Coulomb branches and Quantum cohomology

The symplectic topologist as a representation theorist

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Based on:

- joint work with Eduardo Gonzalez and Cheuk Yu Mak (https://arxiv.org/abs/2202.05785)
- work in progress with Constantin Teleman

3D TQFT Yoga

G is a connected compact Lie group, $G_{\mathbb{C}}$ is its complexification.

Slogan (Teleman '14)

A compact symplectic manifold M with Hamiltonian G-action should define an object $\mathbb{O}_G(M)$ in the Rozansky-Witten 2-category (or 3D B-model) of the universal centralizer of the Langlands dual group $\mathcal{Z}_{G^\vee_{\mathbb{C}}}$.

- More recently, [Bullimore, Dimofte, Gaiotto,Hilburn] and [Teleman] proposed that one can more generally associate an object in the B-model of the Coulomb branch with matter, $\mathrm{Spec}(\mathcal{C}_{G_{\mathbb{C}}}(V))$, for any complex representation V of $G_{\mathbb{C}}$.

3D TQFT Yoga (cont)

- An object of the 2-category is roughly speaking expected to be a holomorphic Lagrangian $\mathcal{L} \subset \text{Spec}(\mathcal{C}_{G_{\mathbb{C}}}(V))$ together with a sheaf of categories $\mathcal{C}$ over $\mathcal{L}$.

Thus, we expect

$$G \curvearrowright M => \mathcal{L}_{M,G}(V) \subset \text{Spec}(\mathcal{C}_{G_{\mathbb{C}}}(V)).$$

Goals of Talk

- *Explain how to define $\mathcal{L}_G(M) := \mathcal{L}_{M,G}(\{0\})$ for the "pure" Coulomb branch.*
- *Sketch implications of this picture for quantum cohomology of GIT quotients.*

Universal centralizer (classical perspective)

Let $K_{\mathbb{C}}$ be a complex semi-simple Lie group and $\mathfrak{k}_{\mathbb{C}}$ its Lie algebra.

- Let $\mathfrak{k}_{\mathbb{C}}^{reg}$ denote the set of regular elements whose stabilizer with respect to the adjoint action has minimal dimension $(r = rank(K_{\mathbb{C}}))$. Set:

$$T^*K_{\mathbb{C}}^{reg} := K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}^{reg}$$

- There is a Hamiltonian action on $T^*K_{\mathbb{C}}^{reg}$ given by $g \cdot (k, \phi) := (gkg^{-1}, \mathrm{Ad}_g(\phi))$. A moment map for this action is

$$\mu(k, \phi) = Ad_k(\phi) - \phi.$$

- The universal centralizer is the holomorphic symplectic reduction

$$\mathcal{Z}_{K_{\mathbb{C}}} := \mu(k, \phi)^{-1}(0)/K_{\mathbb{C}}.$$

Properties of $\mathcal{Z}_{K_{\mathbb{C}}}$

- $\mathcal{Z}_{K_{\mathbb{C}}}$ is a smooth, holomorphic symplectic affine variety.
- The map
$$\tau : \mathcal{Z}_{K_{\mathbb{C}}} \longrightarrow \mathfrak{k}_{\mathbb{C}}//K_{\mathbb{C}} \cong \mathbb{C}^r$$
is a totally integrable system with (holomorphic) Lagrangian fibers. The map
$$\pi : \mathcal{Z}_{K_{\mathbb{C}}} \longrightarrow K_{\mathbb{C}}//K_{\mathbb{C}} \cong T_{\mathbb{C}}/W,$$
(W=Weyl group) also has Lagrangian fibers.
- Because it is holomorphic symplectic, $\mathcal{Z}_{K_{\mathbb{C}}}$ admits a non-commutative deformation ("star quantization") $\mathcal{Z}_{K_{\mathbb{C}},u}$.

Universal centralizer (modern perspective)

The new(er) description we are after is the following:

Theorem (Bezrukavnikov-Finkelberg-Mirkovic '05)

There is an isomorphism of algebras

$$\Gamma(\mathcal{O}_{\mathcal{Z}_{G^\vee_{\mathbb{C}},u}}) \cong \hat{H}_*^{G\times S^1}(\Omega G).$$

In particular, setting $u = 0$, there is an isomorphism of affine varieties: $\mathcal{Z}_{G^\vee_{\mathbb{C}}} \cong \mathrm{Spec}(\hat{H}_*^G(\Omega G))$.

