Bootstrapping photon scattering in 3+1d

Aditya Hebbar

June 3rd, 2022

Based on upcoming work with
K. Haring, D. Karateev, M. Meineri and J. Penedones

CPHT Centre de Physique Théorique

l'X ÉCOLE POLYTECHNIQUE

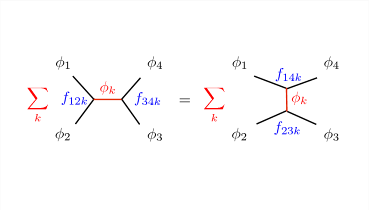
Outline

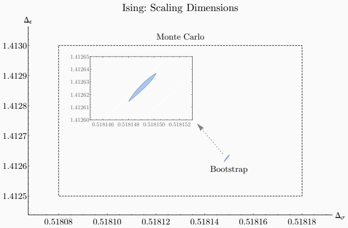
Introduction

Bootstrap Philosophy

- Use consistency of a theory with symmetries, causality, locality and unitarity to constrain the parameters in the theory.

- Motivating example: Conformal bootstrap





[El-Showk et.al 2012]

[Kos, Poland, Simmons-Duffin, Vichi 2016]

The question we want to study: What is the allowed space of photon EFTs which are consistent with Lorentz invariance, locality, causality and unitarity?

Photon Effective Field theory

$$\mathcal{L}_{EFT} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_8 + \mathcal{L}_{10} + \ldots,$$

Assuming parity invariance, there are two independent dimension 8 operators:

$$\mathcal{L}_8 = c_1(F_{\mu\nu}F^{\nu\mu})(F_{\alpha\beta}F^{\beta\alpha}) + c_2 F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu},$$

And three independent dimension 10 operators:

$$\mathcal{L}_{10} = c_3 F_{\alpha\beta}\partial^\beta F_{\mu\nu}\partial^\alpha F^{\nu\rho}F_\rho{}^\mu + c_4(\partial_\alpha F_{\mu\nu}\partial^\alpha F^{\nu\mu})(F_{\rho\sigma}F^{\sigma\rho}) + c_5\partial_\alpha F_{\mu\nu}F^{\nu\rho}\partial^\alpha F_{\rho\sigma}F^{\sigma\mu},$$

and so on at higher orders.

Scalar S-matrix bootstrap (recap)

Scalar scattering amplitude

- Consider $2 \to 2$ scattering of identical scalar particles.
- $\langle p_4, p_3 | S | p_2, p_1 \rangle = \mathbb{1} + i(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) T(s, t, u)$
- Mandelstam invariants

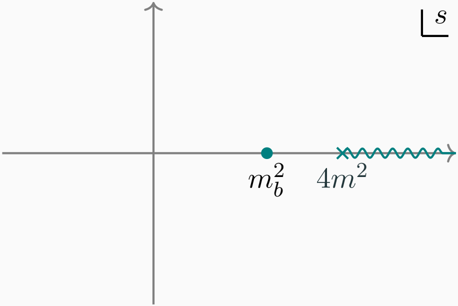
$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2 \quad \text{and} \quad u = (p_1 - p_4)^2$$

$$s + t + u = 4m^2$$

Scalar scattering amplitude: properties

- Crossing: $T(s,t,u)$ is totally symmetric under interchange of its arguments.

- Analyticity: Poles due to (possible) bound states for $0 < s < 4m^2$ and multiparticle cuts starting from $s = 4m^2$.



Scalar scattering amplitude: properties

- Unitarity:

$$|S_l(s)| \leq 1 \quad \forall \quad l \quad \text{and} \quad s \geq 4m^2$$

- Where the partial waves are defined by:

$$S_l(s) = 1 + i\frac{\sqrt{s-4m^2}}{32\pi\sqrt{s}} \int_{-1}^{1} d\cos\theta P_l(\cos\theta) T\left(s, t(s,\cos\theta)\right)$$

