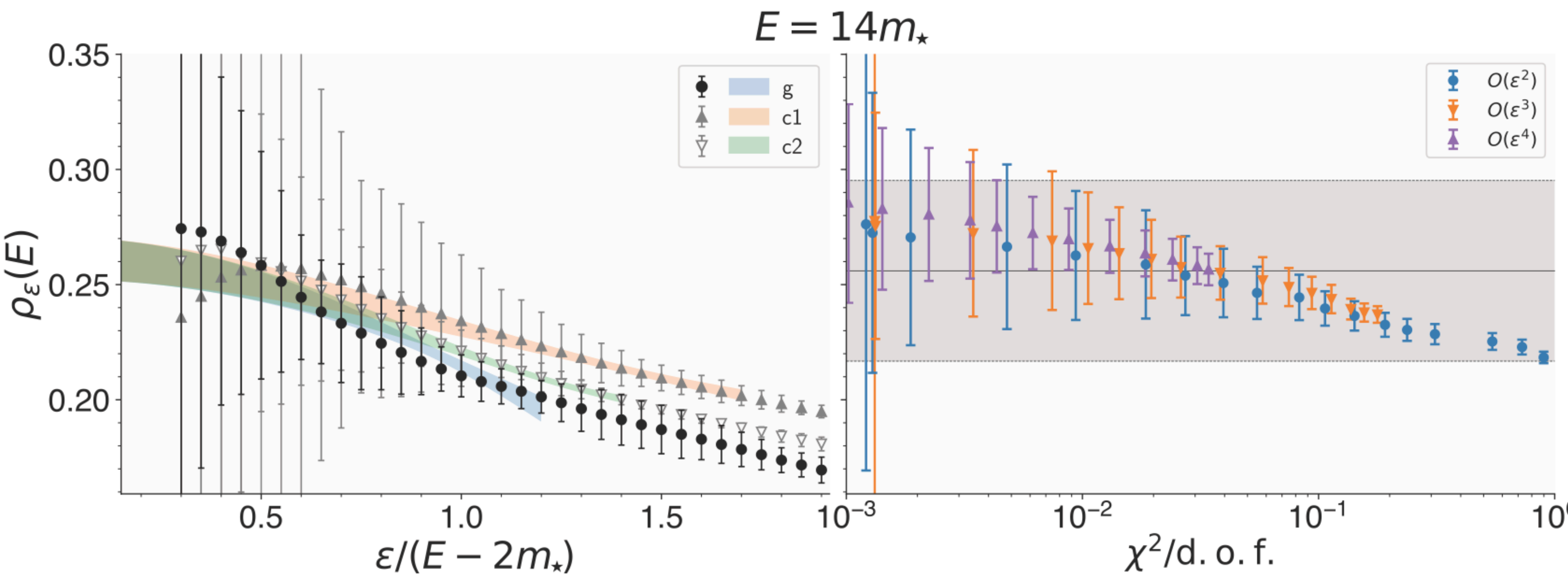
$$\delta_\epsilon^{\rm c1}(x) = \frac{2}{\pi}\frac{\epsilon^3}{(x^2+\epsilon^2)^2}, \qquad \delta_\epsilon^{\rm c2}(x) = \frac{8}{3\pi}\frac{\epsilon^5}{(x^2+\epsilon^2)^3}.$$



