

Towards 3d mirror symmetry

2022-08-22

# Recollections on 2d mirror symmetry

## Topological Twisting

Super-Poincaré Lie algebra

$$\mathfrak{so}(2) \times \mathbb{R}^2 + \pi(s_+^{\oplus 2} \otimes s_-^{\oplus 2})$$

Supercharges

$$Q_A, Q_B$$

$$Q_A^2 = Q_B^2 = 0$$

$$Q$$

2d A-model  
TQFT

$$Q_A = 0$$

2d  $N=(2,2)$   
susy QFT

$$Q_B = 0$$

2D B-model  
TQFT

de Rham cohomology of

$$\{\bar{\partial}\sigma = 0\}$$

Calabi-Yau  $\sigma$ -model

Dolbeault cohomology of

$$\{d\sigma = 0\}$$

Later we will see the fields in the A/B-model can be encoded using the derived stacks

$$T^*_{\text{FIR}} \text{Maps}(C, X)_{\text{dR}}$$

||

$$\text{Set}(C, (T^*_{\text{FIR}} X)_{K^{1/2}})_{\text{dR}}$$

$$T^*_{\text{FIR}} \text{Maps}(C_{\text{dR}}, X)$$

||

$$\text{Maps}(C_{\text{dR}}, T^*_{\text{FIR}} X)$$

By the Atiyah - Segal axioms an nd TQFT is a symmetric monoidal functor

$$Z : (n\text{-Cob}, \amalg) \longrightarrow ((n-1)\text{-Cat}_{\mathbb{C}}, \otimes_{\mathbb{C}})$$

Thus we have invariants

$$Z(M^n) \in \mathbb{C}, \quad Z(M^{n-1}) \in \text{Vect}_{\mathbb{C}}, \quad Z(M^{n-2}) \in \text{Cat}_{\mathbb{C}}, \dots$$

that are functorial under cobordisms.

Ex

$$Z(\overset{\circ}{\bullet} \sqcup \overset{\circ}{\bullet}) \cong Z(\overset{\circ}{\bullet} \circ \overset{\circ}{\bullet}) \xrightarrow{\quad} Z(\overset{\circ}{\bullet})$$


Cobordism Hypothesis  $Z$  can be recovered from  $Z(\text{pt})$

Kontsevich The 2d A and B-models are determined by  $\text{Fuk}(X)$  and  $\text{Coh}(X)$  respectively

There is an involution  $\sigma_{2d}$  of the 2d  $N=(2,2)$  super-Poincaré algebra which exchanges  $Q_A, Q_B$ .

Physically 2d mirror symmetry is the statement that this extends to an involution

$$(-)^v \curvearrowright \text{2d } N=(2,2) \text{ SUSY QFT}$$

such that

$$\tau \cong \sigma(\tau^v)$$

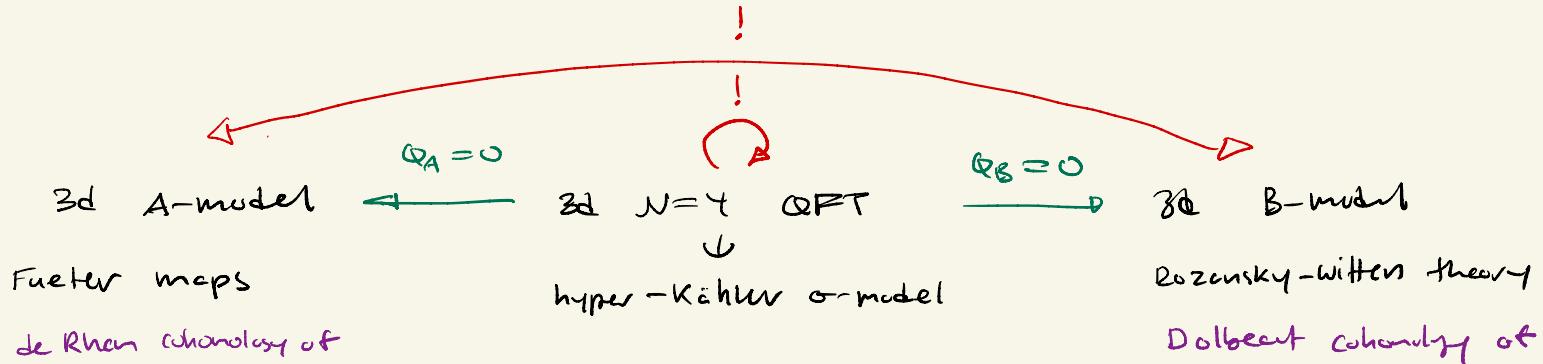
In particular

$$\tau_A \cong (\tau^v)_B \quad \tau_B \cong (\tau^v)_A$$

Ex If  $X, X^v$  are mirror CY manifolds

$$\mathrm{Fk}(X) \cong \mathrm{coh}(X^v) \quad \mathrm{coh}(X) \cong \mathrm{Fk}(X^v)$$

