Gravitating vortices, cosmic strings, and the Kähler–Yang–Mills–Higgs equations

Luis Álvarez-Cónsul

ICMAT & CSIC, Madrid

Workshop on Gauged Maps, Vortices and Their Moduli
Les Diablerets, 22–26 August 2022

Based on joint work with Mario Garcia-Fernandez, Oscar Garcia-Prada, Vamsi Pingali and Chengjian Yao.

Geom. Topol. (2013),
Comm. Math. Phys. (2017),
Pure Appl. Math. Q. (2019),
Math. Ann. (2021),
`arXiv:2201.03455 [math.DG]`,
and further work in progress.

Subject of this talk:

Gauged maps coupled to Kähler metrics

Our first goal: couple gauge theory to gravity in Kähler geometry

Our first goal: couple gauge theory to gravity in Kähler geometry

- M. García-Fernández, PhD Thesis. ICMAT, Madrid, 2009, `arXiv:1102.0985 [math.DG]`
- ___, M. García-Fernández and O. García-Prada, G&T (2013).

1. Gravitating vortices from Kähler–Yang–Mills

Our first goal: couple gauge theory to gravity in Kähler geometry

- M. García-Fernández, PhD Thesis. ICMAT, Madrid, 2009, `arXiv:1102.0985 [math.DG]`
- ___, M. García-Fernández and O. García-Prada, G&T (2013).

Data: M compact (Kählerian) complex manifold
$E \longrightarrow M$ holomorphic vector bundle over M

1. Gravitating vortices from Kähler–Yang–Mills

Our first goal: couple gauge theory to gravity in Kähler geometry

- M. García-Fernández, PhD Thesis. ICMAT, Madrid, 2009, `arXiv:1102.0985 [math.DG]`
- ___, M. García-Fernández and O. García-Prada, G&T (2013).

Data: M compact (Kählerian) complex manifold
$E \longrightarrow M$ holomorphic vector bundle over M

The Kähler–Yang–Mills equations (KYM)

for a Kähler 2-form (or metric) ω on M and a Hermitian metric h on E:

$$i\Lambda_\omega F_h = \mu(E)\,\mathrm{Id}_E$$

$$S_\omega - \alpha\Lambda_\omega^2 \operatorname{Tr} F_h^2 = C$$

- $F_h \in \Omega^2(X)$ curvature 2-form of Chern connection of h,
- $\Lambda_\omega \colon \Omega^i(X) \to \Omega^{i-2}(X)$ contraction with ω, so $\Lambda_g^2 \operatorname{Tr} F_h^2 \in C^\infty(M)$,
- $S_\omega \in C^\infty(M)$ scalar curvature of Kähler metric $g = \omega(\cdot, J\cdot)$,
- $\alpha > 0$ coupling constant, $\mu(E), C \in \mathbb{R}$ determined by the topology.

1. Gravitating vortices from Kähler–Yang–Mills

Data: M compact (Kählerian) complex manifold
$E \longrightarrow M$ holomorphic vector bundle over M

The Kähler–Yang–Mills equations (KYM)

for a Kähler 2-form (or metric) ω on M and a Hermitian metric h on E:

$$i\Lambda_\omega F_h = \mu(E)\,\mathrm{Id}_E$$

$$S_\omega - \alpha\Lambda_\omega^2 \,\mathrm{Tr}\, F_h^2 = C$$

- $F_h \in \Omega^2(X)$ curvature 2-form of Chern connection of h,
- $\Lambda_\omega \colon \Omega^i(X) \to \Omega^{i-2}(X)$ contraction with ω, so $\Lambda_g^2 \,\mathrm{Tr}\, F_h^2 \in C^\infty(M)$,
- $S_\omega \in C^\infty(M)$ scalar curvature of Kähler metric $g = \omega(\cdot, J\cdot)$,
- $\alpha > 0$ coupling constant, $\mu(E), C \in \mathbb{R}$ determined by the topology.

The KYM equations have a **symplectic interpretation** that will be reviewed later in more generality.

Second goal: couple the above equations to a ‘gauged map’

Second goal: couple the above equations to a 'gauged map'

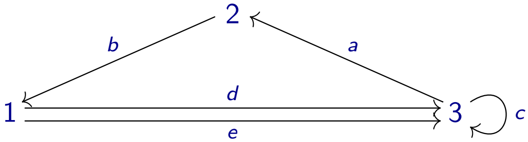
We start with an example: gravitating vortices on quiver bundles.

Second goal: couple the above equations to a 'gauged map'

We start with an example: gravitating vortices on quiver bundles.

Fixed data: X compact (Kählerian) complex manifold,
Q quiver with vertex set Q_0, arrow set Q_1, tail and head maps $t, h\colon Q_1 \to Q_0$.

Example:

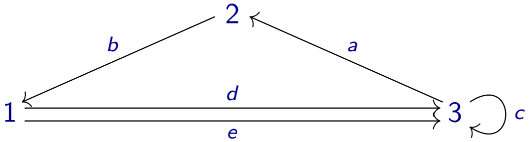


Second goal: couple the above equations to a ‘gauged map’

We start with an example: gravitating vortices on quiver bundles.

Fixed data: X compact (Kählerian) complex manifold,
Q quiver with vertex set Q_0, arrow set Q_1, tail and head maps $t, h\colon Q_1 \to Q_0$.

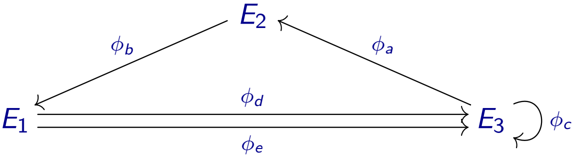
Example:



A *holomorphic Q-bundle* is a pair $(E,\phi)=(\{E_i\}_{i\in Q_0},\{\phi_a\}_{a\in Q_1})$ consisting of:

- $\{E_i\}_{i\in Q_0}$ family of holom. vector bundles on X indexed by the vertices,
- $\{\phi_a\}_{a\in Q_1}$ family of homomorphisms $\phi_a\colon E_{ta}\to E_{ha}$ indexed by the arrows.

Example:

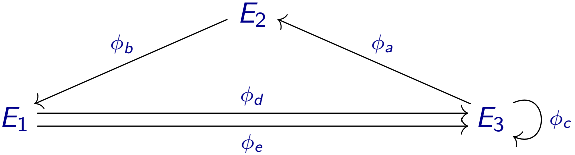


Gravitating vortices on quiver bundles:

A *holomorphic* Q*-bundle* is a pair $(E,\phi)=(\{E_i\}_{i\in Q_0},\{\phi_a\}_{a\in Q_1})$ consisting of:

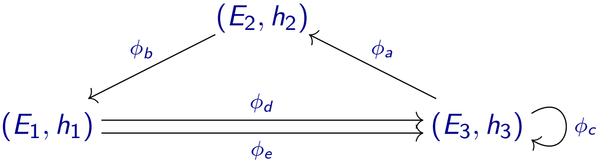
- $\{E_i\}_{i\in Q_0}$ family of holom. vector bundles on X indexed by the vertices,
- $\{\phi_a\}_{a\in Q_1}$ family of homomorphisms $\phi_a : E_{ta} \to E_{ha}$ indexed by the arrows.

