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Big Questions of Permutation Groups

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Big questions of permutation groups – some answered, others open

1. Basic kinds of permutation groups – building blocks
2. New tools: how to use them and the FSGC
3. Answering some old questions – using different building blocks
4. Big questions dictate – need for new theory

Basic permutation groups

Else G is subgroup of direct product of transitive groups induced on the G-orbits in Ω

- Set $\Omega = \{1, \ldots, n\}$ (or infinite)
- **Permutation group** on Ω is a subgroup $G \leq Sym(\Omega)$
- Usually assume G **transitive**: $\forall\, \alpha, \beta \in \Omega\ \exists\ g \in G\ s.t.\ \alpha^g = \beta$
- Important question: does G preserve a nontrivial partition P of Ω ?

- If yes then $\mathbf{G \leq K_1 \wr K_2 < Sym(\Omega)}$, where $K_1 \leq Sym(B), K_2 \leq Sym(P)$ are smaller transtive groups induced on a part B and set of parts P

Basic permutation groups

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- If $\Omega = \{1, \ldots, n\}$ repeat a finite number of times with the smaller transitive groups $\boldsymbol{K_1, K_2, \ldots}$ until the process stops

- **Obtain iterated wreath product embedding**

$$G \leq K_1 \wr K_2 \wr \cdots \wr K_r \leq Sym(\Omega)$$

- and each K_i preserves only trivial point partitions; such K_i are **primitive**

- Primitive permutation groups are traditionally the basic kinds of permutation groups

Works for some families of infinite transitive groups – need some kind of finiteness condition

Basic permutation groups

- Not the only possible "story": could change the "important question" for transitive permutation groups

- Alternative important question: does G have a nontrivial normal subgroup N that is intransitive on Ω ?

- If yes then the set P of $N-$orbits is a $G-$invariant partition and $\mathbf{G \leq K_1 \wr K_2 < Sym(\Omega)}$, where $K_1 \leq Sym(B), K_2 \leq\ Sym(P)$ where B is an N-orbit

- Get $G \leq K_1 \wr K_2 \wr \cdots \wr K_r \leq Sym(\Omega)$ and each K_i is **quasiprimitive** - all nontrivial normal subgroups transitive

Primitive and quasiprimitive groups equally valid as basic groups: choice depends on "context"and "tools"

Tools for working with basic permutation groups

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- Take $\Omega = \{1, \ldots, n\}$ **and (quasi)primitive permutation group** $G \leq Sym(\Omega)$

- **Structure theorems:** identify different types of (quasi)primitive groups
- partition the set of finite (quasi)primitive groups

- **Broad-brush description:** affine, almost simple, diagonal and a product construction applied to each of these types [Peter Cameron's approach]


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O'Nan-Scott Thm Timeline

1870 Jordan thought about various kinds of primitive groups

1960s

Peter Neumann: diagonal groups appear in his Dphil thesis 1964

Rheinhold Baer origins of this case subdivision

1979

Michael O'Nan and Leonard Scott each brought manuscripts containing the "O'Nan-scott Theorem" to the 1979 **Santa Cruz Conference on Finite Groups**

1985/88

Michael Aschbacher and Leonard Scott 1985 and L. G. Kovacs 1986 both independently corrected (TW type)

Liebeck-Praeger-Saxl 1988 self-contained proof – more information on TW

1992

Praeger Similar structure theorem for finite **quasiprimitive** permutation groups

Praeger Schneider

2017: analysis and comparison of Aschbacher/Scott/Kovacs contributions

(book) 2018: summary of different statements, subdivisions of the O'Nan-Scott Theorem

2022

Bailey, Cameron, Praeger, Schneider Diagonal Geometry to explain diagonal groups

Tools: O'Nan—Scott Theorem and its "relatives"

- **Purpose:** to highlight how to apply the finite simple group classification (FSGC):

- **Most types give structure for Ω or G helpful for applications**
  - **Affine type:** $\Omega = V,\ G = NG_0 \leq AGL(V), N$ trans, G_0 irred
    - [all affine quasiprimitive groups are primitive]
  - **Almost simple:** $N = Soc(G)$ nonabelian simple group T
  - **Diagonal:** $N = Soc(G) = T^k$ with T nonabelian simple,
  - $k \geq 2$, and N_α a diagonal subgroup of T^k
  - **Product type:** $G \leq Sym(\Gamma) \wr S_k$ preserving a Cartesian decomposition on Γ^k where $\Omega = \Gamma^k$ (primitive) or Γ^k labels a canonical invariant partition (quasiprimitive)

So what's new?