### 3d mirror symmetry



$$\left\{ \begin{array}{l} \sigma = \delta \sigma \\ = (I \frac{\partial \sigma}{\partial x} + J \frac{\partial \sigma}{\partial y} + K \frac{\partial \sigma}{\partial z}) \end{array} \right\} \xleftarrow{\quad} \left\{ \sigma: M^3 \rightarrow Y, \text{ fermions} \right\} \xrightarrow{\quad} \left\{ \sigma = d \sigma \right\}$$

↓

$\text{Seet}(M_{\text{dR}}^1 \times C, \gamma_{K_C^{1/2}})_{\text{dR}}$

$\text{Maps}(M_{\text{dR}}^1 \times C_{\text{dR}}, Y)$

Expect an equivalence of 2-categories. Work in this direction by

Kapustin, Rozansky, Saulina

Kapustin, Vysotsky, Setter

Tellman

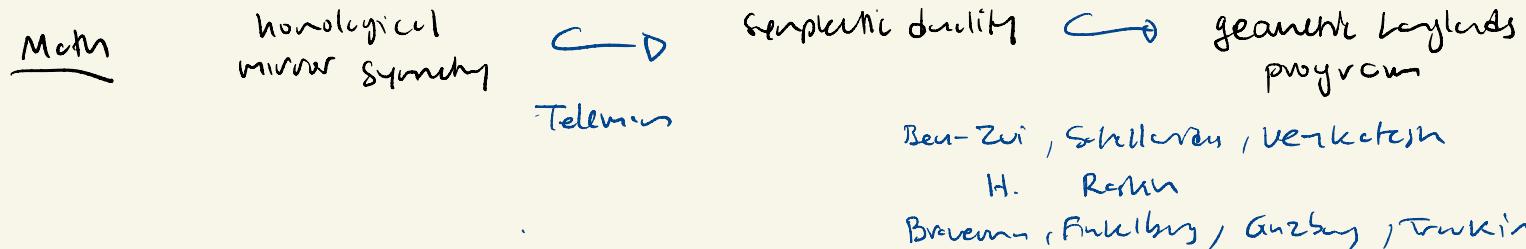
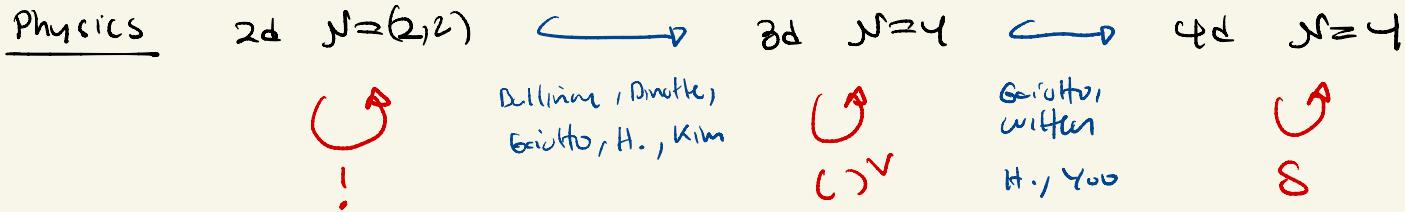
H., Gomberg, Morel-Cee

Doan, Reznikov

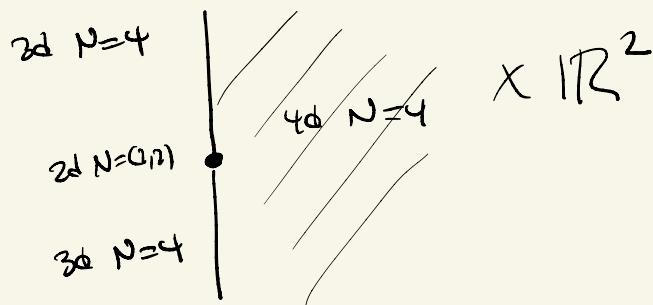
Khoan

To get examples of mirror pairs need to consider gauged  $\sigma$ -models.

# Relationship to other theories



Most of this mathematics comes down to understanding  
simultaneous A/B twists of configurations like



Get relations between

# The gauged hyperkähler $\sigma$ -model

Input

$$G \times \mathrm{Sp}(1)_H \hookrightarrow X \xrightarrow{\mu} \mathfrak{g} \otimes (\mathrm{im} H)_H$$

hyper-Hamiltonian      permitting action on twistor sphere

$$S_H^2 = \{ q_1 I + q_2 J + q_3 K \mid |q|^2 = 1 \}$$

## Bosonic Fields

Principal  $G$ -bundle

$$P \rightarrow M^3$$

connection 1-form

$$A \in \Omega^1(M^3, \mathfrak{g}_P)$$

hypermultiplet scalar

$$S \in \Gamma(M^3, X_P)$$

vectormultiplet scalar

$$\phi \in \Gamma(M^3, \mathfrak{g} \otimes (\mathrm{im} H)_C) \rightarrow \mathrm{Sp}(1)_C$$

## FI parameter

$$t \in \mathfrak{g}^* \otimes (\mathrm{im} H)_H$$

## mass parameter

$$m \in \mathfrak{f} \otimes (\mathrm{im} H)_C$$

$F \hookrightarrow X$  hyper-Hamiltonian  
commutes with  $G \times \mathrm{Sp}(1)_H$

(flavor symmetry)