Example:



A *Hermitian metric* $h=\{h_i\}_{i\in Q_0}$ on a Q-bundle (E,ϕ) is a family of Hermitian metrics h_i on E_i, indexed by the vertices.

Example:



Gravitating vortices on quiver bundles:

A *holomorphic Q-bundle* is a pair $(E,\phi)=(\{E_i\}_{i\in Q_0},\{\phi_a\}_{a\in Q_1})$ consisting of:

- $\{E_i\}_{i\in Q_0}$ family of holom. vector bundles on X indexed by the vertices,
- $\{\phi_a\}_{a\in Q_1}$ family of homomorphisms $\phi_a\colon E_{ta}\to E_{ha}$ indexed by the arrows.

A *Hermitian metric* $h=\{h_i\}_{i\in Q_0}$ on a Q-bundle (E,ϕ) is a family of Hermitian metrics h_i on E_i, indexed by the vertices.

The gravitating quiver vortex equations

Fix constants $\rho\in\mathbb{R}_{>0}$, $\sigma\in\mathbb{R}_{>0}^{Q_0}$ and $\tau\in\mathbb{R}^{Q_0}$. The *gravitating quiver (ρ,σ,τ)-vortex equations* for a pair (ω,h), consisting of a Kähler metric ω on X and a Hermitian metric h on a holomorphic Q-bundle (E,ϕ), are

$$\sigma_i \mathrm{i}\Lambda_\omega F_{h_i}+\sum_{a\in h^{-1}(i)}\phi_a\circ\phi_a^{*h_a}-\sum_{a\in t^{-1}(i)}\phi_a^{*h_a}\circ\phi_a=\tau_i\,\mathrm{Id}_{E_i},$$

$$S_\omega-\rho\sum_{i\in Q_0}\sigma_i\Lambda_\omega^2\,\mathrm{Tr}\,F_{h_i}^2+2\rho\sum_{a\in Q_1}\left(\Delta_\omega+2\left(\frac{\tau_{ha}}{\sigma_{ha}}-\frac{\tau_{ta}}{\sigma_{ta}}\right)\right)|\phi_a|_{h_a}^2=c.$$

How do the gravitating quiver vortex equations arise?

Homogeneous bundles and holomorphic quiver bundles:

Data: H/P flag manifold, where

H connected simply connected semisimple complex Lie group

$P \subset H$ parabolic subgroup

Homogeneous bundles and holomorphic quiver bundles:

Data: H/P flag manifold, where
H connected simply connected semisimple complex Lie group
$P \subset H$ parabolic subgroup

Theorem (Bondal & Kapranov 1990, Hille 1998, ___ & García-Prada, 2003, etc.)

There are a quiver with relations $(Q, \mathcal{R})$ depending on parabolic subgroup $P \subset H$, and an equivalence of categories

$$\left\{ \begin{array}{c} H\text{-equivariant holomorphic} \\ \text{vector bundles on } X \times H/P \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{holomorphic } Q\text{-bundles on } X \\ \text{that satisfy the relations } \mathcal{R} \end{array} \right\}$$

Homogeneous bundles and holomorphic quiver bundles:

Data: H/P flag manifold, where
H connected simply connected semisimple complex Lie group
$P \subset H$ parabolic subgroup

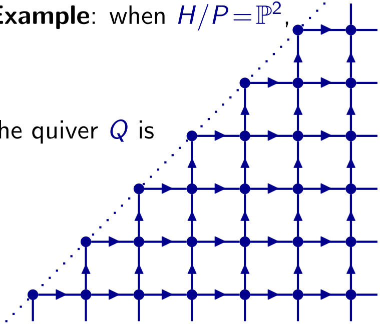
Theorem (Bondal & Kapranov 1990, Hille 1998, ___ & García-Prada, 2003, etc.)

There are a quiver with relations $(Q, \mathcal{R})$ depending on parabolic subgroup $P \subset H$, and an equivalence of categories

$$\left\{ \begin{matrix} H\text{-equivariant holomorphic} \\ \text{vector bundles on } X \times H/P \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{holomorphic } Q\text{-bundles on } X \\ \text{that satisfy the relations } \mathcal{R} \end{matrix} \right\}$$

Example: when $H/P = \mathbb{P}^2$,

the quiver Q is



and the relations are 'commutative diagrams'.

Gravitating vortices from equivariant dimensional reduction:

Data:

- Kähler form $\omega_{H/P}$ on H/P that is invariant under action of maximal compact subgroup $K \subset H$,
- Kähler forms ω on X and $\widetilde{\omega} := \omega \oplus \omega_{H/P}$ on $M = X \times H/P$,
- Holomorphic $(Q, \mathcal{R})$-bundle (E, ϕ) on X, with corresponding H-equivariant holomorphic vector bundle $\widetilde{E}$ on $M = X \times H/P$,
- Hermitian metric $h = \{h_i\}_{i \in Q_0}$ on holomorphic quiver bundle (E, ϕ), with corresponding K-invariant Hermitian metric $\widetilde{h}$ on $\widetilde{E} \to M = X \times H/P$.

Gravitating vortices from equivariant dimensional reduction:

Data:

- Kähler form $\omega_{H/P}$ on H/P that is invariant under action of maximal compact subgroup $K \subset H$,
- Kähler forms ω on X and $\widetilde{\omega} := \omega \oplus \omega_{H/P}$ on $M = X \times H/P$,
- Holomorphic $(Q, \mathcal{R})$-bundle (E, ϕ) on X, with corresponding H-equivariant holomorphic vector bundle $\widetilde{E}$ on $M = X \times H/P$,
- Hermitian metric $h = \{h_i\}_{i \in Q_0}$ on holomorphic quiver bundle (E, ϕ), with corresponding K-invariant Hermitian metric $\widetilde{h}$ on $\widetilde{E} \to M = X \times H/P$.

Then:

Theorem (___, M. Garcia-Fernandez, O. Garcia-Prada, 2019)

There are explicit formulas for the constants $\rho \in \mathbb{R}_{>0}$, $\sigma \in \mathbb{R}_{>0}^{Q_0}$ and $\tau \in \mathbb{R}^{Q_0}$, such that the pair $(\widetilde{\omega}, \widetilde{h})$ is a solution of the Kähler–Yang–Mills equations for $\widetilde{E}$ and $M = X \times H/P$ if and only if (ω, h) is a solution of the gravitating quiver (ρ, σ, τ)-vortex equations for (E, ϕ) and X.

Abelian gravitating vortices:

Particular case of the above theorem:

$H/P = \mathbb{P}^1$	complex projective line, with Kähler form $(4/\tau)\omega_{FS}$,
X	compact Riemann surface,
$L \longrightarrow X$	holomorphic line bundle,
$\phi \in H^0(X, L)$	holomorphic section.

ϕ determines an SU(2)-equivariant rank 2 holomorphic vector bundle $\widetilde{E}$ over $X \times \mathbb{P}^1$:

$$0 \longrightarrow p^*L \longrightarrow \widetilde{E} \longrightarrow q^*\mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0$$

Abelian gravitating vortices:

Particular case of the above theorem:

$H/P = \mathbb{P}^1$	complex projective line, with Kähler form $(4/\tau)\omega_{FS}$,
X	compact Riemann surface,
$L \longrightarrow X$	holomorphic line bundle,
$\phi \in H^0(X, L)$	holomorphic section.

