Quillen metrics: a summary

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Gauged maps, vortices and their moduli

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- $H^\bullet(X,E)$ cohomology of $\left(\Omega^{0,\bullet}(X,E),\overline{\partial}^X\right)$.
- $\lambda = \det H^\bullet(X,E)$ complex line.
- Metrics g^{TX}, g^E determine canonical Quillen metric $\| \, \|_\lambda$ on λ via the ζ-function regularized determinant of the Hodge Laplacian.

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- We will explain the curvature theorem for Quillen metrics.
- We will establish some functorial properties of the Quillen metrics

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- Mathematical physics: Mirror symmetry, GW invariants, BCOV torsion...

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- The above objects are defined up to canonical isomorphism.

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- If $m = 1$, $\tau(v) = \sigma_0 \otimes (v\sigma_0)^{-1}$.

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- One may decide that $\det H$ is Hermitian line bundle $(\det E, \| \|_{\det E})$ (consistent with theory of Knudsen-Mumford).

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- $-\frac{\overline{\partial}\partial}{2i\pi} T^E = c_1(\det E, \| \|_{\det E})$.
- If H (holomorphic) vector bundle, $\frac{\overline{\partial}\partial}{2i\pi} T^E = c_1\left(\det H, g^{\det H}\right) - c_1(\det E, \| \|_{\det E})$.

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- $A_T = A'' + A'_T$.
- $\mathrm{ch}\left(E, g_T^E\right) = \varphi \mathrm{Tr_s}\left[\exp\left(-A_T^2\right)\right]$ Quillen Chern character forms, sum of (p,p) forms (φ $2i\pi$ normalization).

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3. By zeta function reg., form T^E such that $\frac{\overline{\partial}\partial}{2i\pi}T^E = \alpha_\infty - \alpha_0$.

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Remark

In degree 0, T^E is just analytic torsion.

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- $\mathcal{H}^\bullet \simeq H^\bullet(X, E)$ inherits L_2 metric.

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$$\| \, \|_\lambda^2 = | \, |_\lambda^2 \exp(T) \, .$$

- The situation is formally similar to the finite-dimensional case, with $E^\bullet = \Omega^{0,\bullet}(X, E)$.

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- In this case, $R\pi_* E = \sum (-1)^i H^i(X, E)$.

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- $R\pi_* E$ direct image of E.
- In general, $R\pi_* E \in K_\cdot(S)$, i.e. coherent sheaf (Grauert).
- Special case: $H^\bullet(X, E)$ has constant dimension, holomorphic vector bundle on S.
- In this case, $R\pi_* E = \sum (-1)^i H^i(X, E)$.
- In general, $\lambda = \det R\pi_* E$ is a canonical holomorphic line bundle on S (Knudsen-Mumford), $\lambda_s \simeq \det H^\bullet(X_s, E|_{X_s})$.

The curvature theorem

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Locally Kähler: if $U \subset S$ small open set, $\pi^{-1}U$ Kähler.

Bott-Chern classes

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- E v.b. with connections $\nabla^E, \nabla^{E\prime}$, Chern-Simons $\widetilde{P}\left(\nabla^E, \nabla^{E'}\right)$ (defined modulo exact forms) such that

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- Quillen met. vary by Bott-Chern (ext. of Polyakov anomaly formulas): key ing. in proof of curvature t.

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- $\ldots = 0$ if ω^M closed.

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- Making $\epsilon \to 0$ adiabatic limit.
- In the proper sense, D_ϵ^M converges to superconnection A on $\Omega^{0,\bullet}(X, E|_X)$.
- This also explains why $\overline{\partial}\partial$ is the 'adiabatic limit' of Laplacian on S.

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Remark

This result is often misquoted. The results of B-Freed, valid only in the smooth category, are mentioned instead.

The question of functoriality

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- General principle: the ratio of two such Quillen metrics is given by an explicit local formula, compatible with the curvature theorem.
- Simplest example: the anomaly formulas for Quillen metrics.

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- D divisor in Σ, $i : D \to \Sigma$, σ canonical section of D, $E = [-D] \oplus \mathbf{C}$, r restriction to D.

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Theorem (B-Lebeau, 1989)

If g^{TX}, g^{TY} are Kähler metrics, there is a local formula,

$$\log \left(\frac{\| \,\|_{\lambda(E)}}{\| \,\|_{\lambda(F)}} \right)^2 = \int_X T,$$

where T is an explicit 'local' Bott-Chern current on X.

Structure of T

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$$\frac{\overline{\partial}\partial}{2i\pi}T = \mathrm{Td}\left(TY, g^{TY}\right)\mathrm{ch}\left(F, g^{F}\right)\delta_Y - \mathrm{Td}\left(TX, g^{TX}\right)\mathrm{ch}\left(E, g^{E}\right).$$

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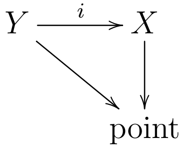
- Formula expresses functoriality with respect to



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- Roughly speaking, we obtain this way an arithmetic formula for Ray-Singer torsion.
- There is an extension to higher Chern classes, involving analytic torsion forms.

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- $|dz|^2 = 2\mathrm{Im}(\tau)$.

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- $c_1\left(H^{1,0}(C_\tau)\right) = \frac{1}{4\pi y'^2} dx' dy' =$ canonical volume form on H.

Analytic torsion as a Kähler potential

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- T^{an} Kähler potential for minus the canonical volume form on $X(1)$.

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- Dedekind η-function $\eta = q^{1/24} \prod_1^{+\infty} (1 - q^n)$ modular form modular form of weight 1/2, $\Delta = (2\pi)^{12} \eta^{24}$ discriminant, .

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- $\|dz\|_\lambda = c \, |\eta|^2$ (Ray-Singer, using Kronecker limit formula).

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- There is a non-vanishing holomorphic function $f(\tau)$ such that $dz \simeq f(\tau)$.
- $\|dz\|_\lambda = |f(\tau)|$.
- The identification of the section dz with $c\eta^2$ can be obtained by showing $\| \|$ extends to $X(1)$ as generalized metric using the embedding formula near the cusp.

The case of $K3$ surfaces (Yoshikawa)

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- Yoshikawa: moduli sp. of $K3$ with an anti-symplectic involution ι, and replaces analytic torsion by equivariant torsion.
- Constructs holomorphic section of λ_ι in the open regular set of the moduli space (Torelli theorem).
- He identifies his invariant, up to a constant, with a Borcherds modular form.

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- BCOV torsion appears in the formulation of mirror symmetry.
- Questions of birational invariance of BCOV torsion.

An important remark

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Remark

In general, the metric defining analytic torsion should be Kähler. However special metrics (like Calabi-Yau) are not needed, not even their existence. This simplifies the study of degeneration (see Yoshikawa).

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Thank you!