The complexified 3d  $N=4$  super-Poincaré algebra is

$$sp(1)_E \times \mathbb{C}^3 \oplus \pi \mathbb{C}_C^2 \otimes \mathbb{C}_H^2 \otimes \mathbb{C}_E^2$$

$$sp(1)_C \times sp(1)_H \cong \text{spin}(\mathbb{C}_C^2 \otimes \mathbb{C}_H^2) \quad R\text{-symmetry}$$

To twist need

$$\Omega \in \pi \mathbb{C}_C^2 \otimes \mathbb{C}_H^2 \otimes \mathbb{C}_E^2 \quad \Omega^2 = 0$$

comuting  $sp(1)'_E \hookrightarrow sp(1)_E \times sp(1)_C \times sp(1)_H \quad \Omega \text{ is scalar for } sp(1)'$

$$U(1) \longrightarrow sp(1)_C \times sp(1)_H \quad |\Omega| = 1$$

Two classes

A  $sp(1)'_E \xrightarrow{\Delta} sp(1)_E \times sp(1)_H \quad U(1) \hookrightarrow sp(1)_C$   
 determined by complex structure on  $H_C$

B  $sp(1)'_E \xrightarrow{\Delta} sp(1)_E \times sp(1)_C \quad U(1) \hookrightarrow sp(1)_H$   
 determined by complex structure on  $X$

3d A-model (3d Seiberg-Witten equation)

Choose a spin structure  $\mathbb{Q} \rightarrow M^3$ .

Redefining fields using  $Sp(1)_E' \xrightarrow{\Delta} Sp(1)_E \times Sp(1)_H$  gives

$$s \in \Gamma(M^3, X_P) \longrightarrow s \in \Gamma(M^3, X_{Q \times P})$$

$$\mu \circ s \in \Gamma(M^3, (g \otimes m|H_H)_P) \longrightarrow \mu \circ s \in \Omega^1(M^3, g)$$

The complex structure on  $H_C$  gives  $m|H_C = \mathbb{R} \oplus \mathbb{C}$

$$\phi \in \Gamma(M^3, g \otimes m|H_C) \rightarrow \sigma \in \Gamma(M^3, g) \quad \psi \in \Gamma(M^3, g_C)$$

The equations  $Q_A = 0$  are

$$0 = \not{A}s + \sigma \cdot s$$

$$0 = \square_A \psi$$

$$0 = *F_A + d_A \sigma + \mu \circ s$$

$$0 = [\epsilon, \epsilon^+] = [\sigma, \psi] = [\sigma, \epsilon^+]$$

On  $M^3 = M^1 \times C + \text{fermions}$  get

Seet  $(M_{de}^1 \times C, (X/\langle \zeta \rangle_{K^{1/2}})_d)_R + \text{stability}$

3a B-model (Rozansky-Witten theory)

Redefining fields using  $\mathrm{Sp}(1)' \xrightarrow{\Delta} \mathrm{Sp}(1)_C \times \mathrm{Sp}(1)_C$  gives

$$\phi \in \Gamma(M^3, \mathrm{ad}(mH)_C) \longrightarrow \phi \in \Omega^3(M^3, \omega)$$

combines to give complex connection

$$A = A + i\phi$$

Our  $\mathrm{U}(1) \subseteq \mathrm{Sp}(1)_H$  makes  $X$  2-shifted derived stack.

The equations  $Q_B = 0$  are

$$0 = F_A$$

$$0 = m_C \circ S$$

$$0 = d_A S$$

$$0 = *d_A *\phi - M_{IR}$$

stability!

In terms of stacks get

$$\mathrm{maps}(M_{IR}^3, X//G_C) + \text{stability}$$

last time: gauged hyperkähler σ-model  $G \times \mathrm{Sp}(1)_H \odot X \xrightarrow{\mu} \check{g} \otimes (\mathrm{m} H)_{\mathbb{H}}$

3d A-model

$$\mathrm{Sp}(n) \cong \mathrm{Sp}(1)_H \quad \mathrm{U}(n)_C \subseteq \mathrm{Sp}(1)_C$$

$$Q \rightarrow M^3 \quad \text{spin-bundle}$$

Bosonic fields

$$P \rightarrow M^3$$

$$A \in \Omega^1(M^3, \omega_P)$$

$$S \in \Gamma(M^3, X_{P \times Q})$$

$$(\sigma, \psi) \in \Gamma(M^3, g \otimes g_C)$$

Equations

$$0 = \not{d} S + \sigma \cdot S$$

$$0 = d_A \varphi = [\sigma, \varphi] = [\varphi, \varphi^+]$$

$$0 = *F_A + d_A \sigma + M \sigma S$$

Fermions

$$\text{Seet } (M_{dR}^1 \times C, (X//G)_{K_C^{1/2}})_{dR} \\ + \text{stability}$$

3d B-model

$$\mathrm{Spin}(n) \cong \mathrm{Sp}(1)_C \quad \mathrm{U}(n)_H \subseteq \mathrm{Sp}(1)_H$$