ϕ determines an SU(2)-equivariant rank 2 holomorphic vector bundle $\widetilde{E}$ over $X \times \mathbb{P}^1$:

$$0 \longrightarrow p^*L \longrightarrow \widetilde{E} \longrightarrow q^*\mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0$$

Proposition (___, M. Garcia-Fernandez, O. Garcia-Prada, 2017).
SU(2)*-invariant solutions of the KYM equations on* $\widetilde{E} \longrightarrow X \times \mathbb{P}^1$ *are equivalent to solutions of:*

Gravitating vortex equations (or 'self-dual Einstein–Maxwell–Higgs eqns')

for a Kähler metric ω on X and a Hermitian metric h on L:

$$i\Lambda_\omega F_h + |\phi|_h^2 = \tau$$

$$S_\omega + \alpha(\Delta_\omega + \tau)(|\phi|_h^2 - \tau) = c$$

Gravitating vortex = solution of the gravitating vortex equations

Einstein–Bogomol'nyi equations & cosmic strings:

- Physics: **Einstein–Bogomol'nyi equations** $\stackrel{\text{def}}{\Longleftrightarrow}$ gravitating vortex equations **with $c = 0$**.

- Solutions of the Einstein–Bogomol'nyi equations $\Longleftrightarrow$ Nielsen–Olesen cosmic strings (1973) in the Bogomol'nyi phase, i.e., solutions of coupled Abelian Einstein–Higgs equations, in the Bogomol'nyi phase, on $\mathbb{R}^{1,1} \times X$ independent of variables in $\mathbb{R}^{1,1}$.

Einstein–Bogomol'nyi equations & cosmic strings:

- Physics: **Einstein–Bogomol'nyi equations** $\overset{\text{def}}{\Longleftrightarrow}$ gravitating vortex equations **with** $c = 0$.
- Solutions of the Einstein–Bogomol'nyi equations $\Longleftrightarrow$ Nielsen–Olesen cosmic strings (1973) in the Bogomol'nyi phase, i.e., solutions of coupled Abelian Einstein–Higgs equations, in the Bogomol'nyi phase, on $\mathbb{R}^{1,1} \times X$ independent of variables in $\mathbb{R}^{1,1}$.
- Cosmic strings are a model (by spontaneous symmetry breaking) for topological defects in the early universe.
- $\alpha = 2\pi G$, $G > 0$ is universal gravitation constant

Physics literature: Linet (1988), Comtet–Gibbons (1988), Spruck–Yisong Yang (1995), Yisong Yang (1995)...

2. Our moduli problem for gauged maps

Fixed data: (X,ω) compact (Kählerian) symplectic manifold,
G compact Lie group with complexification $G^{\mathbb{C}}$,
$E \longrightarrow X$ (C^∞) principal G-bundle,
$(F,\hat{J},\hat{\omega})$ Hamiltonian Kähler G-manifold,
$\mathcal{F} := E \times_G F \to X$ associated fibre bundle.

Moduli problem

Construct and study

$\mathcal{M} := \Big\{$ moduli space of triples (J, A, ϕ), consisting of
- J=complex structure on X,
- A=connection on E,
- ϕ=global section of $\mathcal{F} \to X$,

such that:
- A determines a struct. of holomor. principal $G^{\mathbb{C}}$-bundle on $E^{\mathbb{C}} := E \times_G G^{\mathbb{C}}$ over (X, J),
- (X, J, ω) is a Kähler manifold
- the section $\phi\colon X \to \mathcal{F}$ is holomorphic. $\Big\}$

Standard methods used to construct moduli spaces:

(1) **(Scheme-theoretic) algebraic geometry:** As a quotient algebraic scheme applying Mumford's Geometric Invariant Theory.

(2) **Symplectic geometry:** As a symplectic quotient.

In this talk: method (2).

Applying method (1), one could have considered:

Moduli problem in algebraic geometry:

GIT algebraic construction of

$$\mathcal{M}_{\text{alg}} := \left\{ \begin{array}{l} \text{moduli space of triples } (M, F, \phi) \text{, consisting of} \\ \bullet\, M\text{: polarized complex projective variety,} \\ \bullet\, E \to M\text{: algebraic principal } G^{\mathbb{C}}\text{-bundle over } M, \\ \bullet\, \phi\colon M \to \mathcal{F}\text{: algebraic global section of} \\ \quad \text{associated bundle } \mathcal{F} := E \times_G F \to X. \end{array} \right\}$$

Close links are expected between the moduli spaces $\mathcal{M}$ and $\mathcal{M}_{\mathrm{alg}}$.

This expectation follows mainly from:

- the Theorem of Kempf & Ness for group actions on complex projective manifolds,
- the Theorem of Donaldson, Uhlenbeck & Yau for the Hermitian–Yang–Mills equation, and the generalization to gauged maps on Kähler fibrations by Mundet i Riera,
- the Theorem of Chen, Donaldson & Sun for the Kähler–Einstein equation on complex projective manifolds with $c_1 > 0$.

3. The Kähler–Yang–Mills–Higgs equations

Recall we fixed the following data: (X, ω) compact symplectic manifold

- G compact Lie group with **quadratic** Lie algebra $(\mathfrak{g}, (\cdot, \cdot))$
- $(F, \hat{J}, \hat{\omega})$ Hamiltonian Kähler G-manifold with moment map $\hat{\mu}\colon F \to \mathfrak{g}^*$
- $E \to X$ principal G-bundle with associated bundle $\mathcal{F} := E \times_G F \to X$
- $\alpha, \beta \in \mathbb{R}_{>0}$ coupling constants; $z \in \mathfrak{g}^G$ (invariant under Ad_G-action)

3. The Kähler–Yang–Mills–Higgs equations

Recall we fixed the following data: (X, ω) compact symplectic manifold

G compact Lie group with **quadratic** Lie algebra $(\mathfrak{g}, (\cdot, \cdot))$

$(F, \hat{J}, \hat{\omega})$ Hamiltonian Kähler G-manifold with moment map $\hat{\mu} \colon F \to \mathfrak{g}^*$

$E \to X$ principal G-bundle with associated bundle $\mathcal{F} := E \times_G F \to X$

$\alpha, \beta \in \mathbb{R}_{>0}$ coupling constants; $z \in \mathfrak{g}^G$ (invariant under Ad_G-action)

The Kähler–Yang–Mills–Higgs equations (KYMH)

for a complex structure J on (X, ω), a connection A on E such that $F_A \in \Omega_J^{1,1}(\mathrm{ad}\, E)$, and a section $\phi \in \Gamma(X, \mathcal{F})$ such that $\bar{\partial}_{J,A}\phi = 0$:

$$\alpha \Lambda_\omega F_A + \beta \hat{\mu}(\phi) = z,$$

$$S_J + \beta \Delta_\omega |\hat{\mu}(\phi)|^2 + \alpha \Lambda_\omega^2 (F_A \wedge F_A) - 4\alpha(\Lambda_\omega F_A, z) = c.$$

3. The Kähler–Yang–Mills–Higgs equations

Recall we fixed the following data: (X, ω) compact symplectic manifold

G compact Lie group with **quadratic** Lie algebra $(\mathfrak{g}, (\cdot, \cdot))$