Bulava, MTH, Hansen, Patella, Tantalo (2021), arXiv:*2111.12774*

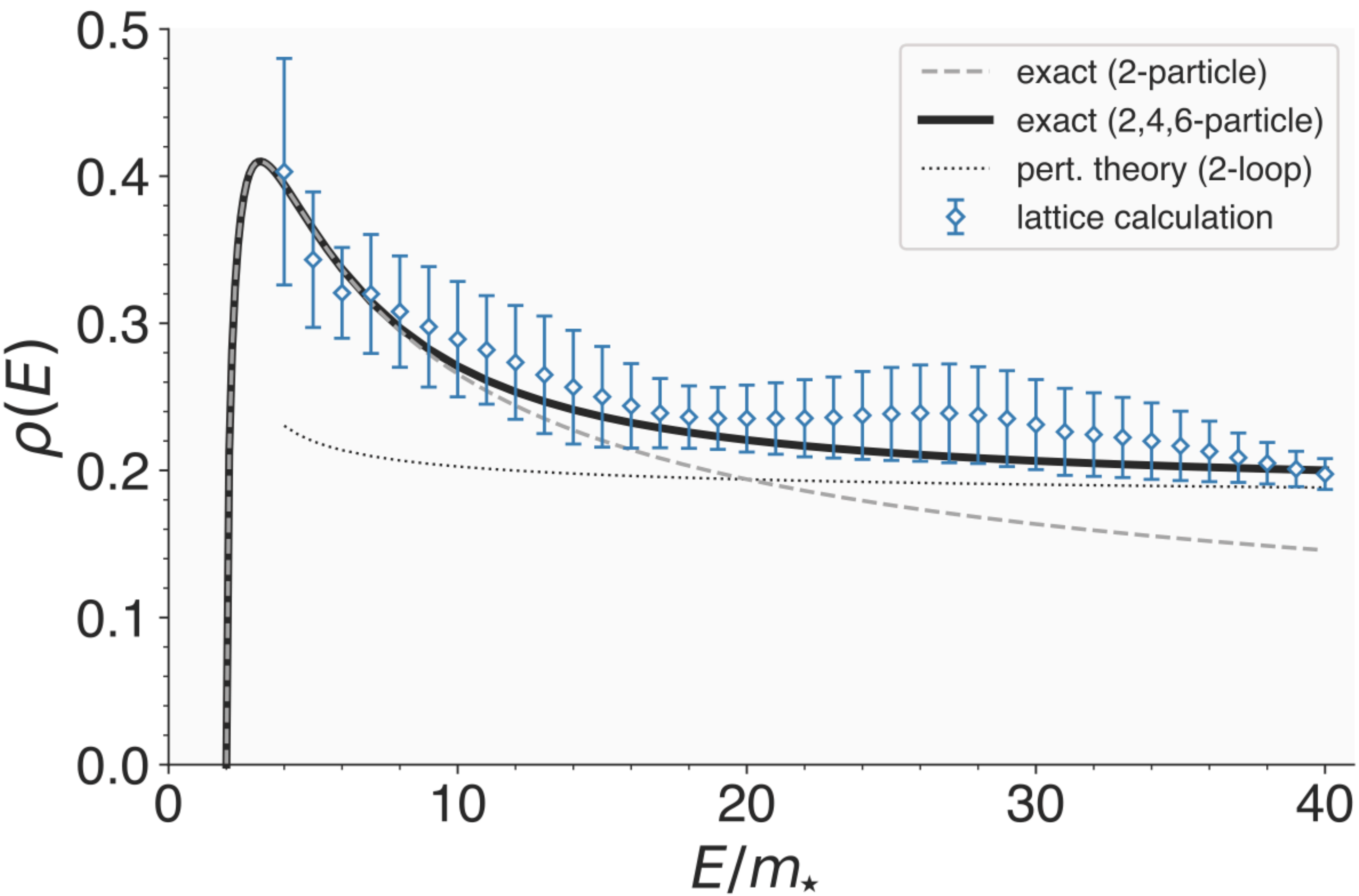
Extrapolating the smearing to zero

$$\rho_\epsilon^{\times}(E) \equiv \int_0^\infty d\omega\, \delta_\epsilon^{\times}(E-\omega)\, \rho(\omega) = \rho(E) + \sum_{k=1}^{\infty} w_k^{\times} a_k(E) \epsilon^k$$



