The L^2 geometry of the moduli space of vortices on the two-sphere in the dissolving limit

Martin Speight (Leeds)
Rene García Lara (Universidad Autonoma de Yucatan)

22/8/22

Rene $\longrightarrow$ 

Vortices on the sphere

- Hermitian line bundle L over $\Sigma = (S^2, g_\Sigma)$, degree n

$$E(\phi, A) = \frac{1}{2}\|\mathrm{d}_A\phi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{8}\|\tau - |\phi|^2\|_{L^2}^2$$

Vortices on the sphere

- Hermitian line bundle L over $\Sigma = (S^2, g_\Sigma)$, degree n

$$E(\phi, A) = \frac{1}{2}\|\mathrm{d}_A\phi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{8}\|\tau - |\phi|^2\|_{L^2}^2$$

- Bogomolny bound: $E \geq \tau\pi n$, equality $\Leftrightarrow$

$$\begin{aligned} \overline{\partial}_A\phi &= 0 && (V1) \\ *F_A &= \frac{1}{2}(\tau - |\phi|^2) && (V2) \end{aligned}$$

Vortices on the sphere

- Hermitian line bundle L over $\Sigma = (S^2, g_\Sigma)$, degree n

$$E(\phi, A) = \frac{1}{2}\|\mathrm{d}_A\phi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{8}\|\tau - |\phi|^2\|_{L^2}^2$$

- Bogomolny bound: $E \geq \tau\pi n$, equality $\Leftrightarrow$

$$\begin{aligned} \overline{\partial}_A\phi &= 0 && (V1) \\ *F_A &= \frac{1}{2}(\tau - |\phi|^2) && (V2) \end{aligned}$$

- Bradlow bound: $\int_\Sigma (V2)$:

$$\begin{aligned} 2\pi n &= \frac{1}{2}\tau|\Sigma| - \frac{1}{2}\|\phi\|_{L^2}^2 \\ \|\phi\|_{L^2}^2 &= \tau|\Sigma| - 4\pi n =: \varepsilon > 0 \end{aligned}$$

The moduli space (Bradlow)

$$\overline{\partial}_A \phi = 0 \qquad (V1), \qquad *F_A = \frac{1}{2}(\tau - |\phi|^2) \qquad (V2)$$

$$M_n = \{\text{solns } (\phi, A) \text{ of (V1), (V2)}\}/\text{gauge group}$$

The moduli space (Bradlow)

$$\bar{\partial}_A \phi = 0 \qquad (V1), \qquad *F_A = \frac{1}{2}(\tau - |\phi|^2) \qquad (V2)$$

$$M_n = \{\text{solns } (\phi, A) \text{ of (V1), (V2)}\}/\text{gauge group}$$

$\varepsilon < 0$ $M_n = \emptyset$	$\varepsilon = 0$ $M_n = \{pt\}$	$\varepsilon > 0$ $M_n \equiv \mathbb{C}P^n$

The moduli space (Bradlow)

$$\bar{\partial}_A \phi = 0 \qquad (V1), \qquad *F_A = \frac{1}{2}(\tau - |\phi|^2) \qquad (V2)$$

$$M_n = \{\text{solns } (\phi, A) \text{ of (V1), (V2)}\}/\text{gauge group}$$

$\varepsilon < 0$	$\varepsilon = 0$	$\varepsilon > 0$
$M_n = \emptyset$	$M_n = \{pt\}$	$M_n \equiv \mathbb{C}P^n$

- $\varepsilon = 0$: $\phi = 0$, $A = \widehat{A}$, const curv

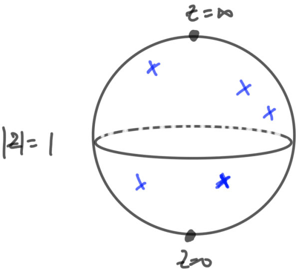
The moduli space (Bradlow)

$$\overline{\partial}_A \phi = 0 \qquad (V1), \qquad *F_A = \frac{1}{2}(\tau - |\phi|^2) \qquad (V2)$$

$$M_n = \{\text{solns } (\phi, A) \text{ of (V1), (V2)}\}/\text{gauge group}$$

$\varepsilon < 0$	$\varepsilon = 0$	$\varepsilon > 0$
$M_n = \emptyset$	$M_n = \{pt\}$	$M_n \equiv \mathbb{C}P^n$