Scalar S-matrix Bootstrap

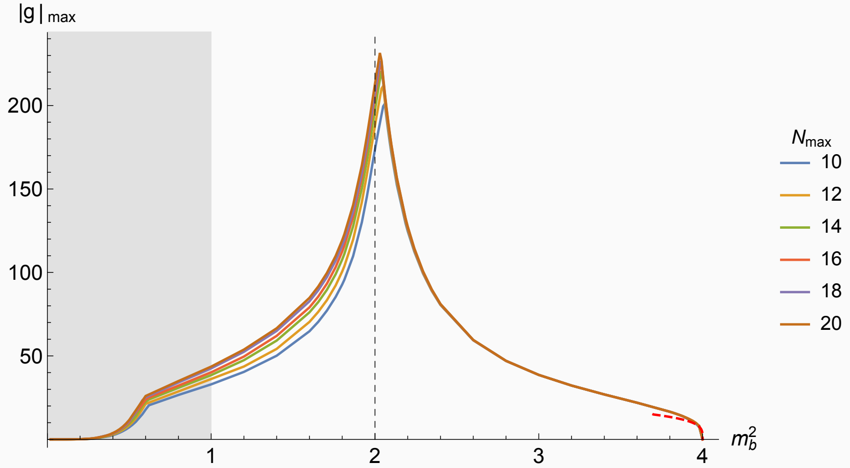
- The idea is to write an ansatz that is manifestly crossing symmetric and has the right analyticity properties.

$$T(s,t,u) = \sum_i \alpha_i \, F_i(s,t,u)$$

- Impose unitarity as a semi-definite positivity constraint:

$$|S_l(s)| \leq 1 \quad \Leftrightarrow \quad \begin{pmatrix} 1 & S_\ell^*(s) \\ S_\ell(s) & 1 \end{pmatrix} \succeq 0$$

Bound on cubic coupling of scalars



[Paulos, Penedones, Toledo, van Rees and Vieira 2017]

Bootstrapping photon scattering

Overview of the kinematics

- We study $2 \to 2$ scattering of photons in $3+1d$.
- No IR divergences since the photons are derivatively coupled.
- Photon is a massless particle and has spin 1 under the massless little group $SO(2)$.
- Hence two helicities for each photon: $\lambda = \pm 1$.
- Multiple functions (16 a priori): $T^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(s,t,u)$

- Analyticity: we assume Landau analyticity.
- Crossing: Schematically

$$T^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(s,t,u) = \mathcal{C}^{\lambda_1'\lambda_2'\lambda_3'\lambda_4'}_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t,u)T^{\lambda_2'\lambda_4'}_{\lambda_1'\lambda_3'}(t,s,u)$$

- Unitarity:

$$\begin{pmatrix} \mathbf{1} & \mathbb{S}_\ell^\dagger(s) \\ \mathbb{S}_\ell(s) & \mathbf{1} \end{pmatrix} \succeq 0$$

Details

One particle states (1PS)

- A photon state is labeled by $|\vec{p}, \lambda = \pm 1\rangle$.
- Lorentz transformation property:

$$|\vec{p}, \lambda\rangle \rightarrow e^{-i\omega\lambda}|\vec{\Lambda p}, \lambda\rangle$$

- Parity

$$\mathcal{P}|\vec{p}, \lambda\rangle = e^{2i\phi\lambda}|-\vec{p}, -\lambda\rangle$$

- Write $S = 1 + iT$
- Consider two particle states which are just tensor products of 1PS:

$$|\kappa_1, \kappa_2\rangle_{id} \equiv |\vec{p}_1, \lambda_1\rangle \otimes_{sym} |\vec{p}_2, \lambda_2\rangle$$

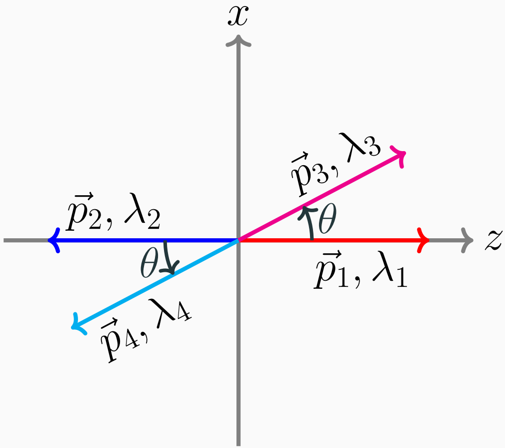
- Scattering amplitudes are matrix elements of the T operator:

$$(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) T^{\lambda_3 \lambda_4}_{\lambda_1 \lambda_2}(p_1, p_2, p_3, p_4) \equiv \langle \kappa_3, \kappa_4 | T | \kappa_1, \kappa_2 \rangle$$