Bulava, MTH, Hansen, Patella, Tantalo (2021), arXiv:*2111.12774*

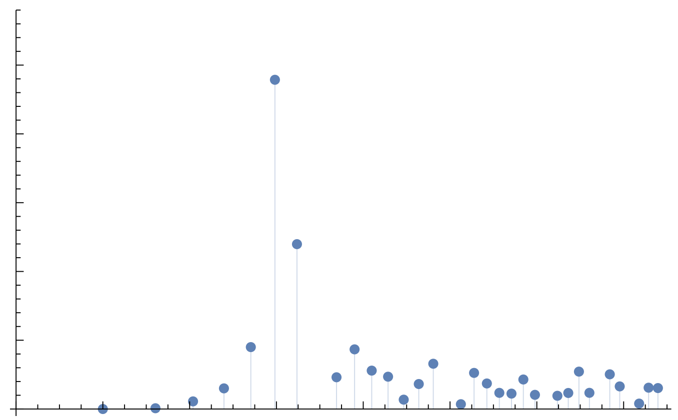
Result



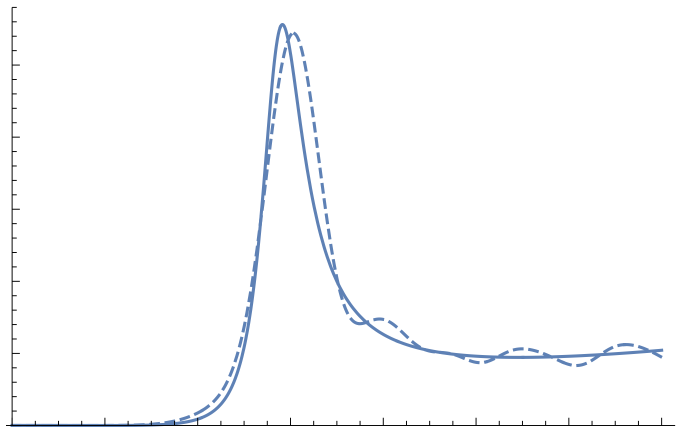
Bulava, MTH, Hansen, Patella, Tantalo (2021), arXiv:*2111.12774*

Spectral summary

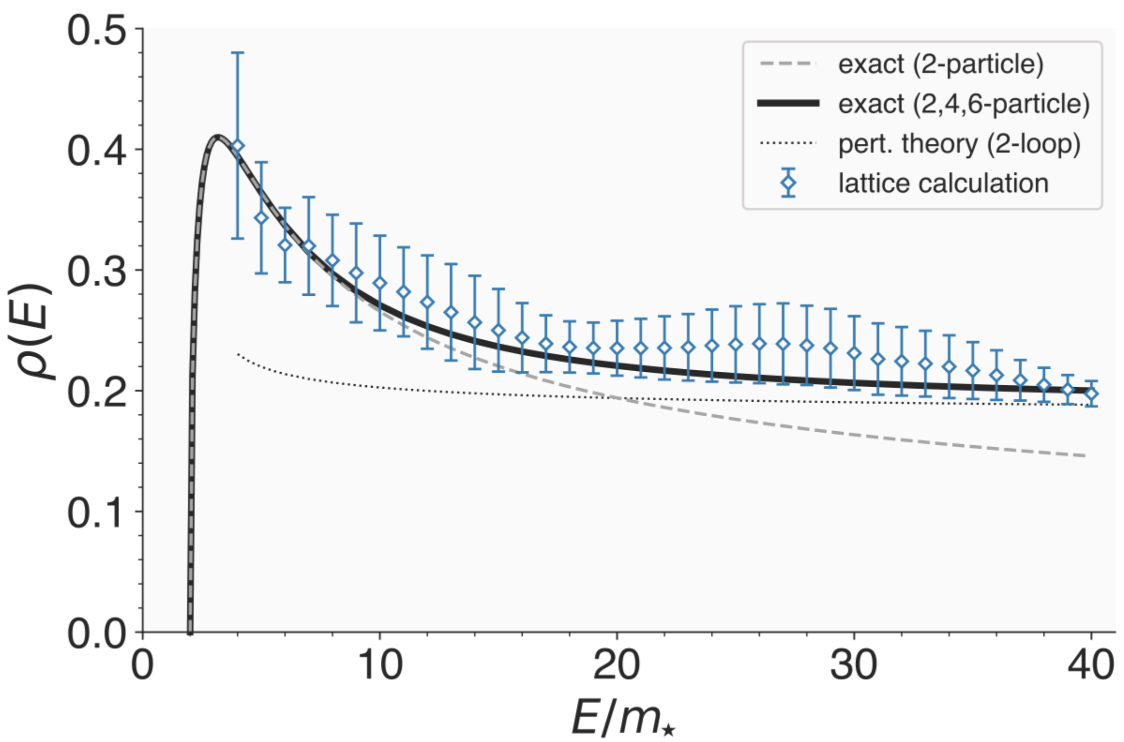
- Cannot solve the inverse problem, we can get $\widehat{\rho}_{L,\Delta}(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho_L(\omega)$
- Smearing is needed anyway to ***suppress volume effects***