- $\varepsilon = 0$: $\phi = 0$, $A = \widehat{A}$, const curv
- $\varepsilon > 0$: $[(\phi, A)]$ uniquely determined by **divisor** (ϕ)



$$\leftrightarrow p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$
$$\leftrightarrow [a_0, a_1, \ldots, a_n]$$

The L^2 metric on M_n

- Any curve of solutions $(\phi(t), A(t))$ of $(V1), (V2)$ defines a tangent vector to M_n at $[(\phi(0), A(0))]$, v say

The L^2 metric on M_n

- Any curve of solutions $(\phi(t), A(t))$ of $(V1), (V2)$ defines a tangent vector to M_n at $[(\phi(0), A(0))]$, v say
- Length of v? Project $(\dot{\phi}(0), \dot{A}(0))$ L^2 $\perp$ gauge orbit through $(\phi(0), A(0))$

$$g_{L^2}(v, v) = \|P(\dot{\phi}(0), \dot{A}(0))\|_{L^2}^2$$

The L^2 metric on M_n

- Any curve of solutions $(\phi(t), A(t))$ of $(V1), (V2)$ defines a tangent vector to M_n at $[(\phi(0), A(0))]$, v say
- Length of v? Project $(\dot{\phi}(0), \dot{A}(0))$ L^2 $\perp$ gauge orbit through $(\phi(0), A(0))$

$$g_{L^2}(v, v) = \|P(\dot{\phi}(0), \dot{A}(0))\|_{L^2}^2$$

- Controls low energy vortex dynamics (Manton, Stuart)

The L^2 metric on M_n

- Any curve of solutions $(\phi(t), A(t))$ of $(V1), (V2)$ defines a tangent vector to M_n at $[(\phi(0), A(0))]$, v say
- Length of v? Project $(\dot{\phi}(0), \dot{A}(0))$ $L^2 \perp$ gauge orbit through $(\phi(0), A(0))$

$$g_{L^2}(v, v) = \|P(\dot{\phi}(0), \dot{A}(0))\|_{L^2}^2$$

- Controls low energy vortex dynamics (Manton, Stuart)
- Kähler. $[\omega_{L^2}]$ known explicitly (Baptista)

$$|M_n| = \frac{\pi^n \varepsilon^n}{n!}$$

The L^2 metric on M_n

- Any curve of solutions $(\phi(t), A(t))$ of $(V1), (V2)$ defines a tangent vector to M_n at $[(\phi(0), A(0))]$, v say
- Length of v? Project $(\dot{\phi}(0), \dot{A}(0))$ $L^2 \perp$ gauge orbit through $(\phi(0), A(0))$

$$g_{L^2}(v, v) = \|P(\dot{\phi}(0), \dot{A}(0))\|_{L^2}^2$$

- Controls low energy vortex dynamics (Manton, Stuart)
- Kähler. $[\omega_{L^2}]$ known explicitly (Baptista)

$$|M_n| = \frac{\pi^n \varepsilon^n}{n!}$$

- Rescale: $g_\varepsilon := \varepsilon^{-1} g_{L^2}$

The L^2 metric on M_n

- g_ε, $\varepsilon \in (0, \infty)$: one parameter family of Kähler metrics on M_n of fixed volume

The L^2 metric on M_n

- g_ε, $\varepsilon \in (0, \infty)$: one parameter family of Kähler metrics on M_n of fixed volume
- Limit $\varepsilon \to \infty$ studied by Mundet i Riera & Romao and (independently) Nagy (2017): vortices become pointlike, g_ε converges to product metric on Σ^n / S_n

The L^2 metric on M_n

- g_ε, $\varepsilon \in (0, \infty)$: one parameter family of Kähler metrics on M_n of fixed volume
- Limit $\varepsilon \to \infty$ studied by Mundet i Riera & Romao and (independently) Nagy (2017): vortices become pointlike, g_ε converges to product metric on Σ^n / S_n
- We're interested in opposite limit, $\varepsilon \to 0$:
  Baptista-Manton conjecture: $\lim_{\varepsilon \to 0} g_\varepsilon =$ Fubini-Study metric

The conjecture

- Equip L with hol structure $\overline{\partial}_L = \overline{\partial}_{\widehat{A}}$

$$H^0(L) = \{\phi \in \Gamma(L) : \overline{\partial}_{\widehat{A}}\phi = 0\} \equiv \mathbb{C}^{n+1}$$