- Lorentz transformation property of amplitudes follows from that of 1PS:

$$T^{\lambda_3 \lambda_4}_{\lambda_1 \lambda_2}(p_1, p_2, p_3, p_4) = e^{-i(\omega_1 \lambda_1 + \omega_2 \lambda_2 - \omega_3 \lambda_3 - \omega_4 \lambda_4)} \times T^{\lambda_3 \lambda_4}_{\lambda_1 \lambda_2}(p'_1, p'_2, p'_3, p'_4)$$

COM frame



$$T^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(s,t,u) \equiv T^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}(p_1^{com}, p_2^{com}, p_3^{com}, p_4^{com})$$

COM frame

$$
\begin{aligned}
p_1^{\mathsf{com}} &= (E, 0, 0, E),\\
p_2^{\mathsf{com}} &= (E, 0, 0, E),\\
p_3^{\mathsf{com}} &= (E, +E\sin\theta, 0, +E\cos\theta),\\
p_4^{\mathsf{com}} &= (E, -E\sin\theta, 0, -E\cos\theta).
\end{aligned}
$$

These parameters can be written in terms of Mandelstam variables

$$
E = \frac{\sqrt{s}}{2},
$$

$$
\sin\theta = \frac{2\sqrt{tu}}{s}, \quad \cos\theta = \frac{t-u}{s}.
$$

$$
T_{\lambda_1\lambda_2}^{\lambda_3\lambda_4}(s,t,u) \equiv T_{\lambda_1\lambda_2}^{\lambda_3\lambda_4}(p_1^{\mathsf{com}}, p_2^{\mathsf{com}}, p_3^{\mathsf{com}}, p_4^{\mathsf{com}})
$$

Counting the number of amplitudes

- Each photon can be either helicity $+1$ or -1.
- Therefore, we have with 16 amplitudes a priori.
- Bose symmetry implies

$$T_{\lambda_1,\lambda_2}^{\lambda_3,\lambda_4}(s,t,u) = T_{\lambda_2,\lambda_1}^{\lambda_4,\lambda_3}(s,t,u)$$

- In addition, we can cross $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ to get

$$T_{\lambda_1,\lambda_2}^{\lambda_3,\lambda_4}(s,t,u) = T_{-\lambda_3,-\lambda_4}^{-\lambda_1,-\lambda_2}(s,t,u)$$

- With these relations we go down to 7 independent amplitudes.

- Next we also impose Parity invariance

$$T^{\lambda_3,\lambda_4}_{\lambda_1,\lambda_2}(s,t,u) = T^{-\lambda_3,-\lambda_4}_{-\lambda_1,-\lambda_2}(s,t,u)$$

- Using these we are left with 5 independent amplitudes:

$$\boxed{\begin{aligned} \Phi_1 &\equiv T^{++}_{++}(s,t,u) \\ \Phi_2 &\equiv T^{--}_{++}(s,t,u) \\ \Phi_3 &\equiv T^{+-}_{+-}(s,t,u) \\ \Phi_4 &\equiv T^{-+}_{+-}(s,t,u) \\ \Phi_5 &\equiv T^{+-}_{++}(s,t,u) \end{aligned}}$$

s-t Crossing: $2 \leftrightarrow 3$ interchange

- $T^{\lambda_3,\lambda_4}_{\lambda_1,\lambda_2}(p_1,p_2,p_3,p_4) = T^{-\lambda_2,\lambda_4}_{\lambda_1,-\lambda_3}(p_1,-p_3,-p_2,p_4)$
- $\Phi_4(t,s,u) = \Phi_1(s,t,u)$
- Φ_2, Φ_3 and Φ_5 are $s-t$ symmetric.

t-u Crossing: $3 \leftrightarrow 4$ interchange

- $T^{\lambda_3,\lambda_4}_{\lambda_1,\lambda_2}(p_1,p_2,p_3,p_4) = T^{\lambda_4,\lambda_3}_{\lambda_1,\lambda_2}(p_1,p_2,p_4,p_3)$
- $\Phi_4(s,t,u) = \Phi_3(s,u,t)$
- Φ_1, Φ_2 and Φ_5 are $t-u$ symmetric.