$(F, \hat{J}, \hat{\omega})$ Hamiltonian Kähler G-manifold with moment map $\hat{\mu}\colon F \to \mathfrak{g}^*$

$E \to X$ principal G-bundle with associated bundle $\mathcal{F} := E \times_G F \to X$

$\alpha, \beta \in \mathbb{R}_{>0}$ coupling constants; $z \in \mathfrak{g}^G$ (invariant under Ad_G-action)

The Kähler–Yang–Mills–Higgs equations (KYMH)

for a complex structure J on (X, ω), a connection A on E such that $F_A \in \Omega^{1,1}_J(\mathrm{ad}\, E)$, and a section $\phi \in \Gamma(X, \mathcal{F})$ such that $\bar{\partial}_{J,A}\phi = 0$:

$$\alpha \Lambda_\omega F_A + \beta \hat{\mu}(\phi) = z,$$

$$S_J + \beta \Delta_\omega |\hat{\mu}(\phi)|^2 + \alpha \Lambda^2_\omega (F_A \wedge F_A) - 4\alpha(\Lambda_\omega F_A, z) = c.$$

- $F_A \in \Omega^2(X, \mathrm{ad}\, E)$ curvature 2-form of A
- $\hat{\mu}(\phi) \in \Gamma(X, \mathrm{ad}\, E)$ via inner product $(\cdot, \cdot)$ on $\mathfrak{g}$
- S_J scalar curvature of Kähler metric $g = \omega(\cdot, J\cdot)$
- $\Lambda_\omega\colon \Omega^i \to \Omega^{i-2}$ contraction with ω
- $(F_A \wedge F_A) \in \Omega^4(X)$, so contraction $\Lambda^2_\omega(F_A \wedge F_A) \in C^\infty(X)$

The Kähler–Yang–Mills–Higgs equations (KYMH)

for a complex structure J, a connection A and a global section ϕ:

$$\alpha\Lambda_\omega F_A + \beta\hat{\mu}(\phi) = z,$$
$$S_J + \beta\Delta_\omega|\hat{\mu}(\phi)|^2 + \alpha\Lambda_\omega^2(F_A \wedge F_A) - 4\alpha(\Lambda_\omega F_A, z) = c.$$

Remarks:

- The constant $c \in \mathbb{R}$ only depends on the cohomology class $[\omega]$ and the topology of X and E.
- We **recover** the Hermitian–Yang–Mills–Higgs equation

  $$\alpha\Lambda_\omega F_A + \beta\hat{\mu}(\phi) = z,$$

  (for fixed J) studied by Mundet i Riera, while the equation

  $$S_J = \text{const.}$$

  of the Donaldson–Tian–Yau theory is **deformed** (for $\alpha, \beta \neq 0$).

4. The symplectic origin of the KYMH equations

Data: ω symplectic form on compact manifold X of $\dim_{\mathbb{R}} X = 2n$

$\pi \colon E \longrightarrow X$ C^∞ principal G-bundle over X

$(F, \hat{J}, \hat{\omega})$ Hamiltonian Kähler G-manifold

4. The symplectic origin of the KYMH equations

Data: ω symplectic form on compact manifold X of $\dim_{\mathbb{R}} X = 2n$
$\pi \colon E \longrightarrow X$ C^∞ principal G-bundle over X
$(F, \hat{J}, \hat{\omega})$ Hamiltonian Kähler G-manifold

Define three ∞-dimensional manifolds:

$\mathcal{J} := \{\text{complex structures } J : TX \to TX \text{ on } (X, \omega)\}$
$\mathcal{A} := \{\text{connections } A \text{ on } E \to X\}$
$\mathcal{S} := \{C^\infty \text{ global sections } \phi \text{ of fibre bundle } \mathcal{F} := E \times_G F \to X\}$

4. The symplectic origin of the KYMH equations

Data: ω symplectic form on compact manifold X of $\dim_{\mathbb{R}} X = 2n$

$\pi\colon E \longrightarrow X$ C^∞ principal G-bundle over X

$(F, \hat{J}, \hat{\omega})$ Hamiltonian Kähler G-manifold

Define three ∞-dimensional manifolds:

$$\mathcal{J} := \{\text{complex structures } J : TX \to TX \text{ on } (X, \omega)\}$$

$$\mathcal{A} := \{\text{connections } A \text{ on } E \to X\}$$

$$\mathcal{S} := \{C^\infty \text{ global sections } \phi \text{ of fibre bundle } \mathcal{F} := E \times_G F \to X\}$$

They have **symplectic forms** $\omega_{\mathcal{J}}, \omega_{\mathcal{A}}$ and $\omega_{\mathcal{S}}$:

$$\omega_{\mathcal{J}}(\Phi_1, \Phi_2) = \frac{1}{2} \int_X \operatorname{Tr}(J\Phi_1\Phi_2)\omega^n/n!,$$

$$\omega_{\mathcal{A}}(a_1, a_2) = \int_X (a_1 \wedge a_2) \wedge \omega^{n-1}/(n-1)!,$$

$$\omega_{\mathcal{S}}(\dot{\phi}_1, \dot{\phi}_2) = \int_X \hat{\omega}(\dot{\phi}_1, \dot{\phi}_2)\omega^n/n!,$$

for all $\Phi_1, \Phi_2 \in T_J\mathcal{J}$, $a_1, a_2 \in T_A\mathcal{A}$, $\phi \in \mathcal{S}$, $\dot{\phi}_1, \dot{\phi}_2 \in T_\phi\mathcal{S}$, where

$$T_J\mathcal{J} = \{\Phi\colon TX \to TX \mid \Phi J + J\Phi = 0, \omega(\Phi\cdot, \cdot) = \omega(\cdot, \Phi\cdot)\},$$

$$T_A\mathcal{A} = \Omega^1(X, \operatorname{ad} E), \qquad T_\phi\mathcal{S} = \Gamma(X, \phi^* V\mathcal{F}).$$

The extended gauge group:

Fujiki–Donaldson:

- $\mathcal{H} := \{\text{Hamiltonian symplectomorphisms } (X,\omega) \to (X,\omega)\}$
- Symplectic action of group $\mathcal{H}$ on $(\mathcal{J},\omega_{\mathcal{J}})$ has moment map $\mu_{\mathcal{J}} : \mathcal{J} \to (\mathrm{Lie}\,\mathcal{H})^*$ such that

$$\mu_{\mathcal{J}}(J) = 0 \Longleftrightarrow S_{J,\omega} = \text{constant}$$

The extended gauge group:

Fujiki–Donaldson:

- $\mathcal{H} := \{\text{Hamiltonian symplectomorphisms } (X,\omega) \to (X,\omega)\}$
- Symplectic action of group $\mathcal{H}$ on $(\mathcal{J}, \omega_{\mathcal{J}})$ has moment map $\mu_{\mathcal{J}} : \mathcal{J} \to (\mathrm{Lie}\,\mathcal{H})^*$ such that

$$\mu_{\mathcal{J}}(J) = 0 \Longleftrightarrow S_{J,\omega} = \text{constant}$$

(Hamiltonian) extended gauge group:

$$\widetilde{\mathcal{G}} := \left\{ \begin{array}{c} G\text{-equivariant automorphisms } g \text{ of } E \text{ covering elements} \\ \check{g} \text{ of } \mathcal{H}, \text{ i.e. the following diagram commutes:} \\ \begin{array}{ccc} E & \xrightarrow[\cong]{g} & E \\ \pi \downarrow & & \downarrow \pi \\ (X,\omega) & \xrightarrow[\cong]{\check{g}} & (X,\omega) \end{array} \end{array} \right\}$$