The conjecture

- Equip L with hol structure $\overline{\partial}_L = \overline{\partial}_{\widehat{A}}$

$$H^0(L) = \{\phi \in \Gamma(L) : \overline{\partial}_{\widehat{A}}\phi = 0\} \equiv \mathbb{C}^{n+1}$$

- $S = \{\widehat{\phi} \in H^0(L) : \|\widehat{\phi}\|_{L^2} = 1\}$, unit sphere

$$\pi : S \to (\mathbb{P}(H^0(L)), g_{FS}), \qquad \widehat{\phi} \mapsto \{c\widehat{\phi}\}$$

g_{FS} **defined** s.t. π a Riem sub

The conjecture

- Equip L with hol structure $\overline{\partial}_L = \overline{\partial}_{\widehat{A}}$

$$H^0(L) = \{\phi \in \Gamma(L) : \overline{\partial}_{\widehat{A}}\phi = 0\} \equiv \mathbb{C}^{n+1}$$

- $S = \{\widehat{\phi} \in H^0(L) : \|\widehat{\phi}\|_{L^2} = 1\}$, unit sphere

$$\pi : S \to (\mathbb{P}(H^0(L)), g_{FS}), \qquad \widehat{\phi} \mapsto \{c\widehat{\phi}\}$$

  g_{FS} **defined** s.t. π a Riem sub
- $f : M_n \to \mathbb{P}(H^0(L))$, $[(\phi, A)] \mapsto \{\psi \in H^0(L) : (\psi) = (\phi)\}$

The conjecture

- Equip L with hol structure $\overline{\partial}_L = \overline{\partial}_{\widehat{A}}$

$$H^0(L) = \{\phi \in \Gamma(L) : \overline{\partial}_{\widehat{A}}\phi = 0\} \equiv \mathbb{C}^{n+1}$$

- $S = \{\widehat{\phi} \in H^0(L) : \|\widehat{\phi}\|_{L^2} = 1\}$, unit sphere

$$\pi : S \to (\mathbb{P}(H^0(L)), g_{FS}), \qquad \widehat{\phi} \mapsto \{c\widehat{\phi}\}$$

  g_{FS} **defined** s.t. π a Riem sub
- $f : M_n \to \mathbb{P}(H^0(L))$, $[(\phi, A)] \mapsto \{\psi \in H^0(L) : (\psi) = (\phi)\}$
- “The” Fubini-Study metric: $g_0 := f^* g_{FS}$

The conjecture

- Equip L with hol structure $\overline{\partial}_L = \overline{\partial}_{\widehat{A}}$

$$H^0(L) = \{\phi \in \Gamma(L) : \overline{\partial}_{\widehat{A}}\phi = 0\} \equiv \mathbb{C}^{n+1}$$

- $S = \{\widehat{\phi} \in H^0(L) : \|\widehat{\phi}\|_{L^2} = 1\}$, unit sphere

$$\pi : S \to (\mathbb{P}(H^0(L)), g_{FS}), \qquad \widehat{\phi} \mapsto \{c\widehat{\phi}\}$$

  g_{FS} **defined** s.t. π a Riem sub
- $f : M_n \to \mathbb{P}(H^0(L))$, $[(\phi, A)] \mapsto \{\psi \in H^0(L) : (\psi) = (\phi)\}$
- "The" Fubini-Study metric: $g_0 := f^* g_{FS}$
- Baptista-Manton conjecture: $\lim_{\varepsilon \to 0} g_\varepsilon = g_0$
- Surprising? Massive gain in symmetry

The theorem

Theorem (JMS, RGL 2022) In the limit $\varepsilon \searrow 0$, g_ε converges in C^0 topology to g_0.

The theorem

Theorem (JMS, RGL 2022) In the limit $\varepsilon \searrow 0$, g_ε converges in C^0 topology to g_0.

More precisely:

There exists $C > 0$ such that, for all $v \in TM_n$ and all $\varepsilon \in (0, 1)$

$$|g_\varepsilon(v, v) - g_0(v, v)| \leq C\varepsilon g_0(v, v)$$

The proof

- Given divisor D, exists $\widehat{\phi} \in S \subset H^0(L)$ with $(\widehat{\phi}) = D$
  Unique up to $\widehat{\phi} \mapsto e^{ic}\widehat{\phi}$