Final crossing constraints

- $\Phi_1(s,t,u) = \Phi_1(s,u,t)$
- $\Phi_3(s,t,u) = \Phi_1(u,t,s)$
- $\Phi_4(s,t,u) = \Phi_1(t,s,u)$
- Φ_2 and Φ_5 fully crossing symmetric.

Partial wave amplitudes

- Introduce the following short-hand notation for COM frame states:

$$|(\boldsymbol{p},\theta,\phi);\lambda_1,\lambda_2\rangle = |\vec{p},\lambda_1\rangle \otimes_{sym} |-\vec{p},\lambda_2\rangle$$

  where $\boldsymbol{p} = |\vec{p}|$.

- Two particle states are reducible:

$$|(\boldsymbol{p},\theta,\phi);\lambda_1,\lambda_2\rangle = \sum_{\ell,\lambda} C_\ell(\boldsymbol{p}) e^{-i(\lambda_1+\lambda_2-\lambda)\phi} d^{\ell}_{\lambda\lambda_{12}}(\theta)|c,\vec{0},\ell,\lambda;\lambda_1,\lambda_2\rangle$$

- In particular

$$|(\boldsymbol{p},0,0);\lambda_1,\lambda_2\rangle = \sum_{\ell} C_\ell(\boldsymbol{p})|c,\vec{0},\ell,\lambda_{12};\lambda_1,\lambda_2\rangle$$

$$|(\boldsymbol{p}',\theta,0);\lambda_3,\lambda_4\rangle = \sum_{\ell,\lambda} C_\ell(\boldsymbol{p}')\, d^{\ell}_{\lambda\lambda_{34}}(\theta)|c,\vec{0},\ell,\lambda;\lambda_3,\lambda_4\rangle$$

Partial wave amplitudes

- Define partial wave amplitudes by the formula:

$$(2\pi)^4\delta^{(4)}(p-p')\delta_{\ell\ell'}\delta_{\lambda\lambda'} \times T_{\ell\,\lambda_1\lambda_2}^{\lambda_3\lambda_4}(s) = \langle c', \vec{p}\,'; \ell', \lambda'; \lambda_3, \lambda_4 | T | c, \vec{p}; \ell, \lambda; \lambda_1, \lambda_2 \rangle$$

- They are derived from the amplitude by the following integral:

$$\boxed{T_{\ell\,\lambda_1,\lambda_2}^{\lambda_3,\lambda_4}(s) \equiv \frac{1}{32\pi}\int d\theta \sin\theta \; d^{\ell}_{\lambda_{12}\lambda_{34}}(\theta) T^{\lambda_3,\lambda_4}_{\lambda_1,\lambda_2}(s, t(\theta))}$$

where $t(\theta) = -\frac{s}{2}(1-\cos\theta)$.

- And hence

$$\Phi_1^\ell(s) \equiv \frac{1}{32\pi}\int d\theta \sin\theta \; d_{00}^\ell(\theta)\Phi_1(s,\theta)$$

$$\Phi_2^\ell(s) \equiv \frac{1}{32\pi}\int d\theta \sin\theta \; d_{00}^\ell(\theta)\Phi_2(s,\theta)$$

$$\Phi_3^\ell(s) \equiv \frac{1}{32\pi}\int d\theta \sin\theta \; d_{22}^\ell(\theta)\Phi_3(s,\theta)$$

$$\Phi_4^\ell(s) \equiv \frac{1}{32\pi}\int d\theta \sin\theta \; d_{2-2}^\ell(\theta)\Phi_4(s,\theta)$$

$$\Phi_5^\ell(s) \equiv \frac{1}{32\pi}\int d\theta \sin\theta \; d_{02}^\ell(\theta)\Phi_5(s,\theta)$$

