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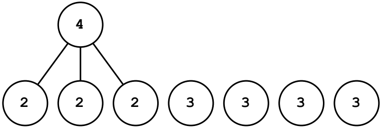

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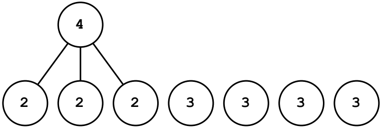

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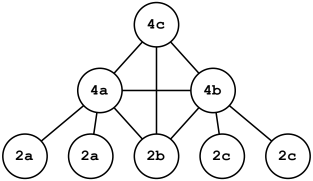

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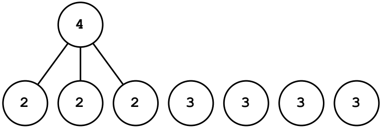

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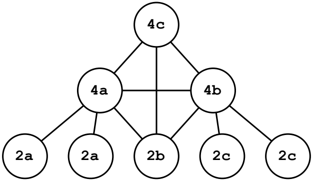

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Theorem (Csákány & Pollák, 1969)

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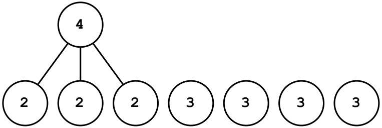

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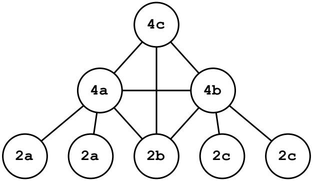

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Open question: Is there a finite non-simple group G with $\mathrm{diam}(\Delta_G) = 4$? If yes, then $G = S \rtimes C_p$ for a non-abelian simple group S and an odd prime p (Csákány & Pollák, 1969).

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The proofs of Shen, Herzog et al. and Ma all involved the prime graph or Gruenberg-Kegel graph of G. The vertices of this graph are the prime divisors of $|G|$, with $p_1 \sim p_2 \iff G$ has an element of order $p_1 p_2$.

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If $S \leqslant M_1$ and $J \leqslant M_2$, then $S \sim M_1 \sim D \sim M_2 \sim J$. Hence $d(S, J) \leqslant 4$.

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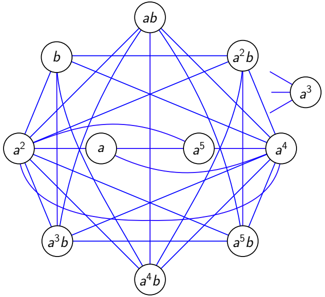

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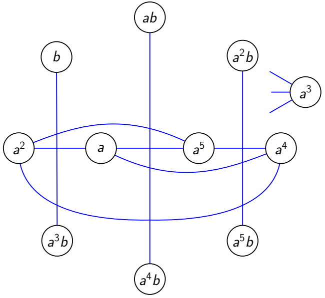

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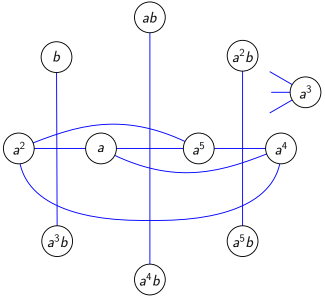

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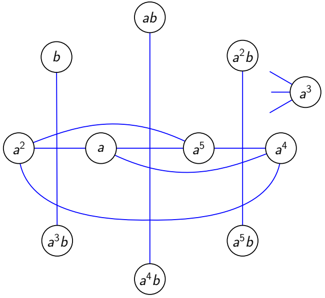

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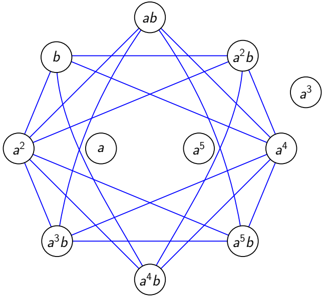

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$\exists$ prime order subgroups $S_1, S_2 \in G$ with $d_{\Delta_G}(S_1, S_2) = 5$.

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Proposition (Cameron, 2022)

If graphs Γ_1 and Γ_2 form a dual pair and $\mathrm{diam}(\Gamma_1) = k$, then $k - 1 \leqslant \mathrm{diam}(\Gamma_2) \leqslant k + 1$.

$\mathrm{diam}(\Delta_G) = 5$, so $4 \leqslant \mathrm{diam}(\Sigma(G)) \leqslant 6$. In fact, $\mathrm{diam}(\Sigma(G)) = 4$.

Similarly, $4 \leqslant \mathrm{diam}(\Xi(G))$. In fact, $\mathrm{diam}(\Xi(G)) = 4$.

The subgraph Γ_G of Δ_G induced by soluble subgroups forms a dual pair with the soluble graph $\Pi(G)$, where $x \sim y \iff \langle x, y \rangle$ is soluble.

$\exists$ prime order subgroups $S_1, S_2 \in G$ with $d_{\Delta_G}(S_1, S_2) = 5$.

Hence $5 \leqslant \mathrm{diam}(\Gamma_G)$ and $4 \leqslant \mathrm{diam}(\Pi(G))$; in fact, $\mathrm{diam}(\Pi(G)) \in \{4, 5\}$ (Burness, Lucchini & Nemmi, 2021+).