Makes  $X$  into 2-shifted symplectic stack

Bosonic fields

$$P \rightarrow M^3$$

$$A = A + i\phi \in \Omega^1(M^3, \omega_C)_P \\ S \in \Gamma(M^3, X_P)$$

Equations

$$0 = d_A S$$

$$0 = F_A$$

$$0 = M_C$$

$$0 = +d_A * \phi + M_{IR}$$

Fermions

$$\text{Maps } (M_{dR}^3, (X//G)) \\ + \text{stability}$$

Natural class of deformations

PI parameters

untwisted

$$t \in X(g) \otimes (\text{im } H)_H$$

$$\mu \rightarrow \mu + t$$

mass parameters

$$FC^0 \times \text{flavor symmetry}$$

$$m \in t_F \otimes (\text{im } H)_C$$

$$\phi \rightarrow \phi + m$$

A-twist

$$t \in \mathfrak{sl}^1(M^3; X(g)_P) \subseteq \mathfrak{sl}(M^3, g_P)$$

$$m = (m_R, m_C) \in -\Delta^0(M^3, t_F \otimes (\mathbb{R} \oplus \mathbb{C}))$$

B-twist

$$t = (t_R, t_C) \in \Delta^0(M^3, g_P \otimes (\mathbb{R} \oplus \mathbb{C}))$$

$$m \in \mathfrak{sl}^1(M^3, t_F)$$

In mirror symmetry applications often break spaces to 4D and get

A-twist

$$t = (t_R, t_C)$$

$$m = (m_R, m_C)$$

B-twist

Resulting moduli spaces on 3 manifolds with transverse holomorphic foliations will be algebraic stacks.

In particular can understand integers  $t_{IR}, m_{IR}$  using the following maps of stacks.

[F.I. parameter]

open

$$X//G \xrightarrow{+_{IR}} X//G$$

[Mass parameter]

Lagrangian counterparts

$$(X//G)^n \xleftarrow{\text{fixed}} X//G^{nt} \xrightarrow{\text{attracting}} X//G$$

given by applying

maps  $\beta_{\text{sm}}$  ( $\rightarrow (X//G)/\text{sm}$ )  
to

$$I = C/C \hookrightarrow A/C \xleftarrow{\quad} C/C = pt$$

## Local operators

One of the most fundamental invariants of a 3d TQFT  
is the  $E_3$  algebra of local operators  $\mathcal{Z}(S^2) \in \text{Vect}_{\mathbb{C}}$

There is an ansatz that says this can be constructed  
by geometric quantization:

$$EOM(S^2) = \left\{ \begin{array}{l} \text{solutions to equations} \\ \text{of motion on } S^2 \end{array} \right\} \text{ is symplectic}$$

$$\mathcal{Z}(S^2) = G \otimes (EOM(S^2))^\top$$

Note In THP setting we have  $B = D \cup_{\overline{D}} D$  for  
sphere. This is not  $\mathbb{R} \times \mathbb{P}^1$ .

Have maps

$$C_0(\text{Conf}_n(\mathbb{R}^3)) \otimes Z(S^2)^{\otimes n} \longrightarrow Z(S^2)$$

For 0-chains get products

$$Z\left(\left(\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \end{array}\right)\right)$$
$$Z(S^2)^{\otimes n} \longrightarrow Z(S^2)$$

since  $\text{Conf}_2(\mathbb{R}^3) \cong S^2$  is connected all products are homotopic.

Also have binary operations of degree 2 corresponding to  $[S^2] \in C_0(\text{Conf}_2(\mathbb{R}^3))$ , This gives Poisson bracket

$$\{, \}: Z(S^2) : Z(S^2)^{\otimes 2} \longrightarrow Z(S^2)$$

B-twist

$$\text{Maps}(\mathbb{B}\mathrm{dR}, X//G) = T[-2]X//G = T^*X//G$$

Geometric quantization gives

$$Z_B(S^2) = \mathcal{Q}[X//G]$$

Turning on a background does deformation quantization

Ex  $\mathcal{Q}[T^*V/G] \rightsquigarrow D_n(V/G)$

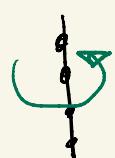
see Ben-Zvi / Nadler loop spaces and  
connections.

## $S^2$ -background

If one chooses  $SO(2) \cong \text{Spin}(3)$  get an  $SO(2)$  action

$$(H_0(SO(2)), *) \curvearrowright Z(S^2)$$

on local operators. Furthermore  $Z(S^2)^{SO(2)}$  has  $n - n + 1$  operations parameterized by  $C_{\bullet}(\text{Conf}_n(\mathbb{R}^3))^{SO(2)} = C_{\bullet}^{SO(2)}(\text{Conf}_n(\mathbb{R}^3))$ .  
0-chains correspond to  $SO(2)$ -invariant configurations



can't move points pair  
one another so non-commutative

The equation

$$[01-\infty] = \text{tr}[S^2] \in H_0^{SO(2)}(\text{Conf}_2(\mathbb{R}^3))$$

tells us that in  $Z(S^2)^{SO(2)}$

$$[a, b] = a \cdot b - b \cdot a = \text{tr}\{a, b\}$$

thus get quantization of  $Z(S^2)$ .

