Imperial College London

The orbital diameter of primitive permutation groups

Kamilla Rekvényi

Young Group Theorists Workshop

Les Diablerets, SwissMAP, 2022 September

$G \leq Sym(\Omega)$ transitive & finite

G acts componentwise on $\Omega \times \Omega$

$\boxed{\text{orbital}}$ orbit of G on $\Omega \times \Omega$

$\Delta = \{(a,a) \mid a \in \Omega\}$
diagonal orbital

non-diagonal orbital
$(a,b)^G$, $a \neq b$

$\boxed{\text{orbital graph}}$

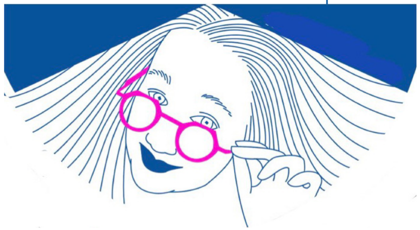
Γ - non-diagonal orbital

$\boxed{\text{vertices}}$

Ω

$\boxed{\text{edges}}$

$a - b \iff \{a,b\} \in \Gamma$



Theorem (Higman)

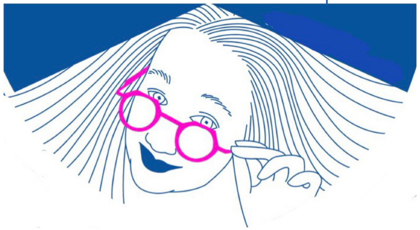
All non-diagonal orbital graphs of a group G are connected

$\iff$

G acts primitively on Ω.

orbital diameter (orbdiam(G))

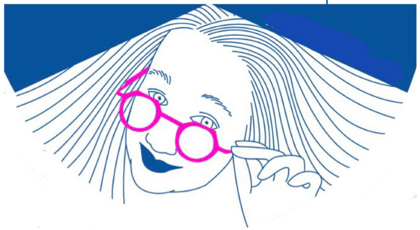
the supremum of the diameters of all orbital graphs



Example 1

G 2-transitive on Ω

Q: What is $\mathrm{orbdiam}(G)$?



Example 1

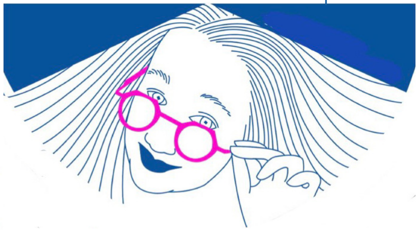
G 2-transitive on Ω

$$\Downarrow$$

$\exists$ unique non-diagonal orbital

$$\Downarrow$$

$$\operatorname{orbdiam}(G) = 1$$

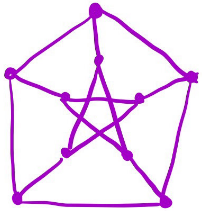


Example 2

$G = S_5$, $I = \{1,2,3,4,5\}$, $\Omega = \{\text{size 2 subsets of } I\}$

orbitals

- Δ
- $\Gamma_1 = \{(A,B) \mid |A \cap B| = 0\}$
- $\Gamma_2 = \{(A,B) \mid |A \cap B| = 1\}$

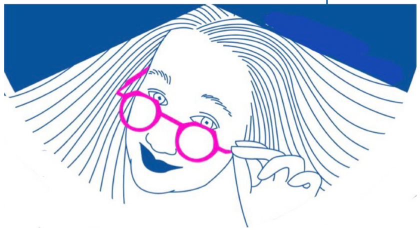


Petersen graph
$\Downarrow$
diam = 2

complement of Petersen graph
$\Downarrow$
diam = 2

alternatively

$\operatorname{orbdiam}(G) \leq \underbrace{\operatorname{rank}(G)}_{\#\text{ orbitals}} - 1 = 2$ in this example



Theorem (Liebeck, MacPherson, Tent)

Classified ∞ families $\mathcal{C}$ of groups such that $\exists t \in \mathbb{N}$, $\forall G \in \mathcal{C}$, $\operatorname{orbdiam}(G) \leq t$.

GOALS

G1 find explicit bounds

G2 classify groups with small orbdiams

Diagonal type

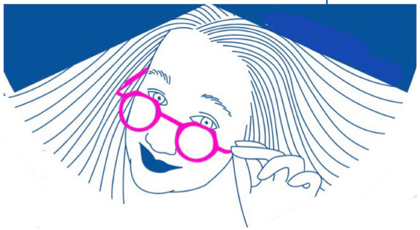
T - simple , $T^k = \overbrace{T \times \dots \times T}^{k}$

$D = \{(a, \dots, a) \mid a \in T\} \cong T$

$\Omega = (T^k : D)$

$T^k \trianglelefteq G \leq N_{Sym(\Omega)}(T^k) \cong T^k . (Out(T) \times S_k)$