Unitarity

Now consider the following three states

$$|1\rangle \equiv \frac{1}{\sqrt{2}}\left(|c,\vec{p},\ell,\lambda;+,+\rangle_{id} - |c,\vec{p},\ell,\lambda;-,-\rangle_{id}\right), \quad \ell \geq 0 \quad (\ell \text{ even}),$$

$$|2\rangle \equiv \frac{1}{\sqrt{2}}\left(|c,\vec{p},\ell,\lambda;+,+\rangle_{id} + |c,\vec{p},\ell,\lambda;-,-\rangle_{id}\right), \quad \ell \geq 0 \quad (\ell \text{ even}),$$

$$|3\rangle \equiv \sqrt{2}\,|c,\vec{p},\ell,\lambda;+,-\rangle_{id}, \quad \ell \geq 1.$$

The state $|1\rangle$ is parity odd while the states $|2\rangle$ and $|3\rangle$ are parity even.

Unitarity implies that the following matrix must be positive semi-definite:

$$\begin{pmatrix} {}_{\mathsf{in}}\langle a'|b\rangle_{\mathsf{in}} & {}_{\mathsf{in}}\langle a'|b\rangle_{\mathsf{out}} \\ {}_{\mathsf{out}}\langle a'|b\rangle_{\mathsf{in}} & {}_{\mathsf{out}}\langle a'|b\rangle_{\mathsf{out}}, \end{pmatrix} \succeq 0$$

The inner products between in and out states lead to partial amplitudes, we have

$$_{out}\langle 1'|1\rangle_{in} = \delta_{\ell\ell'}\delta_{\lambda\lambda'}(2\pi)^4\delta^{(4)}(p-p') \times \left(1 + i\,\left(T_{\ell}{}^{++}_{++}(s) + T_{\ell}{}^{--}_{++}(s)\right)\right),$$

$$_{out}\langle 2'|2\rangle_{in} = \delta_{\ell\ell'}\delta_{\lambda\lambda'}(2\pi)^4\delta^{(4)}(p-p') \times \left(1 + i\,\left(T_{\ell}{}^{++}_{++}(s) - T_{\ell}{}^{--}_{++}(s)\right)\right),$$

$$_{out}\langle 3'|3\rangle_{in} = \delta_{\ell\ell'}\delta_{\lambda\lambda'}(2\pi)^4\delta^{(4)}(p-p') \times \left(1 + 2i\,T_{\ell}{}^{+-}_{+-}(s)\right),$$

$$_{out}\langle 3'|2\rangle_{in} = \delta_{\ell\ell'}\delta_{\lambda\lambda'}(2\pi)^4\delta^{(4)}(p-p') \times \left(2i\,T_{\ell}{}^{+-}_{++}(s)\right).$$

Unitarity conditions

$$\text{even } \ell \geq 0 \quad : \quad \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) + i \begin{pmatrix} 0 & -\Phi_1^{\ell *}(s)+\Phi_2^{\ell *}(s) \\ \Phi_1^{\ell}(s)-\Phi_2^{\ell}(s) & 0 \end{pmatrix} \succeq 0,$$

$$\ell = 0 : \quad \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) + i \begin{pmatrix} 0 & -\Phi_1^{0 *}(s)-\Phi_2^{0 *}(s) \\ \Phi_1^{0}(s)+\Phi_2^{0}(s) & 0 \end{pmatrix} \succeq 0,$$

$$\text{odd } \ell \geq 3 \quad : \quad \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) + 2i \begin{pmatrix} 0 & -\Phi_3^{\ell *}(s) \\ \Phi_3^{\ell}(s) & 0 \end{pmatrix} \succeq 0.$$

and

$$\text{even } \ell \geq 2 \quad : \quad \begin{pmatrix} \mathbb{I}_{2\times 2} & \mathbb{S}_{2\times 2}^{\ell \dagger}(s) \\ \mathbb{S}_{2\times 2}^{\ell}(s) & \mathbb{I}_{2\times 2} \end{pmatrix} \succeq 0,$$

where we have defined

$$\mathbb{I}_{2\times 2} \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \qquad \mathbb{S}_{2\times 2}^{\ell}(s) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + i \begin{pmatrix} \Phi_1^{\ell}(s)+\Phi_2^{\ell}(s) & 2\Phi_5^{\ell}(s) \\ 2\Phi_5^{\ell}(s) & 2\Phi_3^{\ell}(s) \end{pmatrix}.$$