The extended gauge group:

Fujiki–Donaldson:

- $\mathcal{H} := \{\text{Hamiltonian symplectomorphisms } (X,\omega) \to (X,\omega)\}$
- Symplectic action of group $\mathcal{H}$ on $(\mathcal{J}, \omega_{\mathcal{J}})$ has moment map $\mu_{\mathcal{J}} : \mathcal{J} \to (\mathrm{Lie}\,\mathcal{H})^*$ such that

$$\mu_{\mathcal{J}}(J) = 0 \Longleftrightarrow S_{J,\omega} = \text{constant}$$

(Hamiltonian) extended gauge group:

$$\widetilde{\mathcal{G}} := \left\{ \begin{array}{c} G\text{-equivariant automorphisms } g \text{ of } E \text{ covering elements} \\ \check{g} \text{ of } \mathcal{H}, \text{ i.e. the following diagram commutes:} \\ \begin{array}{ccc} E & \xrightarrow[\cong]{g} & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ (X,\omega) & \xrightarrow[\cong]{\check{g}} & (X,\omega) \end{array} \end{array} \right\}$$

Gauge group: $\mathcal{G} := \left\{ \begin{array}{c} G\text{-equivariant automorphisms of } E \\ \text{covering the identity on } X \end{array} \right\}.$

Group extension: $1 \to \mathcal{G} \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathcal{H} \to 1$

The extended gauge group action on $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{S}$:

- $\widetilde{\mathcal{G}}$ acts on $\mathcal{J}$ via $\widetilde{\mathcal{G}} \longrightarrow \mathcal{H}$, $g \longmapsto \check{g}$:

$$g \cdot J := \check{g}_* \circ J \circ \check{g}_*^{-1}.$$

The extended gauge group action on $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{S}$:

- $\widetilde{\mathcal{G}}$ acts on $\mathcal{J}$ via $\widetilde{\mathcal{G}} \longrightarrow \mathcal{H}$, $g \longmapsto \check{g}$:

$$g \cdot J := \check{g}_* \circ J \circ \check{g}_*^{-1}.$$

- $\widetilde{\mathcal{G}}$ acts on $\mathcal{A}$ by sending a connection $A \in \mathcal{A}$, viewed as a splitting (‘vertical projection’)

$$0 \longrightarrow VE \underset{\overset{A}{\curvearrowleft}}{\longrightarrow} TE \longrightarrow \pi^* TX \longrightarrow 0$$

where $\pi : E \to X$ =structure map and VE=vertical bundle, into the composite splitting

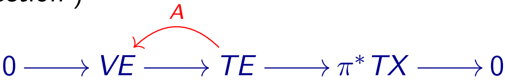
$$0 \longrightarrow \underset{g_*}{VE} \underset{\overset{A}{\curvearrowleft}}{\longrightarrow} \underset{g_*^{-1}}{TE} \longrightarrow \pi^* TX \longrightarrow 0$$

The extended gauge group action on $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{S}$:

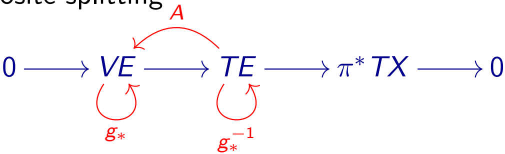
- $\widetilde{\mathcal{G}}$ acts on $\mathcal{J}$ via $\widetilde{\mathcal{G}} \longrightarrow \mathcal{H}$, $g \longmapsto \check{g}$:

$$g \cdot J := \check{g}_* \circ J \circ \check{g}_*^{-1}.$$

- $\widetilde{\mathcal{G}}$ acts on $\mathcal{A}$ by sending a connection $A \in \mathcal{A}$, viewed as a splitting ('vertical projection')



where $\pi\colon E \to X$ =structure map and VE=vertical bundle, into the composite splitting



- $\widetilde{\mathcal{G}}$ acts on $\mathcal{S}$ by sending a section $\phi \in \mathcal{S}$, viewed as a G-equivariant map $\phi\colon E \longrightarrow F$, into the composite

$$g \cdot \phi\colon E \longrightarrow F, \quad e \longmapsto g\phi(eg).$$

Define the **space of triples**

$$\mathcal{T} := \left\{ (J, A, \phi) \in \mathcal{J} \times \mathcal{A} \times \mathcal{S} \;\middle|\; \begin{array}{l} \bullet\, (X, J, \omega) \text{ is Kähler (i.e. } J \in \mathcal{J}), \\ \bullet\, A \text{ induces a structure of holomorphic} \\ \quad \text{principal } G^{\mathbb{C}}\text{-bundle on } E^{\mathbb{C}} := E \times_G G^{\mathbb{C}} \\ \quad \text{over } (X, J) \text{, i.e., } F_A \in \Omega_J^{1,1}(\operatorname{ad} E), \\ \bullet\, \phi \text{ is holomorphic, i.e., } \bar{\partial}_{J,A}\phi = 0 \end{array} \right\}.$$

Define the **space of triples**

$$\mathcal{T} := \left\{ (J, A, \phi) \in \mathcal{J} \times \mathcal{A} \times \mathcal{S} \;\middle|\; \begin{array}{l} \bullet\, (X, J, \omega) \text{ is Kähler (i.e. } J \in \mathcal{J}), \\ \bullet\, A \text{ induces a structure of holomorphic} \\ \quad \text{principal } G^{\mathbb{C}}\text{-bundle on } E^{\mathbb{C}} := E \times_G G^{\mathbb{C}} \\ \quad \text{over } (X, J), \text{ i.e., } F_A \in \Omega^{1,1}_J(\operatorname{ad} E), \\ \bullet\, \phi \text{ is holomorphic, i.e., } \bar{\partial}_{J,A}\phi = 0 \end{array} \right\}.$$

Symplectic form on $\mathcal{T}$: $\omega_{\alpha,\beta} := (\omega_{\mathcal{J}} + 4\alpha\omega_{\mathcal{A}} + 4\beta\omega_{\mathcal{S}})|_{\mathcal{T}}$ for fixed $\alpha, \beta \neq 0$

Diagonal action of extended gauge group $\widetilde{\mathcal{G}}$ on $\mathcal{T} \subset \mathcal{J} \times \mathcal{A} \times \mathcal{S}$

Proposition

- $\widetilde{\mathcal{G}}$-action on $(\mathcal{T}, \omega_{\alpha,\beta})$ has moment map $\mu_{\alpha,\beta} \colon \mathcal{T} \to (\operatorname{Lie} \widetilde{\mathcal{G}})^*$ s.t.

$$\mu^{-1}_{\alpha,\beta}(0) = \{\text{solutions of the KYMH equations}\}.$$

- For $\alpha, \beta > 0$, $(\mathcal{T}, \omega_{\alpha,\beta})$ has a $\widetilde{\mathcal{G}}$-invariant Kähler structure.
- Moduli space $\mathcal{M}_{\alpha,\beta} := \{\text{solutions of KYMH equations}\}/\widetilde{\mathcal{G}}$ is Kähler (away from singularities) for $\alpha, \beta > 0$.