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Tools: O'Nan—Scott Theorem Diagonal geometry

- **Diagonal:** $N = \mathrm{Soc}(G) = T^k$ with T nonabelian simple,
- $k \geq 2$, and N_α a diagonal subgroup of T^k and $\Omega = T^{k-1}$

- **If** $k = 2$ **then** $G \leq Hol(T) = T.Aut(T)$ [well known]
- **If** $k = 3$ **then** $G \leq Aut(L)$ for L Latin square (Cayley table for T)
- **If** $k \geq 4$ **then** $G \leq Aut(D)$ for D a diagonal semilattice of partitions of $\Omega = T^{k-1}$

- **Diagonal geometries defined by axioms** (like a projective plane); **if** $k \geq 4$ **always come from a special group construction**
- **2022 Trans AMS:** Bailey, Cameron, P, Schneider, The geometry of diagonal groups

Big questions answered ??

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- **1980 (FSGC announced): Some questions answered almost immediately – primitive groups.**
  - Classification of 2-transitive groups [Cameron, Hering]
  - Classification of primitive rank 3 groups [Liebeck, Saxl]
  - Classn primitive groups, $|\Omega|$ odd [Liebeck, Saxl, Kantor]
- **Some very surprising results – primitive groups**
  - For almost all n, only primitive subgroups of S_n are S_n and A_n [Cameron, Neumann, Teague, 1982]
- **What about quasiprimitive groups?**
  - Quasiprimitive rank 3 [Devillers et al 2011]
  - For almost all n, only quasiprimitive subgroups of S_n are S_n and A_n [Heath-Brown, Praeger, Shalev, 2005]

Open problem: Classify the quasiprimitive groups of odd degree

Applications: many questions arise

- **Applications may be in algebra, or number theory, or geometry, or combinatorics, or …**

- **How to reduce these problems to "basic" cases:**
  - how to decide whether primitive or quasiprimitive reduction is appropriate/possible?
  - What if neither possible? maybe need different fundamental theory to apply FSGC?

- **How to finish:** what if a reduction is possible but we don't know enough about FSG's to complete a solution?

Applications: a case study

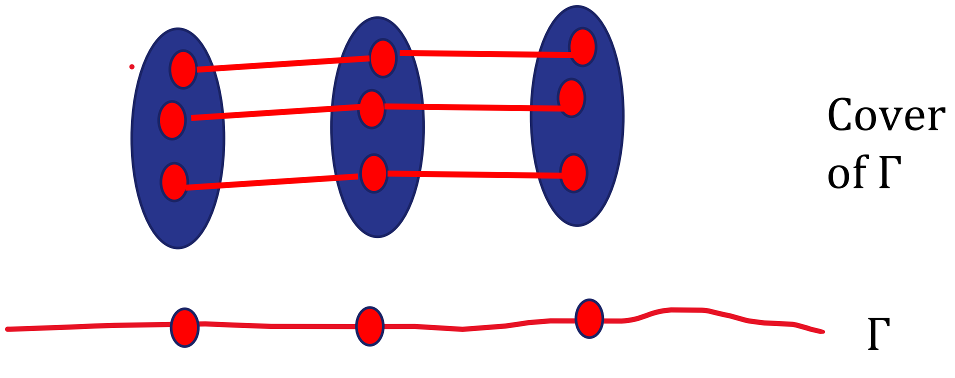
How quasiprimitive groups became important

- **2-arc-transitive graphs** Γ : $\mathrm{Aut}(\Gamma)$ transitive on 2-arcs

(α, β, γ) with $\{\alpha, \beta\}, \{\beta, \gamma\}$ both edges

- **Babai 1985: each regular graph has a 2-arc-transitive cover**

Regular: vertices have same valency
Fact: most regular graphs have trivial automorphism group while 2-arc-transitive graphs have lots of symmetry
Babai's deduction: 2-arc-transitive graphs are `wild'
But hey: I didn't about Babi's construction until 1993



1993 Studying 2-arc-transitive graphs – seeking a reduction

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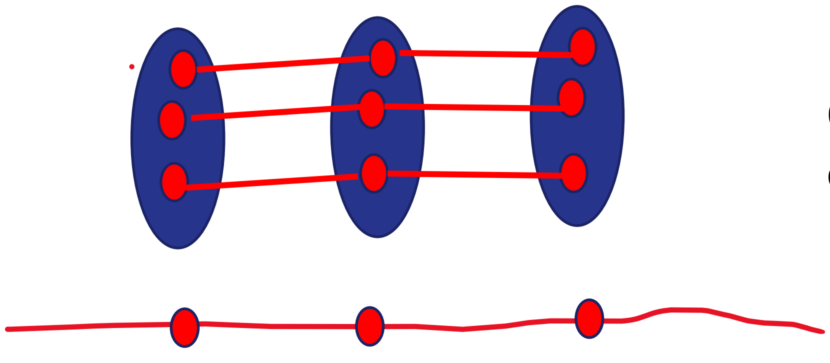
- **Finite graph** Γ : $G \leq \mathrm{Aut}(\Gamma)$ transitive on 2-arcs – Γ not bipartite
- **Intransitive normal subgroup** : $1 \neq N < G$ so at least 3 N-orbits
- **Normal quotient graph** Γ_N : vertices are N-orbits; join two if at least one edge between
- **Theorem:** Γ **cover of** Γ_N **and** G/N **transitive**

On 2-arcs of Γ_N

Reduction to basic case: choose N maximal intransitive, then G/N is both quasiprimitive on Γ_N

Problem: nothing was known about quasiprimitive permutation groups so I had to develop the theory



What was the outcome?