- We assume $\Phi_i(s,t,u)$ analytic for

$$s\ ,\ t\ , u\ \in \mathcal{C} - [0,\infty)$$

- We take the following ansatz for the amplitudes

$$\Phi_i(s,t,u) = \sum_{a,b,c}^{N_{max}} \alpha^i_{abc}\rho_s^a\rho_t^b\rho_u^c$$

where $\rho_z \equiv \frac{1-\sqrt{-z}}{1+\sqrt{-z}}$ maps the z cut plane to the interior of the unit disc.

- We impose constraints on α^i_{abc} such that

$$\Phi_i(s,\cos\theta) \xrightarrow{s\to 0} \Phi_i^{EFT}(s,\cos\theta)$$

Photon amplitude low energy expansion

$$\Phi_1^{EFT}(s,t,u) = g_2 s^2 + 4 g_3 s^3 + g_4 s^4 + g_4' s^2 tu$$
$$+ \left(\beta_{11} s^2 + \beta_{12} tu\right) \log\left(-s\sqrt{g_2}\right)$$
$$+ \beta_{13}\left(t^2 \log\left(-t\sqrt{g_2}\right) + u^2 \log\left(-u\sqrt{g_2}\right)\right) \Big] + O(s^5),$$

$$\Phi_2^{EFT}(s,t,u) = f_2(s^2+u^2+t^2) + f_3 stu$$
$$+ f_4(s^2+t^2+u^2)^2 + \beta_2\left(s^4 \log\left(-s\sqrt{g_2}\right)\right.$$
$$\left. + t^4 \log\left(-t\sqrt{g_2}\right) + u^4 \log\left(-u\sqrt{g_2}\right)\right) + O(s^5),$$

$$\Phi_5^{EFT}(s,t,u) = h_3 stu + O(s^5).$$

Relation to the Lagrangian coefficients

- $g_2 = 2(4c_1 + 3c_2)$
- $f_2 = 2(4c_1 + c_2)$
- $g_3 = 4c_4$
- $f_3 = 6(c_3 + 2c_4 - c_5)$
- $h_3 = \frac{3}{2}c_3$

Bounds on EFT coefficients

What is the allowed space in the Wilson Coefficients?