See Secondary Products in supersymmetric field theories  
Beem, Ben-Zvi, Bullimore / Dimofte, Mekareka

## A-twist

Let  $\mathbb{B} = \mathbb{D} \cup_{\mathbb{D}} \mathbb{D}$ . since  $K_{\mathbb{B}} = G_{\mathbb{B}}$  have moduli space

$$\text{Maps}(\mathbb{B}, T^*V/G)_{dR} \cong T^*\text{Maps}(\mathbb{B}, V/G)_{dR}$$

Geometric quantization ansatz gives BFN construction

$$Z_A(S^2) \cong (H^0(\text{Maps}(\mathbb{B}, V/G)), \star) \quad \xleftarrow{\text{to turn on background & take } H^*(\cdot)}$$

More explicitly

$$\begin{aligned} \text{Maps}(\mathbb{B}, V/G) &= \text{Maps}(\mathbb{D}, V/G) \times \text{Maps}(\mathbb{D}, V/G) \\ &\quad \text{Maps}(\mathbb{D}, V/G) \\ &= \left\{ ((P_1, s_1), (P_2, s_2), \sigma) \mid \begin{array}{l} \sigma: P_1|_{\mathbb{D}} \cong P_2|_{\mathbb{D}} \\ \sigma(s_1) = s_2 \end{array} \right\} \\ &\quad \text{Hecke transformation} \end{aligned}$$

Product comes from composition of Hecke transformations.

$$\begin{array}{c} 1 \\ \overbrace{\quad\quad\quad}^0 \quad 2 \\ \overbrace{\quad\quad\quad}^0 \quad 3 \end{array} \quad \mathbb{B} \times \mathbb{B} \xrightarrow{i_{12} \times i_{23}} \frac{\mathbb{D} \cup \mathbb{D} \cup \mathbb{D}}{\mathbb{B} \quad \mathbb{B}} \xleftarrow{i_{13}} \mathbb{B}$$

Analytically, this can be understood as follows

Bogomolny equations

$$F_A = * d\sigma$$



Given  $\mathbb{R}^3 \cong \mathbb{R}_+ \times \mathbb{Q}_z$  have

$$[D_+, D_{\bar{z}}] = 0 \quad D_+ = D_+ + i\sigma$$

If  $\sigma$ -shifts on  $[t_0, t_1]$  parallel transport along  $D_+$  (scattering) gives iso between holomorphic bundles on  $t_0 \times \mathbb{C}, t_1 \times \mathbb{C}$

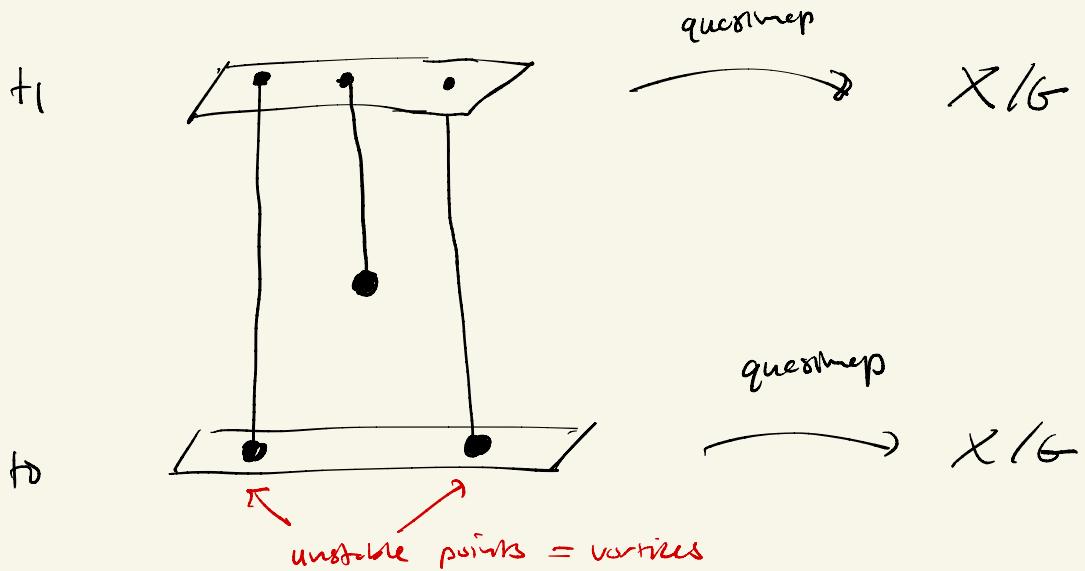
If  $\sigma$  singular at  $(0,0)$  get Hecke transformation instead.

Ex Dirac monopole  $G = U(1)$

$$\sigma = -\frac{i}{2r} k \quad k \in \mathbb{Z} \quad (P, A) = (\pi^*(\alpha_k), \pi^* A_k)$$
$$\pi: \mathbb{R}^3 - \{0\} \longrightarrow S^2 = \mathbb{C}P^1$$

scattering is multiplication by  $z^k$

When you incorporate  $X$  get



so monopoles create/destroy vortices.

Can be used to compute partition functions in 2d GLSMs in terms of representation theory of  $Z_4(\mathbb{S}^2)$ .