- The integrable system is now given by the projection $\tau : \mathrm{Spec}(\hat{H}_*^G(\Omega G)) \longrightarrow \mathrm{Spec}(H^*(BG))$.
- The algebra $\hat{H}_*^{\hat{G}}(LG/G)$ is Morita equivalent to $\hat{H}_*^{\hat{T}}(LG/T)$, which is a matrix algebra over it.

Quantum cohomology

Let (M^{2n}, ω) be a monotone closed symplectic manifold (that is $[\omega] = \lambda[c_1(M)] \in H^2(M)$ with $\lambda > 0$), equipped with a Hamiltonian action of a torus T.

- Let $QH^*_{S^1 \times T}(M)$ denote the quantum cohomology of M which is equivariant with respect to the T-action and loop rotation. As a vector space this is given by

$$QH^*_{S^1 \times T}(M) := H^*_T(M)[q^{\pm 1}, u]$$

  where q is the Novikov variable, and u is the positive generator of $H^*(BS^1)$.

This vector space carries much structure, the most elementary pieces of which are as follows:

- The reduction modulo u is the ordinary quantum cohomology $QH^*_T(M)$, which carries an equivariant quantum product.
- The full equivariant quantum cohomology $QH^*_{S^1 \times T}(M)$ carries a quantum connection $\nabla_{q\partial_q}$, which differentiates in the direction of the Novikov variable.

Module structures

Theorem (Gonzalez-Mak-P '22)

There is a module action

$$\mathcal{S} : \hat{H}_*^{S^1 \times T}(LG/T) \otimes QH^*_{S^1 \times T}(M) \longrightarrow QH^*_{S^1 \times T}(M)$$

From here, we get a map

$$\mathcal{S}_G : \hat{H}_*^{S^1 \times G}(\Omega G) \otimes QH^*_{S^1 \times G}(M) \longrightarrow QH^*_{S^1 \times G}(M)$$

Corollary

The support of $QH^*_G(M)_{|q=1}$ as a coherent sheaf over $BFM(G^\vee_{\mathbb{C}})$ is a (possibly singular) holomorphic Lagrangian subvariety $\mathcal{L}_G(M) \hookrightarrow \mathcal{Z}_{G^\vee_{\mathbb{C}}}$.

Remark

This result uses Gabber's famous result on the "involutivity of characteristics" for modules over a deformation quantization.

Examples

- Let M be a monotone toric variety acted on by T. There is a combinatorially defined "Hori-Vafa" superpotential $W_{HV} : T^\vee_{\mathbb{C}} \longrightarrow \mathbb{C}$. Then $\mathcal{L}_G(M)$ is given by

$$\mathrm{graph}(dW_{HV}) \subset T^* T^\vee_{\mathbb{C}}.$$

- (Conjectural) Let $M = G/T$, then there is an embedding of the classical Toda system $T^* T^\vee_{\mathbb{C}} \hookrightarrow \mathcal{Z}_{G^\vee_{\mathbb{C}}}$ (this involves an alternative "Toda" realization of $\mathcal{Z}_{G^\vee_{\mathbb{C}}}$ as a Hamiltonian reduction of $T^* G^\vee_{\mathbb{C}}$ by $N^\vee \times N^\vee$). Then $\mathcal{L}_G(G/T)$ is given by a cotangent fiber in $T^* T^\vee_{\mathbb{C}}$.