The proof

- Given divisor D, exists $\widehat{\phi} \in S \subset H^0(L)$ with $(\widehat{\phi}) = D$
  Unique up to $\widehat{\phi} \mapsto e^{ic}\widehat{\phi}$
- Pseudovortex: $(\sqrt{\varepsilon}\widehat{\phi}, \widehat{A})$
  Satisfies (V1) vacuously. Satisfies $\int_\Sigma (V2)$

The proof

- Given divisor D, exists $\widehat{\phi} \in S \subset H^0(L)$ with $(\widehat{\phi}) = D$
  Unique up to $\widehat{\phi} \mapsto e^{ic}\widehat{\phi}$
- Pseudovortex: $(\sqrt{\varepsilon}\widehat{\phi}, \widehat{A})$
  Satisfies (V1) vacuously. Satisfies $\int_\Sigma (V2)$
- Deform this to obtain true vortex with $(\phi) = D$:

$$(\phi, A) = (\sqrt{\varepsilon}\widehat{\phi}e^{u/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u)$$

for some smooth $u : \Sigma \to \mathbb{R}$.

The proof

- Given divisor D, exists $\widehat{\phi} \in S \subset H^0(L)$ with $(\widehat{\phi}) = D$
  Unique up to $\widehat{\phi} \mapsto e^{ic}\widehat{\phi}$
- Pseudovortex: $(\sqrt{\varepsilon}\widehat{\phi}, \widehat{A})$
  Satisfies (V1) vacuously. Satisfies $\int_\Sigma (V2)$
- Deform this to obtain true vortex with $(\phi) = D$:

$$(\phi, A) = (\sqrt{\varepsilon}\widehat{\phi}e^{u/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u)$$

  for some smooth $u : \Sigma \to \mathbb{R}$.
  - Satisfies (V1) automatically
  - Satisfies (V2) iff

$$\Delta u - \frac{\varepsilon}{|\Sigma|} + \varepsilon|\widehat{\phi}|^2 e^u = 0.$$

The proof

- Given divisor D, exists $\widehat{\phi} \in S \subset H^0(L)$ with $(\widehat{\phi}) = D$
  Unique up to $\widehat{\phi} \mapsto e^{ic}\widehat{\phi}$
- Pseudovortex: $(\sqrt{\varepsilon}\widehat{\phi}, \widehat{A})$
  Satisfies (V1) vacuously. Satisfies $\int_\Sigma (V2)$
- Deform this to obtain true vortex with $(\phi) = D$:

$$(\phi, A) = (\sqrt{\varepsilon}\widehat{\phi}e^{u/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u)$$

  for some smooth $u : \Sigma \to \mathbb{R}$.
  - Satisfies (V1) automatically
  - Satisfies (V2) iff

$$\Delta u - \frac{\varepsilon}{|\Sigma|} + \varepsilon|\widehat{\phi}|^2 e^u = 0.$$

- Energy estimate, elliptic estimate, Sobolev $\Rightarrow$

$$\|u\|_{C^0} \leq C\varepsilon.$$

  Vortices are uniformly well approximated by pseudovortices (for small ε)

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

Unpacking that a little. . .

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\Delta u_0 - \frac{\varepsilon}{|\Sigma|} + \varepsilon|\widehat{\phi}|^2 e^{\overline{u}} e^{u_0} = 0.$$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker\Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1}\int_\Sigma u$

$$\int_\Sigma \left(\Delta u_0 - \frac{\varepsilon}{|\Sigma|} + \varepsilon|\widehat{\phi}|^2 e^{\overline{u}} e^{u_0}\right) = 0.$$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$0 - \varepsilon + \varepsilon \int_\Sigma |\widehat{\phi}|^2 e^{\overline{u}} e^{u_0} = 0.$$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\overline{u} = \ln\left(\int_\Sigma |\widehat{\phi}|^2 e^{u_0}\right).$$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\overline{u} = \ln\left(\int_\Sigma |\widehat{\phi}|^2 e^{u_0}\right).$$

- Monotonicity, $\|\widehat{\phi}\|_{L^2} = 1$: $|\overline{u}| \leq \|u_0\|_{C^0}$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\overline{u} = \ln\left(\int_\Sigma |\widehat{\phi}|^2 e^{u_0}\right).$$

- Monotonicity, $\|\widehat{\phi}\|_{L^2} = 1$: $|\overline{u}| \leq \|u_0\|_{C^0}$
- To bound $\|u\|_{C^0}$ it suffices to bound $\|u_0\|_{C^0}$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\overline{u} = \ln\left(\int_\Sigma |\widehat{\phi}|^2 e^{u_0}\right).$$