5. Obstructions

Dual point of view: fix holomorphic data and vary metric data.

Fix: • J=complex structure on compact manifold X,
• $\Omega \in H^{1,1}(X)$ Kähler cohomology class,
• A =connection on E such that curvature $F_A \in \Omega^{1,1}_J(\mathrm{ad}\, E)$,
so (J, A) corresponds to a complex structure I on $E^{\mathbb{C}} = E \times_G G^{\mathbb{C}}$ s.t. $(E^{\mathbb{C}}, I) \to (X, J)$ is holomorphic principal $G^{\mathbb{C}}$-bundle,
• holomorphic section ϕ of $\mathcal{F} = E^{\mathbb{C}} \times_{G^{\mathbb{C}}} F \to (X, J)$.

Vary: pair of 'metrics' $b = (\omega, h)$ in space

$$B_I := \left\{ (\omega, h) \;\middle|\; \begin{array}{l} \bullet\, \omega \in \Omega \quad \text{symplectic form such that } (X, J, \omega) \text{ is Kähler}, \\ \bullet\, \text{section } h\colon X \to E^{\mathbb{C}}/G, \text{ viewed as a 'Hermitian metric'}, \\ \text{i.e., a reduction of } E^{\mathbb{C}} \text{ to a principal } G\text{-subbundle } E_h \subset E^{\mathbb{C}}. \end{array} \right\}$$

5. Obstructions

Fix: • J=complex structure on compact manifold X,

• $\Omega \in H^{1,1}(X)$ Kähler cohomology class,

• A =connection on E such that curvature $F_A \in \Omega^{1,1}_J(\text{ad } E)$,

so (J, A) corresponds to a complex structure I on $E^{\mathbb{C}} = E \times_G G^{\mathbb{C}}$ s.t. $(E^{\mathbb{C}}, I) \to (X, J)$ is holomorphic principal $G^{\mathbb{C}}$-bundle,

• holomorphic section ϕ of $\mathcal{F} = E^{\mathbb{C}} \times_{G^{\mathbb{C}}} F \to (X, J)$.

Vary: pair of 'metrics' $b = (\omega, h)$ in space

$$B_I := \left\{ (\omega, h) \;\middle|\; \begin{array}{l} \bullet\, \omega \in \Omega \quad \text{symplectic form such that } (X, J, \omega) \text{ is Kähler,} \\ \bullet \text{ section } h\colon X \to E^{\mathbb{C}}/G, \text{ viewed as a 'Hermitian metric',} \\ \text{i.e., a reduction of } E^{\mathbb{C}} \text{ to a principal } G\text{-subbundle } E_h \subset E^{\mathbb{C}}. \end{array} \right\}$$

View the KYMH equations as equations in the unknowns $b = (\omega, h) \in B_I$:

$$\alpha \Lambda_\omega F_h + \beta \hat{\mu}(\phi) = z,$$
$$S_\omega + \beta \Delta_\omega |\hat{\mu}(\phi)|^2 + \alpha \Lambda^2_\omega (F_h \wedge F_h) - 4\alpha(\Lambda F_h, z) = c.$$

Theorem (___, M. Garcia-Fernandez, O. Garcia-Prada, 2013)

B_I is a **symmetric space**, i.e. it has a canonical affine connection ∇ s.t.

- torsion $T_\nabla = 0$
- $\nabla R_\nabla = 0$, where R_∇ is the curvature

Theorem (___, M. Garcia-Fernandez, O. Garcia-Prada, 2013)

B_I is a **symmetric space**, i.e. it has a canonical affine connection ∇ s.t.

- torsion $T_\nabla = 0$
- $\nabla R_\nabla = 0$, where R_∇ is the curvature

A key fact for the symmetric-space construction is an **isomorphism**

$$\begin{aligned} \mathrm{Lie}\,\widetilde{\mathcal{G}}_b &\xrightarrow{\cong} T_b B_I \\ \zeta &\longmapsto v_\zeta := \begin{pmatrix} \text{infinitesimal action of} \\ I\zeta \in \mathfrak{X}(E_h) \text{ on } b \in B_I \end{pmatrix} \\ \zeta_v &\longleftarrow\!\shortmid v \end{aligned}$$

where

- $b = (\omega, h) \in B_I$,
- $\widetilde{\mathcal{G}}_b =$ extended gauge group of (X, ω) and E_h,
- $\mathrm{Lie}\,\widetilde{\mathcal{G}}_b =$ (Lie algebra of $\widetilde{\mathcal{G}}_b$) $\subset \mathfrak{X}(E_h)$,

Futaki invariant:

Use the family of moment maps for the $\widetilde{\mathcal{G}}_b$-action on $\mathcal{T}_b$ (for fixed α, β),

$$\mu_b \colon \mathcal{T}_b \to (\mathrm{Lie}\, \widetilde{\mathcal{G}}_b)^*, \quad \forall b \in B_I,$$

to define a **1-form σ_I on B_I** by

$$\sigma_I(v) := -\langle \mu_b(J, A, \phi), \zeta_v \rangle, \quad \forall b \in B_I, v \in T_b B_I.$$

Futaki invariant:

Use the family of moment maps for the $\widetilde{\mathcal{G}}_b$-action on $\mathcal{T}_b$ (for fixed α, β),

$$\mu_b\colon \mathcal{T}_b \to (\operatorname{Lie} \widetilde{\mathcal{G}}_b)^*, \quad \forall b \in B_I,$$

to define a **1-form σ_I on B_I** by

$$\sigma_I(v) := -\langle \mu_b(J, A, \phi), \zeta_v \rangle, \quad \forall b \in B_I, v \in T_b B_I.$$

Use 1-form σ_I to define a character, called the (α, β)-**Futaki invariant**,

$$\mathcal{F}_I\colon \operatorname{Lie}\operatorname{Aut}(E^{\mathbb{C}}, I, \phi) \longrightarrow \mathbb{C}, \quad \zeta \longmapsto \mathrm{i}\sigma_I(Y_{\zeta|b}) + \sigma_I(Y_{I\zeta|b})$$

(where "Y_ζ" means infinitesimal action of ζ) of

$$\operatorname{Lie}\operatorname{Aut}(E^{\mathbb{C}}, I, \phi) := \begin{pmatrix} \text{Lie algebra of infinitesimal automorphisms of holom.} \\ \text{principal } G^{\mathbb{C}}\text{-bundle } (E^{\mathbb{C}}, I) \to (X, J) \text{ preserving } \phi \in \mathcal{S} \end{pmatrix}.$$

Futaki invariant:

Use the family of moment maps for the $\widetilde{\mathcal{G}}_b$-action on $\mathcal{T}_b$ (for fixed α, β),

$$\mu_b \colon \mathcal{T}_b \to (\operatorname{Lie} \widetilde{\mathcal{G}}_b)^*, \quad \forall b \in B_I,$$

to define a **1-form** σ_I **on** B_I by

$$\sigma_I(v) := -\langle \mu_b(J, A, \phi), \zeta_v \rangle, \quad \forall b \in B_I, v \in T_b B_I.$$