Seidel operators

The starting point is a classical construction of Seidel. For simplicity, let σ be a co-character $\sigma : S^1 \longrightarrow T$ (at this point could be an arbitrary element of $\Omega Ham(M, \omega)$ but not later on). We obtain a fiber bundle $E(\sigma) \longrightarrow \mathbb{C}P^1$ by gluing two copies of a disc

$$D_0^2 \times M \bigsqcup D_\infty^2 \times M / \sim$$
$$(x, e^{2\pi i\theta}) \sim (\sigma(\theta)x, e^{2\pi i\theta})$$

The divisors at $0, \infty$ are canonically diffeomorphic to M. Given a section class $A_\sigma \in H_2(E(\sigma), \mathbb{Z})$ we can form the moduli of two pointed sections $\bar{\mathcal{M}}_{0,2}(E(\sigma), A_\sigma)$. Using these moduli spaces, we can define a "push-pull operation"

$$Z \longrightarrow ev_{\infty,*}[Z \times_{ev_0} \bar{\mathcal{M}}_{0,2}(E(\sigma), A_\sigma)]q^{c_1^{vert}(A_\sigma)}$$

which gives rise to a $\mathbf{k}[q^{\pm}]$-linear operator

$$S_\sigma^{(0)} : QH^*(M) \longrightarrow QH^*(M)$$

Shift operators

The algebraic properties of Seidel operators are

1. $S_{\sigma_1} \cdot S_{\sigma_2} = S_{\sigma_1+\sigma_2}$.
2. The map $\sigma \longrightarrow S_\sigma(1)$ induces a ring homomorphism $\mathbf{k}[\mathcal{X}(T)] \longrightarrow QH^*(M)$.

Later on [Okounkov-Maulik] used the same idea to define shift-operators

$$S_\sigma : QH^*_{S^1\times T}(M) \longrightarrow QH^*_{S^1\times T}(M).$$

These have slightly different algebraic properties:

1. S_σ is a "σ-twisted" homomorphism (with respect to the equivariant parameters).
2. S_σ commutes with the quantum connection.

Shift operators(cont)

Let $A_\sigma : \hat{T} \longrightarrow \hat{T}$ take $(a, t) \longrightarrow (a, \sigma^{-1}(a)t)$.

- The natural $S^1 \times T$ action on $E(\sigma)$ is "twisted" meaning that at M_0 it acts by $\rho_{id}(a, t, m) = t \cdot m$ (the obvious $S^1 \times T$ action) but on M_∞ by $\rho_\sigma(a, t, m) = \rho_{id} \circ (A_\sigma, id_M)$. Thus the moduli space naturally produces a correlator:

$$C_\sigma : QH^*_{\rho_\sigma}(M) \longrightarrow QH^*_{\rho_{id}}(M).$$

from which $S_\sigma = C_\sigma \circ A^*_\sigma$

- This "asymmetry" leads to the module structure over the non-commutative ring, for example if $T = S^1$, and s is the variable for the Seidel operator, it says that

$$sh = (h + u)s \tag{1}$$

Parameterized shift operators

- Let U be the "universal Seidel space" over LG

$$U := LG \times M \times D_0^2 \bigsqcup LG \times M \times D_\infty^2 / \sim$$

where the equivalence relation is given by $(\gamma, x, e^{2\pi i\theta})_0 \sim (\gamma, \gamma(\theta)(x), e^{2\pi i\theta})_\infty$.

- Define an $S^1 \times T \times T$ action on the charts

$$(\tau, g, h) \cdot (\gamma, x, re^{2\pi i\theta})_0 = (g\gamma(\cdot - \tau)h^{-1}, hx, re^{2\pi i(\theta+\tau)})_0$$
$$(\tau, g, h) \cdot (\gamma, x, re^{2\pi i\theta})_\infty = (g\gamma(\cdot - \tau)h^{-1}, gx, re^{2\pi i(\theta+\tau)})_\infty$$

Let U/T be the quotient of U by the last T-action of $S^1 \times T \times T$. It is a free quotient and we have an induced Seidel space bundle

$$\pi_{U/T} : U/T \longrightarrow LG/T \tag{2}$$

(P.S. Continued)

- Consider the $(S^1 \times T)$-invariant subspaces of U/T

$$S_{T,0} = (LG \times M \times \{0\})/T \subset (LG \times M \times D_0^2)/T \tag{3}$$
$$S_{T,\infty} = (LG \times M \times \{0\})/T \subset (LG \times M \times D_\infty^2)/T \tag{4}$$