- Monotonicity, $\|\widehat{\phi}\|_{L^2} = 1$: $|\overline{u}| \leq \|u_0\|_{C^0}$
- To bound $\|u\|_{C^0}$ it suffices to bound $\|u_0\|_{H^2}$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\overline{u} = \ln\left(\int_\Sigma |\widehat{\phi}|^2 e^{u_0}\right).$$

- Monotonicity, $\|\widehat{\phi}\|_{L^2} = 1$: $|\overline{u}| \leq \|u_0\|_{C^0}$
- To bound $\|u\|_{C^0}$ it suffices to bound $\|\Delta u_0\|_{H^2}$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\overline{u} = \ln\left(\int_\Sigma |\widehat{\phi}|^2 e^{u_0}\right).$$

- Monotonicity, $\|\widehat{\phi}\|_{L^2} = 1$: $|\overline{u}| \leq \|u_0\|_{C^0}$
- To bound $\|u\|_{C^0}$ it suffices to bound $\|\Delta u\|_{L^2}$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\overline{u} = \ln\left(\int_\Sigma |\widehat{\phi}|^2 e^{u_0}\right).$$

- Monotonicity, $\|\widehat{\phi}\|_{L^2} = 1$: $|\overline{u}| \leq \|u_0\|_{C^0}$
- To bound $\|u\|_{C^0}$ it suffices to bound $\|F_A - F_{\widehat{A}}\|_{L^2}$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\overline{u} = \ln\left(\int_\Sigma |\widehat{\phi}|^2 e^{u_0}\right).$$

- Monotonicity, $\|\widehat{\phi}\|_{L^2} = 1$: $|\overline{u}| \leq \|u_0\|_{C^0}$
- To bound $\|u\|_{C^0}$ it suffices to bound $\|F_A\|_{L^2}$

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\overline{u} = \ln \left(\int_\Sigma |\widehat{\phi}|^2 e^{u_0} \right).$$

- Monotonicity, $\|\widehat{\phi}\|_{L^2} = 1$: $|\overline{u}| \leq \|u_0\|_{C^0}$
- To bound $\|u\|_{C^0}$ it suffices to bound E

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

$$\|u\|_{H^2}^2 = \int_\Sigma u^2 + |\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2$$

- SEE: for all smooth $u \perp_{L^2} \ker \Delta$ (i.e. with $\int_\Sigma u = 0$),

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\overline{u} = \ln\left(\int_\Sigma |\widehat{\phi}|^2 e^{u_0}\right).$$

- Monotonicity, $\|\widehat{\phi}\|_{L^2} = 1$: $|\overline{u}| \leq \|u_0\|_{C^0}$
- To bound $\|u\|_{C^0}$ it suffices to bound $\qquad E = \tau\pi n$

Unpacking that a little...

- So... $\|u\|_{C^0} \leq C$

Unpacking that a little...

- So... $\|u\|_{C^0} \leq C$
- But

$$\Delta u_0 - \frac{\varepsilon}{|\Sigma|} + \varepsilon|\widehat{\phi}|^2 e^u = 0$$

Unpacking that a little...

- So... $\|u\|_{C^0} \leq C$
- But

$$\begin{aligned} & \Delta u_0 - \frac{\varepsilon}{|\Sigma|} + \varepsilon|\widehat{\phi}|^2 e^u = 0 \\ \Rightarrow \quad & \|\Delta u_0\|_{L^2} \leq C\varepsilon \end{aligned}$$

Unpacking that a little...

- So... $\|u\|_{C^0} \leq C$
- But

$$\begin{aligned} & \Delta u_0 - \frac{\varepsilon}{|\Sigma|} + \varepsilon |\widehat{\phi}|^2 e^u = 0 \\ \Rightarrow \quad & \|\Delta u_0\|_{L^2} \leq C\varepsilon \\ \Rightarrow \quad & \|u\|_{C^0} \leq C\varepsilon \end{aligned}$$

Unpacking that a little...

- So... $\|u\|_{C^0} \leq C$
- But

$$\begin{aligned} & \Delta u_0 - \frac{\varepsilon}{|\Sigma|} + \varepsilon|\widehat{\phi}|^2 e^u = 0 \\ \Rightarrow \quad & \|\Delta u_0\|_{L^2} \leq C\varepsilon \\ \Rightarrow \quad & \|u\|_{C^0} \leq C\varepsilon \end{aligned}$$

- Vortices are uniformly well approximated by pseudovortices (for small ε)

Unpacking that a little...