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1. **My O'Nan-Scott style theorem for finite quasiprimitive permutation groups – and very useful**
2. **If** $G \leq \mathrm{Aut}(\Gamma)$ transitive on 2-arcs, quasiprimitive on vertices **then only half of the "quasiprimitive types" possible for G**
3. **Approach useful:** could classify all affine examples, and construct many new families
4. **Normal quotient method + theory of quasiprimitive groups**: very appropriate for studying many families of edge-transitive graphs

Further applications: what kinds of group problems arise?

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- **Group factorisation problems:**
  - Graph Γ=(V, E) $\mathrm{A} = \mathrm{Aut}(\Gamma)$, say A transitive on arcs
- **"Regular embedding"** Γ **into a surface:** Subgroup $\mathrm{G} < A$ preserving the surface is transitive (in fact regular) on flags (incident vertex-edge-face triples)
  - Factorisation $\mathrm{A} = \mathrm{G}\,\mathrm{A_v}$ with $\mathrm{G} \cap A_v = G_v$ dihedral.
- **Decide if** Γ **is a "Cayley graph" :**
  - $\exists\, G < A$ with G regular on V
  - Factorisation $\mathrm{A} = \mathrm{G}\,\mathrm{A_v}$ with $\mathrm{G} \cap A_v = 1$

- **Work on Factorisation problems in progress [Liebeck, Praeger. Li, Xia]**
- **Many applications where factorisations arise, e.g. in algebra**

Applications in algebraic number theory

If time: discuss two threads
- Derangements (fixed point free permutations)
- Group coverings

- **Galois groups of field extensions – many deep results – (look for Bob Guralnick's name)**
- **Derangements:**
  - 1981 Fein, Kantor, Schacher:
  - Each transitive $G \leq Sym(\Omega)$ contains a derangement of prime power order
  - Proof needs FSGC
- **Fulman and Guralnick (sequence of papers):**
  - $\exists$ absolute constant c such that, if G finite simple group

  And $G \leq Sym(\Omega)$ transitive, then proportion of derangements in G is at least c

Applications in algebraic number theory

- **Group Coverings: sample problem**
- **2011+ Bubboloni, Praeger, Spiga:**
  - Let $f(x) \in Z[x]$ with k distinct irreducible factors, all of degree >1 and let $G = Gal\_Q(f)$
  - Then G = set theoretic union of proper subgroups from k conjugacy classes of G [stabilisers of roots of f]
  - Most Galois groups are symmetric groups S_n
  - Q: how small can k be relative to n [linear in n]
  - **Bubboloni, Spiga :** almost simple Galois groups G

If time: discuss two threads

- Derangements (fixed point free permutations)
- Group coverings

Last comments: problems suggested by combinatorial evidence

- One example only:
- Question: when do all g elements of a primitive $G \leq \mathrm{Sym}(\Omega)$ have at least one regular cycle? [cycle of length |g|]
- Pablo Spiga asked this: with a lot of experimental evidence
- 2016, 2017 Spiga (with Guest, Giudici, Praeger); always unless $\Omega = \Gamma^r$ and $A_m^r \leq G \leq S_m \wr S_r$ where S_m action on Γ is on $k-$subsets of $\{1, \ldots, m\}$

Enjoy

Wish you all the best

- For this week!
- Have a fantastic time

References for a few results

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2006 CEP Seminormal and subnormal subgroup lattices for transitive permutation groups,
J. Australian Math. Soc. **80**, 45-63.

Explains how primitive and quasiprimitive groups are equally valid as basic groups

2006 Peter M Neumann The concept of primitivity in group theory and the second memoir of Galois
Arch. Hist. Exact Sci. **60** (2006) 379-429

Explains that Galois' concept of primitive may have been closer to quasiprimitive

2018 CEP and Csaba Schneider Permutation groups and cartesian decompositions

London Math. Soc. Lecture Note Series, Vol. 449, Cambridge University Press

Chapter 7 explains versions of the O'Nan-Scott Theorem for primitive and quasiprimitive groups. In particular Section 7.6.1 compares notation for five different versions available in the literature.

2017 CEP and Csaba Schneider The contribution of L. G. Kovacs to the theory of permutation groups.
J. Aust. Math. Soc. **102,** 20-33

Section x discusses the results in Aschbacher-Scott and Kovacs about the O'Nan-Scott Theorem

References for recent results

1985 L. BABAI, 'Arc transitive covering digraphs and their eigenvalues', *J. Graph Theory* **9** (1985) 363-370.

1993 CEP An O'Nan-Scott Theorem for finite quasiprimitive permutation groups, and an application to 2-arc transitive graphs, *J. London Math. Soc.*(2) **47**, 227-239.