The Orbital Diameter
Imperial College London
Kamilla REKVÉNYI
2022. September



Young Group Theorists Conference
Les Diablerets

$$\mathrm{orbdiam}(G) \overset{\text{connection}}{\sim} c(T) \leq cn(T)$$

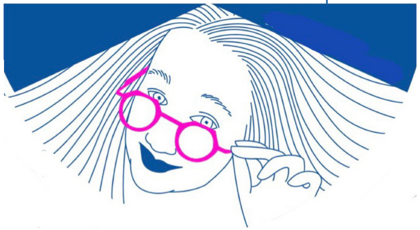
conjugacy width: $c(T)$

covering number: $cn(T)$

$$C = \{t^{\pm a} \mid a \in T\}$$

$$C^{cn(T)} = T$$

$$1 \cup C \cup \ldots \cup C^{c(T)} = T$$



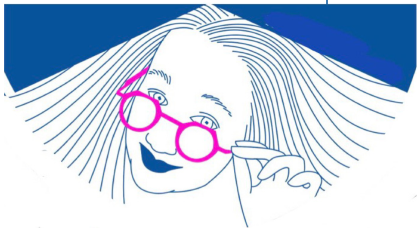
$\boxed{\text{Theorem (R'22)}}$ G1

$$X \underset{\text{primitive}}{\leq} S_k$$

1. $\frac{1}{2}(k-1)\,c(T) \leq \operatorname{orbdiam}(T^k X)$

2. $\operatorname{orbdiam}(T^k S_k) \leq 24(k-1)^2 c^2(T)$

3. $\operatorname{orbdiam}(T^2) = c(T)$



Conjugacy width

Theorem T – simple group of Lie type
r – Lie rank of T

R'22

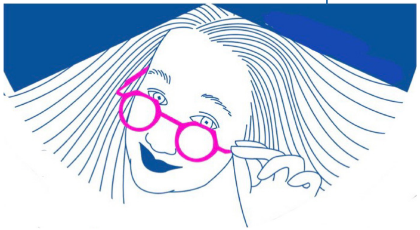
$r - 3 \leq c(T) \leq cn(T) \leq dr$ for some $d \in \mathbb{N}$.

Ellers, Gordeev, Herzog

Arad, Chillag, Moran $\quad cn(T) = 2 \Leftrightarrow T = J_1$

R'22 $\quad c(T) = 3 \Leftrightarrow cn(T) = 3 \Leftrightarrow T$ is one of
6 ∞-te families
13 sporadic groups
or A_7

e.g. $PSL_2(q)$
Monster



$\boxed{\text{Theorem (R'22)}}$ G2

Let $G = T^k X$, $X \leq S_k$ primitive

$\operatorname{orbdiam}(G) = 2 \iff k = 2$ & $c(T) = 2$

so $T \cong J_1$

$\operatorname{orbdiam}(G) = 3 \iff k = 2$ & $c(T) = 3$

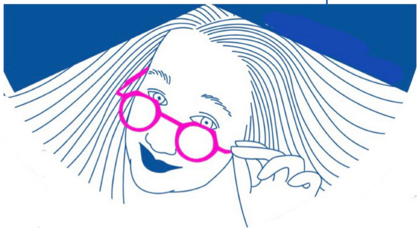
so T is in our list from the previous slide

$\operatorname{orbdiam}(G) = 4 \implies k = 3$ & $T \cong J_1$

or

$k = 2$ & $c(T) = 4$

in which case $\operatorname{orbdiam}(G) = 4$.



Affine type

$G = V G_0$

$\Omega = V = V_n(q)$

$G_0 \leq GL(V)$ acts irreducibly on V

Orbitals $\Gamma = \{0, a\}^G$, $a \in V$

$$\mathrm{diam}(\Gamma) = \min\{ k \mid \text{every } v \in V \text{ is a sum of at most } k \text{ vectors in } \pm a^{G_0} \}$$

$\boxed{\text{Theorem (R.)}}$ G1

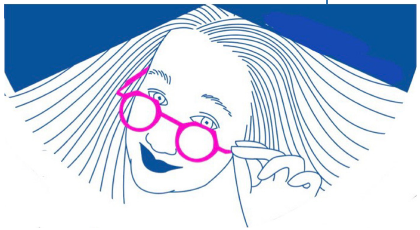
$G \leq AGL(V)$, G_0 almost quasisimple
V irred. in char. p

(1) $G_0 \trianglerighteq A_\ell$ $\quad \mathrm{orbdiam}(G) \geq \dfrac{\ell-5}{2\log_2 \ell}$

(2) $G_0 \trianglerighteq G_\ell(r) \in \mathrm{Lie}(p')$ $\quad \mathrm{orbdiam}(G) \geq \dfrac{r^{\ell-1}}{(\ell+1)^3 \log_2 r}$

(3) $G_0 \trianglerighteq G_\ell(r) \in \mathrm{Lie}(p)$ $\quad \mathrm{orbdiam}(G) \geq \left\lfloor \dfrac{\ell}{2} \right\rfloor$

or V is the natural module

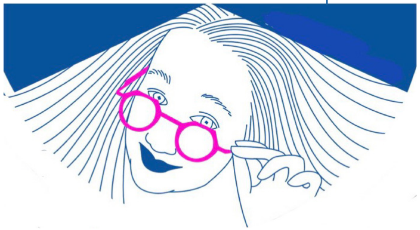


$\boxed{\text{Theorem (R)}}$ G2 — in progress!

For 3. & $G_\ell(r)$ untwisted

orbdiam $\le 2 \iff (G_\ell(r), V)$ is one of the following:

- $(G_2(r), V_6(r))$ r even (rank 2)
- $(G_2(r), V_7(r))$ r odd (rank 4)
- $(A_4(r), V_{10}(r))$ (rank 3)
- $(D_5(r), V_{16}(r))$ (rank 3)
- $(B_3(r), V_8(r))$ (rank 3)
- $(B_4(r), V_{16}(r))$ (rank 4)
- (classical, natural module)



Thank You for your attention!