- So... $\|u\|_{C^0} \leq C$
- But

$$\begin{aligned} & \Delta u_0 - \frac{\varepsilon}{|\Sigma|} + \varepsilon|\widehat{\phi}|^2 e^u = 0 \\ \Rightarrow \quad & \|\Delta u_0\|_{L^2} \leq C\varepsilon \\ \Rightarrow \quad & \|u\|_{C^0} \leq C\varepsilon \end{aligned}$$

- Vortices are uniformly well approximated by pseudovortices (for small ε)
- Now we need to estimate the *metric*

The proof (cont)

- Take a curve of vortex solutions

$$(\phi(t), A(t)) = (\sqrt{\varepsilon}\widehat{\phi}(t)e^{u(t)/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u(t))$$

The proof (cont)

- Take a curve of vortex solutions
$$(\phi(t), A(t)) = (\sqrt{\varepsilon}\widehat{\phi}(t)e^{u(t)/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u(t))$$
- $\|\widehat{\phi}(t)\|^2_{L^2} \equiv 1 \Rightarrow \langle \widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$

The proof (cont)

- Take a curve of vortex solutions

$$(\phi(t), A(t)) = (\sqrt{\varepsilon}\widehat{\phi}(t)e^{u(t)/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u(t))$$

- $\|\widehat{\phi}(t)\|^2_{L^2} \equiv 1 \Rightarrow \langle \widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$
- WLOG $\langle i\widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$

The proof (cont)

- Take a curve of vortex solutions

$$(\phi(t), A(t)) = (\sqrt{\varepsilon}\widehat{\phi}(t)e^{u(t)/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u(t))$$

- $\|\widehat{\phi}(t)\|_{L^2}^2 \equiv 1 \Rightarrow \langle \widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$
- WLOG $\langle i\widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$
- Then $g_0(v, v) = \|\dot{\widehat{\phi}}(0)\|_{L^2}^2$

The proof (cont)

- Take a curve of vortex solutions

$$(\phi(t), A(t)) = (\sqrt{\varepsilon}\widehat{\phi}(t)e^{u(t)/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u(t))$$

- $\|\widehat{\phi}(t)\|_{L^2}^2 \equiv 1 \Rightarrow \langle \widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$
- WLOG $\langle i\widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$
- Then $g_0(v, v) = \|\dot{\widehat{\phi}}(0)\|_{L^2}^2$
- $(\dot{\phi}, \dot{A})$ controlled by $\dot{u}$

$$\Delta u - \frac{\varepsilon}{|\Sigma|} + \varepsilon|\widehat{\phi}|^2 e^u = 0.$$

The proof (cont)

- Take a curve of vortex solutions

$$(\phi(t), A(t)) = (\sqrt{\varepsilon}\widehat{\phi}(t)e^{u(t)/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u(t))$$

- $\|\widehat{\phi}(t)\|_{L^2}^2 \equiv 1 \Rightarrow \langle \widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$
- WLOG $\langle i\widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$
- Then $g_0(v, v) = \|\dot{\widehat{\phi}}(0)\|_{L^2}^2$
- $(\dot{\phi}, \dot{A})$ controlled by $\dot{u}$

$$\Delta \dot{u} + \varepsilon|\widehat{\phi}|^2 e^u \dot{u} = -2\varepsilon e^u h(\widehat{\phi}, \dot{\widehat{\phi}})$$

The proof (cont)

- Take a curve of vortex solutions

$$(\phi(t), A(t)) = (\sqrt{\varepsilon}\widehat{\phi}(t)e^{u(t)/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u(t))$$

- $\|\widehat{\phi}(t)\|^2_{L^2} \equiv 1 \Rightarrow \langle \widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$
- WLOG $\langle i\widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$
- Then $g_0(v, v) = \|\dot{\widehat{\phi}}(0)\|^2_{L^2}$
- $(\dot{\phi}, \dot{A})$ controlled by $\dot{u}$

$$\Delta\dot{u} + \varepsilon|\widehat{\phi}|^2 e^u \dot{u} = -2\varepsilon e^u h(\widehat{\phi}, \dot{\widehat{\phi}})$$

- "SEE for $\boxed{\Delta + \varepsilon|\widehat{\phi}|^2 e^u}$ " $\Rightarrow \|\dot{u}\|_{H^2} \leq C\varepsilon\|\dot{\widehat{\phi}}\|_{L^2}$