Ex (Braverman) Equivalent  $J$ -function of G/B in terms of Whittaker modules for  $U(g)$ .

$$\underline{\text{Ex}} \quad G = \mathbb{C}^\times \quad , \quad X = 0$$

$$\text{Maps}(B, O/G) = \text{Bun}_G(B)$$

$$= G(\mathbb{Z}) \setminus \underbrace{\{(p, \sigma) \mid \sigma : p|_B \cong p^{\text{inv}}|_B\}}_{\text{Gr}_G = G(\mathbb{Z})/G(\mathbb{Z}) \text{ affine Grassmann}}$$

$$\sim \mathbb{C}^\times \setminus \prod_{k \in \mathbb{Z}} z^k$$

$$Z_A(s^2) = H_{\mathbb{C}^\times}(pt) \left\{ [z^k] \mid k \in \mathbb{Z} \right\}$$

!!  
 $m^k$

composition law

$$\mathbb{C}^\times \setminus \mathbb{Z} \times \mathbb{C}^\times \setminus \mathbb{Z} \xrightarrow{\Delta} \mathbb{C}^\times \setminus \mathbb{Z} \times \mathbb{Z} \xrightarrow{+} \mathbb{C}^\times \setminus \mathbb{Z}$$

so get group ring of  $\mathbb{Z}$  over  $H_{\mathbb{C}^\times}(pt)$

$$\mathbb{C}[\mathbb{Q}]_{[m^{\pm 1}]} \cong \mathbb{C}[T^* \mathbb{C}^\times]$$

$\nearrow$   
This is  $Z_B(s^2)$  for  $G = 1$ ,  $X = T^* \mathbb{C}^\times$ . Simplest minor symmetry.

$$\underline{\text{Ex}} \quad G = \mathbb{C}^\times, \quad X = T^*\mathbb{C}$$

will use

$$\begin{array}{ccc} \text{Mors}(B, V/G) & \xhookrightarrow{i} & \{(P_1, P_2, s_1, \sigma)\} \\ \text{vector} & \nearrow & \downarrow \\ \text{bundle} & & \text{Bun}_G(B) \\ & & \text{Bun}_G(B) \end{array}$$

$$z^* i_* : H_0(\text{Mors}(B, V/G)) \longrightarrow H_0(\text{Mors}(B, BG))$$

ring homomorphism

$$\underline{\text{Ex}} \quad \coprod_k z^k \times z^k \mathbb{C}(z) \cap \mathbb{C}(z) \xrightarrow{\quad} \coprod_k z^k \times z^k \mathbb{C}(z) \xrightarrow{z} \coprod z^k \times \mathbb{C}$$

$$V^k = [z^k \mathbb{C}(z) \cap \mathbb{C}(z)]$$

$$\mathbb{C}[q] \{V^k \mid k \in \mathbb{Z}\} \longrightarrow \mathbb{C}[q][m^{\pm 1}]$$

$$V^k \longmapsto c_1(z^k \mathbb{C}(z) / z^k \mathbb{C}(z) \cap \mathbb{C}(z)) m^k$$

$$= \begin{cases} m^k & k \geq 0 \\ q^{-k} m^k & k < 0 \end{cases}$$

Easy to use this to see

$$\mathbb{Q}[\varphi] \{ v^k \mid k \in \mathbb{Z} \} \xrightarrow{\sim} \mathbb{Q}[x, y]$$

$$\varphi \longmapsto xy$$

$$v^1 = m^1 \longmapsto x$$

$$v^{-1} = \varphi m^{-1} \longmapsto y$$

This is consistent with

3d minor

$$G = U(1), X = T^* \mathbb{C} \quad \longleftrightarrow \quad G = 1, X = T^* \mathbb{C}$$

Now want to go deeper into TQFT

$E_2$ -categories of line operators

$$Z_A(\mathcal{S}') \sim D\text{-mod} \left( \text{Maps}(\mathbb{D}, V(G)) \right)$$

Algebraic approximation  
to true answers

$$Z_B(\mathcal{S}') \sim QCoh \left( \text{Maps}(\mathbb{D}_{dR}, V(G)) \right)$$

(de Rham/Betti)

Thm (Beilinson/Drinfeld) Local Langlands for  $\mathbb{C}^\times$

$$(D\text{-mod} (\text{Maps}(\mathbb{D}, \mathbb{C}^\times)), \star) \cong (QCoh (\text{Flot}_{\mathbb{C}^\times} (\mathbb{D})), \otimes)$$

Thm (Rapoport/Zink) de Rham Tate's thesis

$$D\text{-mod} (\text{Maps}(\mathbb{D}, \mathbb{C})) \cong QCoh (\text{Maps}(\mathbb{D}_{dR}, \mathbb{C}/\mathbb{C}^\times))$$

compatibly with Beilinson/Drinfeld.

