Almost Elusive Groups

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Notation and Definitions

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- $\Delta(G)$ is the set of derangements.

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Theorem (Fein, Kantor and Schacher, 1981)

There always exists an element of prime power order in $\Delta(G)$.

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Theorem (Giudici,2003)

Let $G \leqslant \mathrm{Sym}(\Omega)$ be an elusive primitive permutation group. Then $(G, H) = (\mathrm{M}_{11}, \mathrm{PSL}_2(11))$.

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Now assume G is primitive. That is H is a maximal subgroup of G.

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Almost simple: $G_0 \leqslant G \leqslant \mathrm{Aut}(G_0)$ with G_0 a non-abelian simple group.

Affine: Let $V = (\mathbb{F}_p)^d$ with p-prime. Here $G = V{:}H$ with $H \leqslant \mathrm{GL}_d(p)$ and G acts on V via affine transformations.

Primitive Classification

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Examples

- $G = S_n$ with $\Omega = \{\{i, j\} \mid 1 \leqslant i < j \leqslant n\}$ and $n = 2^m + 1$ a Fermat prime or $n - 1 = 2^m - 1$ is a Mersenne prime.

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- $G = \mathrm{PGL}_2(q)$, H is of type P_1 (the stabiliser of a 1-dimensional subspace of $V = (\mathbb{F}_q)^2$) with $q = p = 2^m - 1$ a Mersenne prime.
- $G = p^d{:}\mathrm{GL}_1(p^d)$.

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Let $G \leqslant \mathrm{Sym}(\Omega)$ be a primitive almost simple group with socle G_0 and point stabiliser H.

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- We say $x \in G$ is a derangement if and only if $x^G \cap H = \emptyset$,
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Theorem (H,2021)

Suppose G_0 is a group of Lie type and $\pi(G_0) - \pi(H_0) \leqslant 1$. Then (G_0, H) belongs to a know list of cases.

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Lemma

An element $(v, h) \in G$ is a derangement of order p if and only if the following all hold:

(i) $h \in H$ *is either trivial or of order p;*

(ii) $v \in \ker(h^{p-1} + \cdots + h + 1)$*; and*

(iii) $v \notin \mathrm{im}(h - 1)$.

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Note

The nonzero vectors in V form a unique conjuagcy class of derangements of prime order p.

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Lemma (H,2021)

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a 2-transitive affine permutation group of degree p^d. Then G is almost elusive if and only if one of the following holds

(i) *p does not divide $|H|$; or*

(ii) *$|H|$ and d are both divisible by p, and every $h \in H$ of order p has Jordan form $[J_p^{d/p}]$ on V.*

Thank you for listening