The proof (cont)

- Take a curve of vortex solutions

$$(\phi(t), A(t)) = (\sqrt{\varepsilon}\widehat{\phi}(t)e^{u(t)/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u(t))$$

- $\|\widehat{\phi}(t)\|^2_{L^2} \equiv 1 \Rightarrow \langle \widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$
- WLOG $\langle i\widehat{\phi}, \dot{\widehat{\phi}} \rangle_{L^2} = 0$
- Then $g_0(v, v) = \|\dot{\widehat{\phi}}(0)\|^2_{L^2}$
- $(\dot{\phi}, \dot{A})$ controlled by $\dot{u}$

$$\Delta \dot{u} + \varepsilon|\widehat{\phi}|^2 e^u \dot{u} = -2\varepsilon e^u h(\widehat{\phi}, \dot{\widehat{\phi}})$$

- "SEE for $\boxed{\Delta + \varepsilon|\widehat{\phi}|^2 e^u}$ " $\Rightarrow \|\dot{u}\|_{H^2} \leq C\varepsilon \|\dot{\widehat{\phi}}\|_{L^2}$
- Good enough to bound $|g_\varepsilon - g_0|$.

Convergence of spec Δ

- Spectrum of Δ on (M, g)

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \leq \cdots$$

Convergence of spec Δ

- Spectrum of Δ on (M, g)

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \leq \cdots$$

- **Corollary** (JMS,RGL): There exists $C, \varepsilon_* > 0$ such that, for all $k \in \mathbb{Z}^+$ and all $\varepsilon \in (0, \varepsilon_*)$,

$$\left| \frac{\lambda_k(g_\varepsilon)}{\lambda_k(g_0)} - 1 \right| \leq C\varepsilon$$

Convergence of spec Δ

- Spectrum of Δ on (M, g)

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \leq \cdots$$

- **Corollary** (JMS,RGL): There exists $C, \varepsilon_* > 0$ such that, for all $k \in \mathbb{Z}^+$ and all $\varepsilon \in (0, \varepsilon_*)$,

$$\left| \frac{\lambda_k(g_\varepsilon)}{\lambda_k(g_0)} - 1 \right| \leq C\varepsilon$$

- Spectrum of M_n converges uniformly to spectrum of FS!

Convergence of spec Δ

- Spectrum of Δ on (M, g)

$$0 = \lambda_0(g) < \lambda_1(g) \le \lambda_2(g) \le \lambda_3(g) \le \cdots$$

- **Corollary** (JMS,RGL): There exists $C, \varepsilon_* > 0$ such that, for all $k \in \mathbb{Z}^+$ and all $\varepsilon \in (0, \varepsilon_*)$,

$$\left| \frac{\lambda_k(g_\varepsilon)}{\lambda_k(g_0)} - 1 \right| \le C\varepsilon$$

- Spectrum of M_n converges uniformly to spectrum of FS!
- Surprising this follows from only C^0 convergence!

$$\Delta = -g^{ij}\left(\frac{\partial^2}{\partial x_i \partial x_j} + \Gamma_{ij}^k \frac{\partial}{\partial x_k}\right)$$

Open questions

- Urakawa-Bando (1983): for any finite dimensional subspace $V \subset C^\infty(M)$
$$\Lambda_g(V) := \sup\left\{\frac{\|\mathrm{d}f\|_{L^2}^2}{\|f\|_{L^2}^2} : f \in V\backslash\{0\}\right\}.$$
Then
$$\lambda_k(g) = \inf\{\Lambda_g(V) : \dim V = k+1\}$$

Open questions

- Urakawa-Bando (1983): for any finite dimensional subspace $V \subset C^\infty(M)$

  $$\Lambda_g(V) := \sup \left\{ \frac{\|\mathrm{d}f\|_{L^2}^2}{\|f\|_{L^2}^2} : f \in V \backslash \{0\} \right\}.$$

  Then

  $$\lambda_k(g) = \inf\{\Lambda_g(V) : \dim V = k + 1\}$$

- Corollary easily follows

Open questions

- Convergence of geodesics? Need $g_\varepsilon \to g_0$ in C^1
- Convergence of curvature? Need $g_\varepsilon \to g_0$ in C^2
- n-dependence of the bounds?
- Leading correction to g_0?
- Higher genus? Much more subtle (Manton, Romao)