Use 1-form σ_I to define a character, called the (α, β)-**Futaki invariant**,

$$\mathcal{F}_I \colon \operatorname{Lie} \operatorname{Aut}(E^{\mathbb{C}}, I, \phi) \longrightarrow \mathbb{C}, \quad \zeta \longmapsto \mathrm{i}\sigma_I(Y_{\zeta|b}) + \sigma_I(Y_{I\zeta|b})$$

(where "Y_ζ" means infinitesimal action of ζ) of

$$\operatorname{Lie} \operatorname{Aut}(E^{\mathbb{C}}, I, \phi) := \begin{pmatrix} \text{Lie algebra of infinitesimal automorphisms of holom.} \\ \text{principal } G^{\mathbb{C}}\text{-bundle } (E^{\mathbb{C}}, I) \to (X, J) \text{ preserving } \phi \in \mathcal{S} \end{pmatrix}.$$

Theorem (____, M. Garcia-Fernandez, O. Garcia-Prada, 2019)

The map $\mathcal{F}_I$ is independent of $b \in B_I$. It defines a character of $\operatorname{Lie} \operatorname{Aut}(E, I, \phi)$. Furthermore, if there exists a solution $b \in B_I$ of the Kähler–Yang–Mills–Higgs equations for (X, J) and $(E^{\mathbb{C}}, I)$, then $\mathcal{F}_I = 0$.

Formal notion of stability in terms of geodesics (following Donaldson):

Definition

$(J, A, \phi) \in \mathcal{T}$ is **geodesically (semi)stable** if

$$\lim_{t\to\infty} \sigma_I(\dot{b}_t) > 0 \, (\geq 0)$$

for every non-constant geodesic ray b_t $(0 \leq t < \infty)$ on (B_I, ∇).

Formal notion of stability in terms of geodesics (following Donaldson):

Definition

$(J, A, \phi) \in \mathcal{T}$ is **geodesically (semi)stable** if

$$\lim_{t \to \infty} \sigma_I(\dot{b}_t) > 0 \, (\geq 0)$$

for every non-constant geodesic ray b_t $(0 \leq t < \infty)$ on (B_I, ∇).

Technical problems (work in progress):

- One does not expect geodesics on B_I are smooth (cf. Lempert & Vivas 2013, Darvas & Lempert 2012).
- However one expects (B_I, ∇) is convex by 'weak geodesics' (Chen 2000, Błocki 2013, Chu, Tosatti & Weinkove 2017, etc.), so the above formal picture should be enhanced to include weak geodesics (cf. Chen & Cheng 2021, Darvas & Lu 2020).

Formal notion of stability in terms of geodesics (following Donaldson):

Definition

$(J, A, \phi) \in \mathcal{T}$ is **geodesically (semi)stable** if

$$\lim_{t\to\infty} \sigma_I(\dot{b}_t) > 0\,(\geq 0)$$

for every non-constant geodesic ray b_t $(0 \leq t < \infty)$ on (B_I, ∇).

Technical problems (work in progress):

- One does not expect geodesics on B_I are smooth.
- However one expects (B_I, ∇) is convex by 'weak geodesics', so the above formal picture should be enhanced to include weak geodesics.

Matsushima–Lichnerowicz for the KYMH equations:

Theorem (____, M. Garcia-Fernandez, O. Garcia-Prada, 2019)

Assume $H^1(X, \mathbb{R}) = 0$. If $(E^{\mathbb{C}}, I, \phi)$ admits a solution of the KYMH with $\alpha > 0$ and $\beta > 0$, then the Lie algebra $\mathrm{Lie\,Aut}(E^{\mathbb{C}}, I, \phi)$ is reductive.

Proof based on the moment-map interpretation of KYMH.

6. Return to the abelian gravitating vortex equations

Recall we fix the following data:

X	compact Riemann surface,
$L \longrightarrow X$	holomorphic line bundle,
$\phi \in H^0(X, L)$	non-zero holomorphic section.

Gravitating vortex equations (or 'self-dual Einstein–Maxwell–Higgs eqns') for Kähler metric ω on X and a Hermitian metric h on L:

$$\begin{aligned} i\Lambda_\omega F_h + |\phi|_h^2 - \tau &= 0 \\ S_\omega + \alpha(\Delta_g + \tau)(|\phi|_h^2 - \tau) &= c \end{aligned}$$

- $\tau > 0$, $\alpha > 0$ real parameters
- c is determined by the topology

Combining integration of the two equations:

$$c = \frac{2\pi}{\mathrm{vol}_\omega(X)}\left(\chi(X) - 2\alpha\tau c_1(L)\right)$$

Therefore the Einstein–Bogomol'nyi equations (i.e. $c = 0$) can only have solutions on the Riemann sphere (as $\alpha, \tau, \deg L \geq 0$):

$$c = 0 \Longrightarrow \chi(X) > 0 \Longrightarrow X = \mathbb{P}^1$$

The Einstein–Bogomol'nyi equations (genus $g = 0$):

Existence Theorem (Yisong Yang, 1995, 1997)

Let $D = \sum n_i p_i$ be an effective divisor on $\mathbb{P}^1$ corresponding to a pair (L, ϕ) s.t. $c = 0$ and $N := \sum n_i = c_1(L) < \frac{\mathrm{vol}_\omega(X)}{4\pi}\tau$.
Then the Einstein–Bogomol'nyi equations on $(\mathbb{P}^1, L, \phi)$ have solutions if

$$n_i < \frac{N}{2} \text{ for all } i. \qquad (*)$$

A solution also exists if $D = \frac{N}{2}p_1 + \frac{N}{2}p_2$, with $p_1 \neq p_2$ and N even.

Yang (1995) mentions $(*)$ *"is a technical restriction on the local string number. It is not clear at this moment whether it may be dropped"*.

Striking fact: Yang's "technical restriction" has an **algebro-geometric meaning**, for the natural action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathrm{Sym}^N \mathbb{P}^1 = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(N))$ (*binary quantics* [Sylvester 1882]):

$$n_i < \frac{N}{2} \text{ for all } i \iff D \in \mathrm{Sym}^N \mathbb{P}^1 \text{ is GIT stable}$$

$$D = \frac{N}{2}p_1 + \frac{N}{2}p_2 \iff D \in \mathrm{Sym}^N \mathbb{P}^1 \text{ is strictly GIT polystable}$$

It is natural to make the following:

Conjecture (___ , García-Fernández, García-Prada & Pingali)

The converse of Yang's theorem also holds:

existence of cosmic strings $\iff$ GIT-polystability.

A proof of the conjecture is not complete yet. Previous attempts were wrong, but seem to contain useful partial calculations to relate the existence of solutions to the Hilbert–Mumford weights. Technical difficulties mainly come from non-smoothness of the geodesics on B_l. Joint work in progress with García-Fernández, García-Prada & Pingali.

Positive results (genus $g = 0$):

Theorem (___, Garcia-Fernandez, Garcia-Prada, Pingali & Yao, 2022)

If ϕ has only one zero, then there are no solutions of the gravitating vortex equations for $(\mathbb{P}^1, L, \phi)$.