In H., Dimits, Gomes, Geracie we should  
 line operators  $L$   
 wrapped boundary  $B \in Z(S^1)$   
 conditions

$$\text{Hom}(L_1, L_2) = \left\{ \begin{array}{l} \text{local} \\ \text{operators} \end{array} \right.$$

$$\left. \begin{array}{c} L_1 \\ L_2 \end{array} \right\}$$

$$\text{Hom}(L, B) = \left\{ \begin{array}{l} \text{local} \\ \text{operators} \end{array} \right.$$

$$\left. \begin{array}{c} L \\ B \end{array} \right\}$$

Ex Let  $\mathbb{1}$  be trivial operator

$$\begin{aligned} \text{Hom}(\mathbb{1}, \mathbb{1}) &= \{ \text{local operators} \} \\ &= Z(S^2) \end{aligned}$$

In general get vector bundles  
 on  $M_H/M_G$ .

Ex  $G \times M$  hamiltonian  
 gives  $B$  for pure gauge theory

$$\begin{array}{c} \text{Hom}(\mathbb{1}, \mathbb{1}) \\ \cong \\ \text{Coh}_{\mathcal{U}} \end{array} \subset \begin{array}{c} \text{Hom}(\mathbb{1}, B) \\ \cong \\ \text{SH}_G^0(M) \end{array}$$

Telteman, Pomerolano  
 Bullimore, Dimits/Giusto, H.  
 H., Kennitzer, Weekes

Expect analytic models

$$Z_A(S^1) = \text{FS}_{LG}(L^{T^*V} \times \text{Lag}_c, W = \int_{S^1} p d_A q)$$

$$Z_B(S^1) = \text{MF}_{LG}(L^{T^*V} \times \text{Lag}_c, W = \int_{S^1} p d_A q)$$

Can also consider open string invariants on  $[0,1]$  with holomorphic Lagrangian boundary conditions to get morphism spaces in  $Z_A(\text{pt})$ ,  $Z_B(\text{pt})$ .

other models using vertex algebras

Costello, Gaiotto, Creutzig, Dimofte, Beilin, Nie

Given holomorphic Lagrangians

$$Y \xrightarrow{0} X = T^* Y \xleftarrow{dW} Y$$

Kapustin - Rozensky - Saulina

$$\text{Hom}_B(0, dW) \cong MF(Y, W)$$

Dolan - Pezchikov

$$\text{Hom}_A(0, W) \cong FS(Y, W)$$

Not quite enough for  
3d mirror symmetry

We will see that  
A-type 2-category is  
generated by cotangent  
fiber and needs wrapping

Teleman ICM (specialized at  $G = \mathbb{C}^\times$ )

$\mathcal{T}^* B \mathbb{C}^\times$

A]

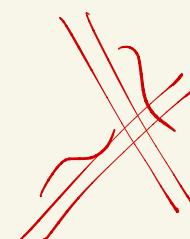
$S^1 - \text{cat}$

||

$\{(e, S^1 \xrightarrow{\rho} \text{Hom}(e, e))\}$

||

$\{(e, C(\mathbb{Z} S^1) = \mathbb{C}[i] \rightarrow \text{End}(1e))\}$



$\mathcal{T}^* \mathbb{C}^\times$

$2 \text{Loc}(\mathbb{C}^\times)$

||

$\{ \pi_1 \mathbb{C}^\times \rightarrow \text{Hom}(e, e) \}$

||

$\{ (e, e \cong e) \}$

B]

$(\text{Rep}(\mathbb{C}^\times), \otimes) - \text{mod}^{(2)}$

||

$\{(e, \text{Rep}(\mathbb{C}^\times) \rightarrow \text{Hom}(e, e))\}$

||

$\{ (e, e \cong e) \}$

$2 \text{Can}(\mathbb{C}^\times)$

||

$\{ (e, \mathbb{C}^{\alpha, \alpha^{-1}} \rightarrow \text{End}(1e)) \}$

More geometrically expect

$$T^* \mathbb{C}^X$$

$$T^+ BC^X$$

$$\text{Hom}_A(T_1^*\mathbb{C}^X, T_1\mathbb{C}^X) \cong \text{Hom}_B(0, 0)$$

SLI

KRS

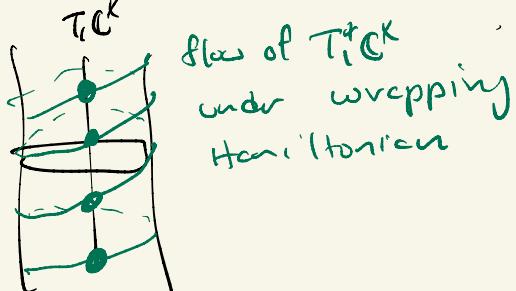
$$(\text{Qcoh}(BC^X), \otimes)$$

!!