- Let $LG \times_T M$ denote the quotient of $LG \times M$ by the relation $(\gamma, x) \sim (\gamma g^{-1}, gx)$ for $g \in T$. The subspace $S_{T,0}$ is $(S^1 \times T)$-equivariantly isomorphic to $LG \times_T M$ with the action

$$(\tau, g) \cdot [\gamma, x] = [g\gamma(\cdot - \tau), x] \tag{5}$$

On the other hand, $S_{T,\infty}$ is $S^1 \times T$-equivariantly isomorphic to $(LG/T) \times M$ with the action

$$(\tau, g) \cdot (\gamma, x) = (g\gamma(\cdot - \tau), gx)$$

(P.S. continued)

- By counting appropriate $S^1 \times T$-equivariant parametrized moduli space in U/T and composing it with the map induced from the projection $(LG/T) \times M \longrightarrow M$, we obtain a map

$$\mathcal{C}_M : \hat{H}_*^{S^1 \times T}(LG \times_T M)[q^{\pm 1}] \longrightarrow \hat{H}_*^{S^1 \times T}(M)[q^{\pm 1}] \qquad (6)$$

  where $\mathcal{C}$ stands for correlation.
- There is a Kunneth map:

$$\mathcal{Q} : \hat{H}_*^{S^1 \times T}(LG/T) \otimes \hat{H}_*^{S^1 \times T}(M)[q^{\pm 1}] \longrightarrow \hat{H}_*^{S^1 \times T}(LG \times_T M)[q^{\pm 1}] \qquad (7)$$

- Finally, we define the paramaterized shift operator by

$$S := \mathcal{C} \circ \mathcal{Q}. \qquad (8)$$

Proving properties by localization

Once the shift operator has been constructed, many of its properties can be deduced from the $G = T$ case (appropriately twisted by the Weyl group) by a form of localization.

- Consider the algebraic affine flag variety, which is the quotient space $L_{poly}G/T \subset LG/T$. The fixed points of the remaining $S^1 \times T$ action on $L_{poly}G/T$ are given by points of the form $\sigma[w]$, where $\sigma \in \mathcal{X}(T)$ and $w \in N_G(T)$ so $[w] \in W$ is a Weyl element. So by the localization theorem,

$$F_R \otimes_R \hat{H}_*^{\hat{T}}(LG/T) \cong \bigoplus_{\sigma[w]\in\tilde{W}} F_R \cdot [\sigma[w]]_{\hat{T}} \tag{9}$$

- Similarly, $H^*_{\hat{T}}(M)$ is a free R-module and hence injects into $H^*_{\hat{T}}(M) \hookrightarrow F_R \otimes H^*_{\hat{T}}(M)$.

Proving properties by localization(cont)

- In the fixed point basis, there is an explicit multiplication law given by

$$(f_1 e_{\tilde{w}_1}) *_{fp} (f_2 e_{\tilde{w}_2}) = f_1(\mathcal{A}_{\tilde{w}_1}(f_2))(e_{\tilde{w}_1 \tilde{w}_2}) \tag{10}$$

where $f_1, f_2 \in F_R$ and $\tilde{w}_1 := \sigma_1[w_1], \tilde{w}_2 := \sigma_2[w_2] \in \tilde{W}$. Thus to prove the module property, we only have to verify that these relations hold (after tensoring with F_R).

- We also have a direct generalization of "commutation with the connection" in the non-abelian context.

Theorem

For any $\alpha \in \hat{H}_*^{S^1 \times T}(LG/T)$,

$$\mathcal{S}_\alpha \circ \nabla_{q\partial_q} = \nabla_{q\partial_q} \circ \mathcal{S}_\alpha$$

Quantum cohomology of symplectic quotients

Definition

A moment map will be called balanced if

1. $[c_1^G(TM)] = [\omega^G]$, where ω^G is the closed equivariant extension of ω determined by the moment map.
2. G-acts freely on $\mu^{-1}(0)$.

In this case, $M//G$ is again monotone. We can ask:

Question

Supposed $M//G$ is normalized. Is there a formula for $QH^*(M//G)$ in terms of $QH_G^*(M)$ and the action by non-abelian shift operators?

There is a tautological "Kostant" section L_u of the Toda projection τ.