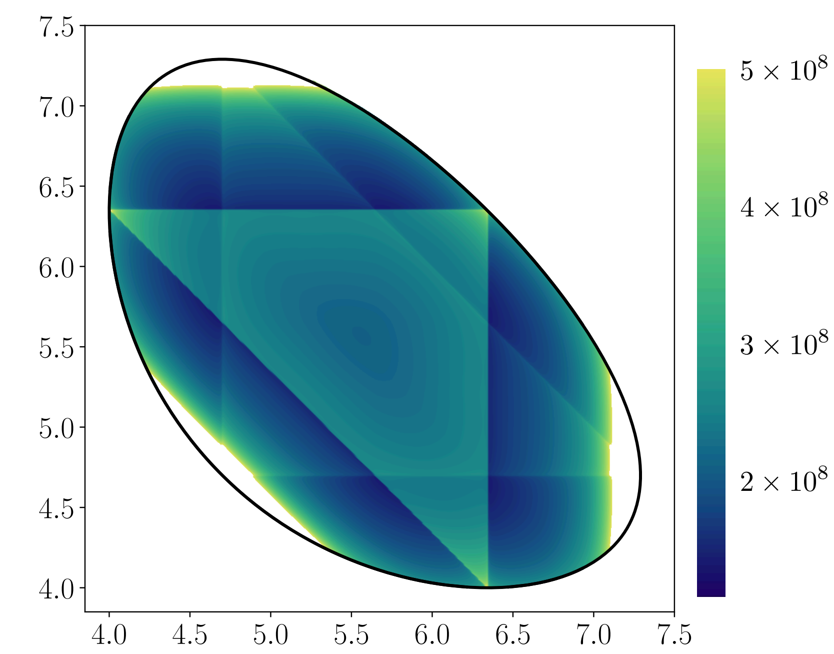
$1/L \ll \Delta \ll \mu_{\text{physical}}$



- Generalized Backus-Gilbert takes $\widehat{\delta}_\Delta^{\text{target}}(\bar{\omega},\omega)$ as input
- *Succesful implementation in O(3) model*
- Not discussed:
  - Spectral functions → scattering amplitudes and (semi)-inclusive rates
  - Formal understanding of volume effects



Two strategies… Conclusion

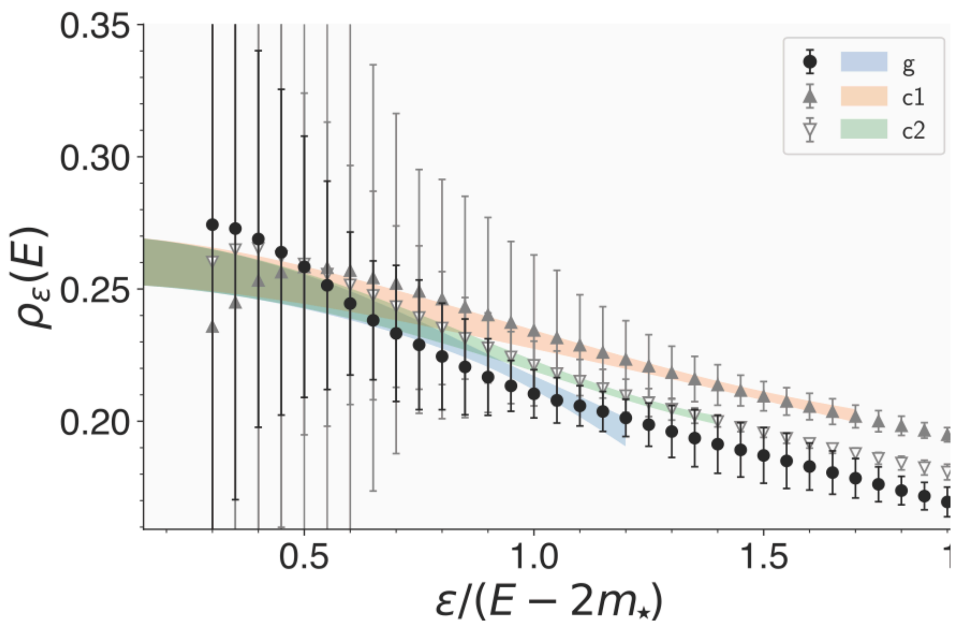


☑ Finite-volume as a tool

- Relate energies and matrix elements
- Tested and highly successful approach
- Limitations:
  - modeling/parametrizing in order to fit
  - need formalism for all open channels at a given energy

☑ Spectral function method

- More direct/natural in a sense (my opinion)
- No new formalism needed for any energies, channels
- Limitations:
  - Tricky volume effects
  - Difficult inverse problem



Thanks for listening!… questions?