$$(\text{Rep}(\mathbb{C}^X), \otimes)$$

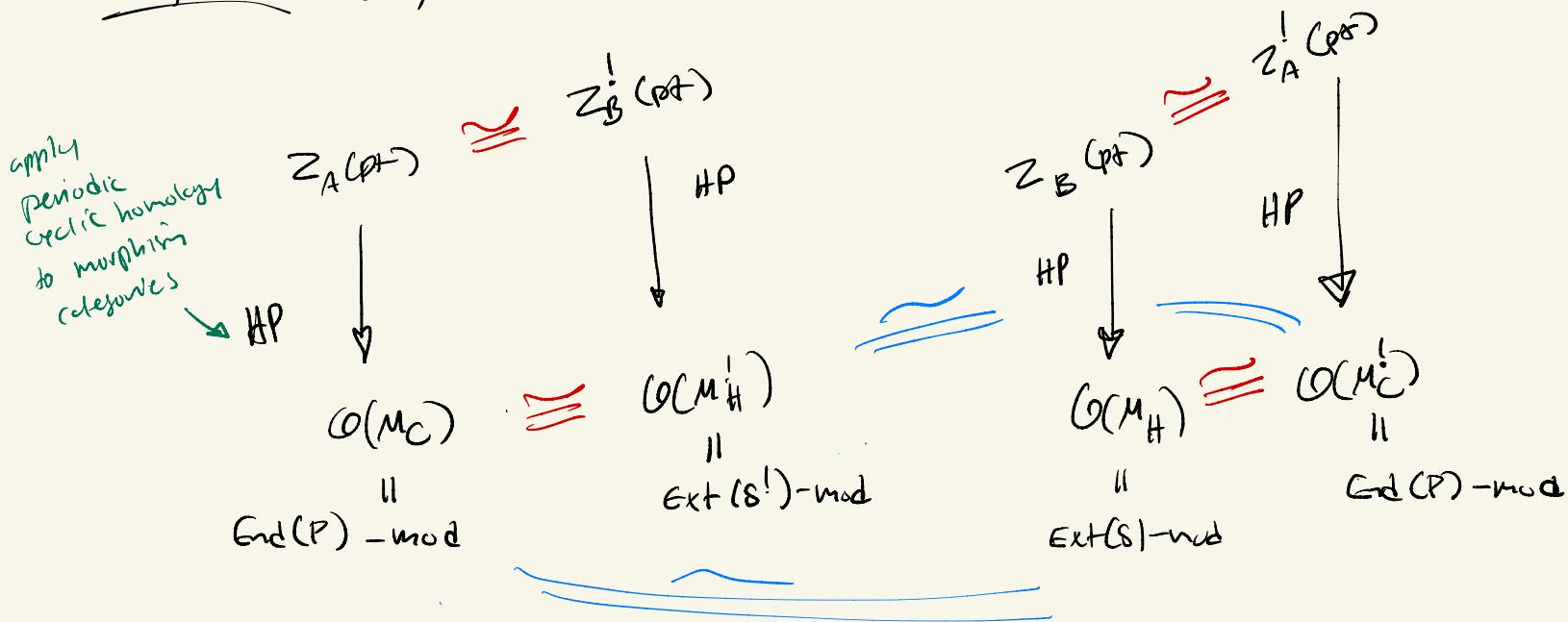
Thus need  $\mathbb{Z} = \# \text{fixed } \mathbb{C}^X\text{-rep intersection points } T_1\mathbb{C}^X \cap T_1\mathbb{C}^X$ .

Need wrapping



Suppose we have a theory  $\mathcal{T}$  and its mirror  $\mathcal{T}'$

Conjecture (Ho, Ganzege based on Bullimore, Dimofte, van der Heijden, H.)



Then (Ganzege, Ho, Mezzi-Gee)

For abelian theories can construct 2-categories satisfying these properties.

Simplest case: Betti Tore's thesis

$$X = \mathbb{C}/\mathbb{C}^X \amalg BC^X \rightarrow \mathbb{C}/\mathbb{C}^X = Y$$

$$Z_A(\tau^* \mathbb{Q})$$

$$\begin{array}{c} \parallel \\ \text{Cat}_{\mathbb{C}} \end{array}$$

turn on  $m_R$



$$\text{Perverse Schubert } (\mathbb{Q}, 0)$$

$$S \amalg$$

$$\left\{ \begin{array}{ccc} \mathbb{E} & \xrightarrow{\substack{\hookleftarrow \\ \perp \\ R}} & \mathbb{E} \\ & \text{mashup} & \end{array} \middle| \begin{array}{l} T_{\mathbb{E}} = \text{fib}(n) \\ T_{\mathbb{E}} = \text{cotib}(e) \end{array} \right\}$$

$$\begin{array}{c} \curvearrowleft \\ S^1 \end{array}$$

$$\begin{array}{c} ! \\ \overbrace{\quad\quad\quad} \end{array}$$

$$Z_B(\tau^* \mathbb{C}/\mathbb{C}^X)$$

$$\begin{array}{c} \parallel \\ \text{Cat}_{\mathbb{Q}} \end{array}$$

turn on  $t_R$

$$\begin{array}{c} ! \\ \overbrace{\quad\quad\quad} \end{array}$$

$$(\text{coh } (X \times_Y X), *) - \text{mod}^{(2)}$$

$$S \amalg$$

$$\left( \begin{pmatrix} \text{coh } (\mathbb{C}/\mathbb{C}^X) & \text{coh } (BC^X) \\ \text{coh } (BC^X) & \text{coh } (\mathbb{C}^U \otimes \mathbb{C}/\mathbb{C}^X) \end{pmatrix}, * \right) - \text{mod}^{(2)}$$



$$\begin{array}{c} \overbrace{\quad\quad\quad} \\ S-\text{dual} \end{array}$$

$$(\text{coh } (BC^X), *)$$