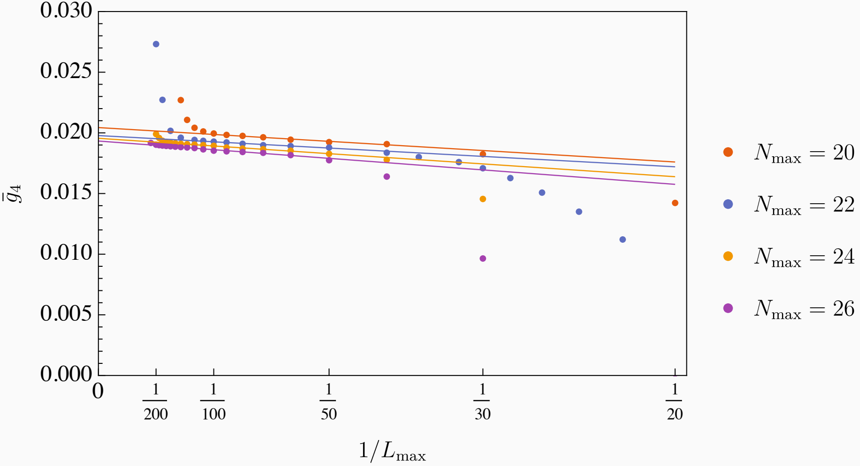
- Analytic bounds imply

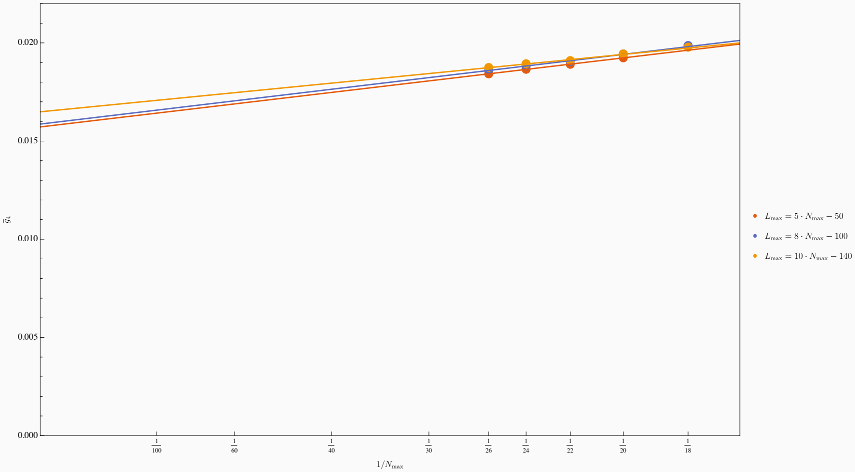
$$g_2 \geq 0 \quad \text{and} -1 \leq \frac{f_2}{g_2} \leq 1$$

- Turns out we can get any intermediate value by weakly coupling to parity odd or even resonances.
- A nice sanity check on the numerics:

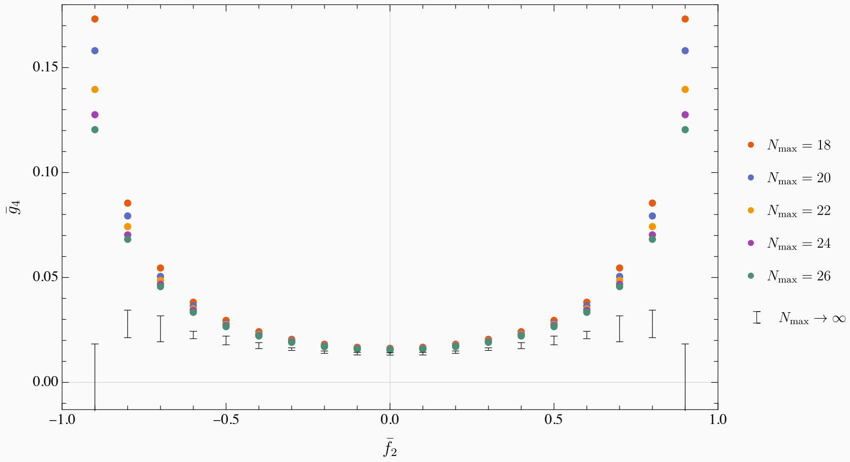
g_2	$\min(\bar{f}_2)$	$\max(\bar{f}_2)$
10^{-2}	-0.998710	0.998710
1	-0.998619	0.998619
10^{2}	-0.991237	0.991237

Estimates of bound on $\bar{g}_4 \equiv \frac{g_4}{g_2^2}$ for fixed $\bar{f}_2 = \frac{-3}{11}$