Proof. **This is a consequence of Matsushima–Lichnerowicz.** Since $L = \mathcal{O}(N)$ with $N > 0$, in suitable homogeneous coordinates $[x_0, x_1]$ of $\mathbb{P}^1$, $\phi \in H^0(\mathbb{P}^1, L) \cong \mathrm{S}^N(\mathbb{C}^2)^*$ is the homogeneous polynomial $\phi = x_0^N$. One can show that this implies that the group $\mathrm{Aut}(\mathbb{P}^1, L, \phi)$ is not reductive: $\mathrm{Aut}(\mathbb{P}^1, L, \phi) = \mathbb{C}^* \rtimes \mathbb{C}$. □

This theorem settles a conjecture of Yang (1997) in the affirmative.

Positive results (genus $g = 0$):

Theorem (____, Garcia-Fernandez, Garcia-Prada, Pingali & Yao, 2022)

If ϕ has only one zero, then there are no solutions of the gravitating vortex equations for $(\mathbb{P}^1, L, \phi)$.

Proof. **This is a consequence of Matsushima–Lichnerowicz.** Since $L = \mathcal{O}(N)$ with $N > 0$, in suitable homogeneous coordinates $[x_0, x_1]$ of $\mathbb{P}^1$, $\phi \in H^0(\mathbb{P}^1, L) \cong S^N(\mathbb{C}^2)^*$ is the homogeneous polynomial $\phi = x_0^N$. One can show that this implies that the group $\mathrm{Aut}(\mathbb{P}^1, L, \phi)$ is not reductive: $\mathrm{Aut}(\mathbb{P}^1, L, \phi) = \mathbb{C}^* \rtimes \mathbb{C}$. □

This theorem settles a conjecture of Yang (1997) in the affirmative.

Theorem (____, Garcia-Fernandez, Garcia-Prada, Pingali & Yao, 2022)

If there are solutions of the gravitating vortex equations for $(\mathbb{P}^1, L, \phi)$ corresponding to an effective divisor $D = n_1 p_1 + n_2 p_2$ with $p_1 \neq p_2$, then $n_1 = n_2$.

Proof. One calculates the (α, β)-Futaki invariant explicitly on $\mathbb{P}^1$, finding that in this case $\mathcal{F}_I \neq 0$ if $n_1 \neq n_2$.

Existence results for genus $g \geq 2$:

Theorem (___, M. Garcia-Fernandez, O. Garcia-Prada, V. P. Pingali, 2021)

Suppose the compact Riemann surface X has genus $g \geq 2$, the holomorphic line bundle $L \to X$ has degree $N > 0$ and $\phi \in H^0(X, L)$ is non-zero. Let τ be a real constant such that $0 < N < \tau/2$. Define

$$\alpha_* := \frac{2g-2}{2\tau(\tau/2 - N)} > 0.$$

Then the set of $\alpha \in \mathbb{R}$ for which the gravitating vortex equations have smooth solutions of volume 2π is open and contains the closed interval $[0, \alpha_*]$. Furthermore, the solution is unique for $\alpha \in [0, \alpha_*]$.

Proof by the continuity method: openness is based on the moment-map interpretation of the equations, and closedness requires a priori estimates that depend on the bound α_*.

7. Deformation of solutions of the KYMH equations in the weak coupling limit

The KYMH equations for $\alpha = \beta$ in the unknowns $b = (\omega, h) \in B_I$:

$$\Lambda_\omega F_h + \hat{\mu}(\phi) = z, \qquad (\star a)$$

$$S_\omega + \alpha \left(\Delta_\omega |\hat{\mu}(\phi)|^2 + \Lambda_\omega^2 (F_h \wedge F_h) - 4(\Lambda_\omega F_h, z) \right) = c. \qquad (\star b)$$

- $(\star a)$ was solved by Mundet i Riera (2000) using 'z-stability' for 'simple pairs' $(E^{\mathbb{C}}, \phi)$ via a Hitchin–Kobayashi correspondence.
- $(\star b)$ for $\alpha = 0$ is the famous cscK metric condition $S_\omega = c$.

We study now $(\star)$ in the '**weak coupling limit**' $\mathbf{0 < |\alpha| \ll 1}$.

7. Deformation of solutions of the KYMH equations in the weak coupling limit

The KYMH equations for $\alpha = \beta$ in the unknowns $b = (\omega, h) \in B_I$:

$$\Lambda_\omega F_h + \hat{\mu}(\phi) = z, \qquad (\star a)$$

$$S_\omega + \alpha \left(\Delta_\omega |\hat{\mu}(\phi)|^2 + \Lambda_\omega^2 (F_h \wedge F_h) - 4(\Lambda_\omega F_h, z) \right) = c. \qquad (\star b)$$

- $(\star a)$ was solved by Mundet i Riera (2000) using 'z-stability' for 'simple pairs' $(E^{\mathbb{C}}, \phi)$ via a Hitchin–Kobayashi correspondence.
- $(\star b)$ for $\alpha = 0$ is the famous cscK metric condition $S_\omega = c$.

We study now $(\star)$ in the '**weak coupling limit**' $\mathbf{0 < |\alpha| \ll 1}$.

Theorem (___, M. Garcia-Fernandez, O. Garcia-Prada, 2019)

Suppose $(E^{\mathbb{C}}, \phi)$ is a simple pair, $\exists$ cscK metric ω_0 on X with cohomology class $[\omega_0] = \Omega_0$ and $\nexists$ non-zero Hamiltonian Killing vector fields on X. If (E^c, ϕ) is z-stable with respect to ω_0, then $\exists$ open neighbhd $U \subset \mathbb{R} \times H^{1,1}(X, \mathbb{R})$ of $(0, \Omega_0)$ such that $\forall (\alpha, \Omega) \in U$ $\exists$ a solution $b = (\omega, h)$ of $(\star)$ with coupling constant α such that $[\omega] = \Omega$.

7. Deformation of solutions of the KYMH equations in the weak coupling limit

The KYMH equations for $\alpha = \beta$ in the unknowns $b = (\omega, h) \in B_I$:

$$\Lambda_\omega F_h + \hat{\mu}(\phi) = z, \qquad (\star a)$$

$$S_\omega + \alpha \left(\Delta_\omega |\hat{\mu}(\phi)|^2 + \Lambda_\omega^2 (F_h \wedge F_h) - 4(\Lambda_\omega F_h, z)\right) = c. \qquad (\star b)$$

Theorem (___, M. Garcia-Fernandez, O. Garcia-Prada, 2019)

Suppose $(E^{\mathbb{C}}, \phi)$ is a simple pair, $\exists$ cscK metric ω_0 on X with cohomology class $[\omega_0] = \Omega_0$ and $\nexists$ non-zero Hamiltonian Killing vector fields on X. If (E^c, ϕ) is z-stable with respect to ω_0, then $\exists$ open neighbhd $U \subset \mathbb{R} \times H^{1,1}(X, \mathbb{R})$ of $(0, \Omega_0)$ such that $\forall (\alpha, \Omega) \in U$ $\exists$ a solution $b = (\omega, h)$ of $(\star)$ with coupling constant α such that $[\omega] = \Omega$.

Proof. By the implicit function theorem in Banach spaces, combined with the above Theorem of Mundet i Riera (2000) and the moment map interpretation of the cscK metric condition $S_\omega = c$. □

Thank you!