Conjecture

The quantum cohomology of a normalized symplectic quotient

$$QH^*(M//G)_{|q=1} \cong QH^*_G(M)_{|q=1} \otimes_{\hat{H}^G_*(\Omega G)} \mathcal{O}_{L_u}$$

When $G = (S^1)^r$, this concretely says that

$$QH^*(M//G)_{|q=1} \cong \frac{QH^*_G(M)_{|q=1}}{(z_i = 1)}$$

where $z_1, \cdots, z_r$ are the Seidel operators.

Remark

In the case when M is a toric Fano manifold and T is a sub-torus, such a description can be deduced from (equivariant versions of) Batyrev's formula for quantum rings.

S^1-case

Theorem (P-Teleman, in progress)

Given an S^1 action on M with balanced moment map, $\mu : M \longrightarrow \mathbb{R}$ then $QH^*_{S^1}(M)$ is a free module over $\mathbb{C}[q^{\pm}, z^{\pm}]$ with rank $\dim(H^*(M//S^1, \mathbb{C}))$.

- Kirwan studied Morse theory of $\frac{1}{2}\mu^2$. In Hamiltonian Floer cohomology, this corresponds to using $\frac{1}{2}\epsilon\mu^2$.
- The basic idea of the proof is to use S^1-equivariant Hamiltonian Floer cohomology of $H_K := \frac{1}{2}K\mu^2$ as $K \longrightarrow \infty$.

Some comments on the proof

- The Floer complexes are all isomorphic in that there are natural isomorphisms $CF^*_{S^1}(M; H_K) \cong CF^*_{S^1}(M; H_{K'})$ (and indeed these are all isomorphic to S^1-equivariant $QH^*_{S^1}(M)$.)
- However the time-one periodic orbits of the Hamiltonian vector field change quite a bit, indeed there are more and more periodic orbits which appear near $\mu = 0$.
- So we as a first approximation take some $K_i \longrightarrow \infty$ and consider

$$CF^*_{S^1}(M; H) := \text{hocolim}_i \, CF^*_{S^1}(M; H_i) \tag{11}$$

where $H_i = 1/2K_i\mu^2$

- The problem is that this contains generators corresponding to other orbit sets other than the desired ones near zero (e.g. fixed points).
- So we want to filter this Floer complex so as to exclude these undesired orbits. The key to doing this is the so called monotone index of a capped periodic orbit which is defined to be

$$mix(x, [u]) = \deg(x, [u]) - 2A_{H_K}(x, [u]). \tag{12}$$

It is independent of the capping class $[u]$.

Lemma

For any $(x, [u]) \in \mathcal{X}(M; H_K)$, $mix(x, [u]) \geq K\mu^2 + C_0$ *for some constant* C_0 *independent of* K.

Take $\delta_i \longrightarrow 0$ such that $K_i\delta_i \longrightarrow \infty$. Let $\mathcal{F}_{\geq p}CF^*_{S^1}(M;H)$ be the subcomplex generated by orbits $(x,[u])$ which satisfy :

$$A_{H_i}(x,[u]) \geq p - K_i\delta_i.$$

Define

$$CF^*_{S^1}(M;H)^{(p)} := \sigma_{<p}\mathcal{F}_{\geq p}CF^*_{S^1}(M;H) \tag{13}$$

be the chains of degree $< p$. Set

$$\widetilde{CF}^*_{S^1}(M;H) := \mathrm{holim}_p\, CF^*_{S^1}(M;H)^{(p)} \tag{14}$$

Theorem

The q-adic filtration on $\widetilde{CF}^*_{S^1}(M;H)$ gives rise to a convergent spectral sequence with E_1 page

$$E_1 = H^*(M//S^1, \mathbb{Q}) \otimes \mathbb{C}[q^{\pm}, z^{\pm}]$$

and which converges to $QH^*_{S^1}(M)$. This spectral sequence collapses at E_1.

A vortex question

In fact, note that there is already a ring homomorphism

$$QK : QH^*_G(M) \longrightarrow QH^*(M//G). \tag{15}$$

For $G = T$, the main conjecture can (essentially) be reduced to the following purely vortex question:

Conjecture

Trivialize $G \cong (S^1)^r$ and let $z_1, \cdots, z_r$ denote each of the Seidel operators. Suppose the G-action is free on $\mu^{-1}(0)$, then

$$QK(z_i) = 1$$

where QK is the morphism from (15).