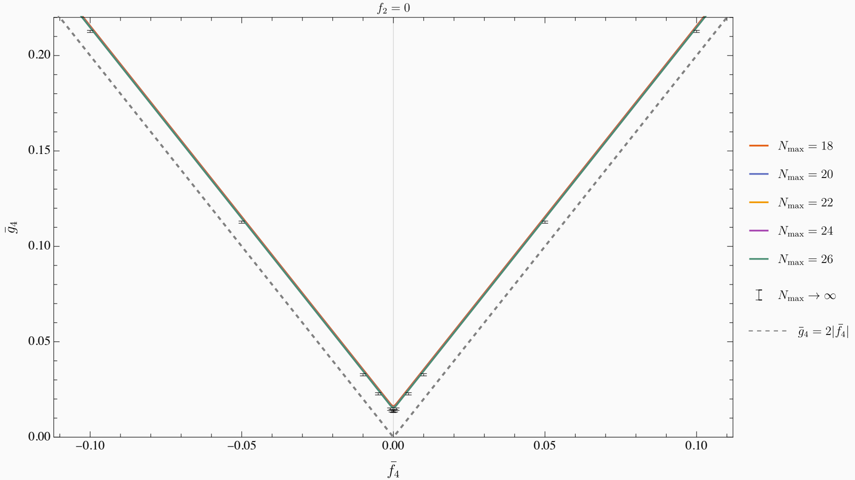



0.020
0.015
0.010
0.005
0.000
$\bar{g}_4$
$\frac{1}{100}$
$\frac{1}{60}$
$\frac{1}{40}$
$\frac{1}{30}$
$\frac{1}{26}$
$\frac{1}{24}$
$\frac{1}{22}$
$\frac{1}{20}$
$\frac{1}{18}$
$1/N_{\max}$
$L_{\max} = 5 \cdot N_{\max} - 50$
$L_{\max} = 8 \cdot N_{\max} - 100$
$L_{\max} = 10 \cdot N_{\max} - 140$

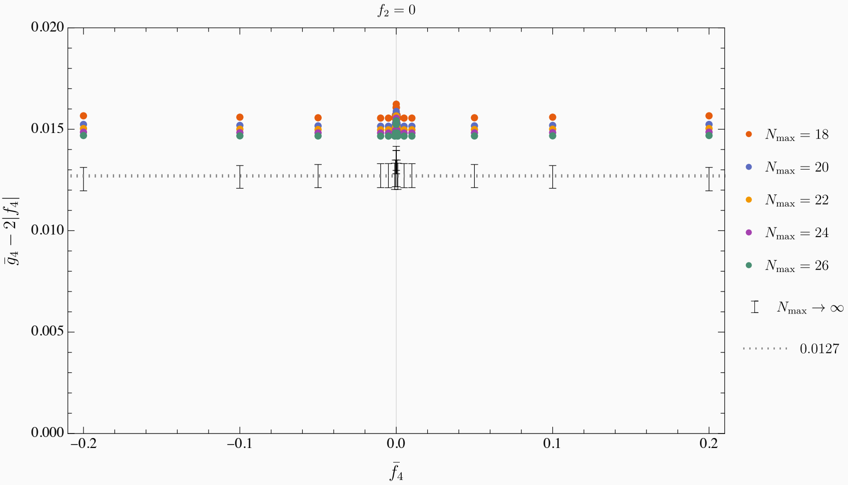
$\bar{g}_4$ vs $\bar{f}_2$



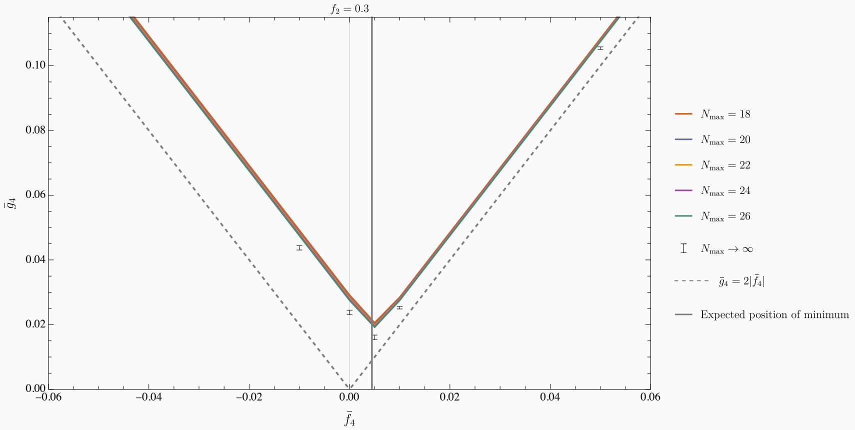
$\bar{g}_4$ vs $\bar{f}_4$ at $\bar{f}_2 = 0$



$\bar{g}_4 - 2|f_4|$ vs $\bar{f}_4$ at $\bar{f}_2 = 0$



$\bar{g}_4$ vs $\bar{f}_4$ at $\bar{f}_2 = 0.3$



Future directions/Open Questions

- Numerics difficult for massless particles - perhaps a better ansatz is necessary.
- What about Electron-Photon scattering or Graviton scattering in 4d? How to handle Faddeev-Kulish states?
- Graviton scattering in 5d (avoids IR divergences)?
- Incorporating particle production?

Thank You!