Möbius Function of finite classical groups

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Young Group Theorists workshop: exploring new connections

SRS SwissMAP Research Station in Les Diablerets

6 September 2022

Möbius Function

Möbius Function

$\mu : \mathbb{N} \to \mathbb{Z}$ such that

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1 \cdot \ldots \cdot p_r \\ 0 & \text{if } p^2 \mid n \text{ for some prime } p \end{cases}$$

prime factors $p_i \neq p_j \ \forall i \neq j$

August Ferdinand Möbius (1790–1868)

Möbius function

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- $\mu(x,y) = -\sum_{x<z\leq y} \mu(z,y)$ if $x < y$

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$x \leq y$ in $\mathbb{N} \iff x \mid y \rightsquigarrow \mu(n) = \mu(1,n)$, $n \in \mathbb{N}$

Möbius function of a finite group

Let G be a finite group and $\mathcal{L}(G)$ its subgroup lattice.

The Möbius function of G is given by

$\mu_G : \mathcal{L}(G) \longrightarrow \mathcal{L}(G)$ such that

$$\begin{cases} \mu_G(G) = 1 \\ \sum_{H \leq K \leq G} \mu_G(K) = 0 \quad \forall\, H < G \end{cases}$$

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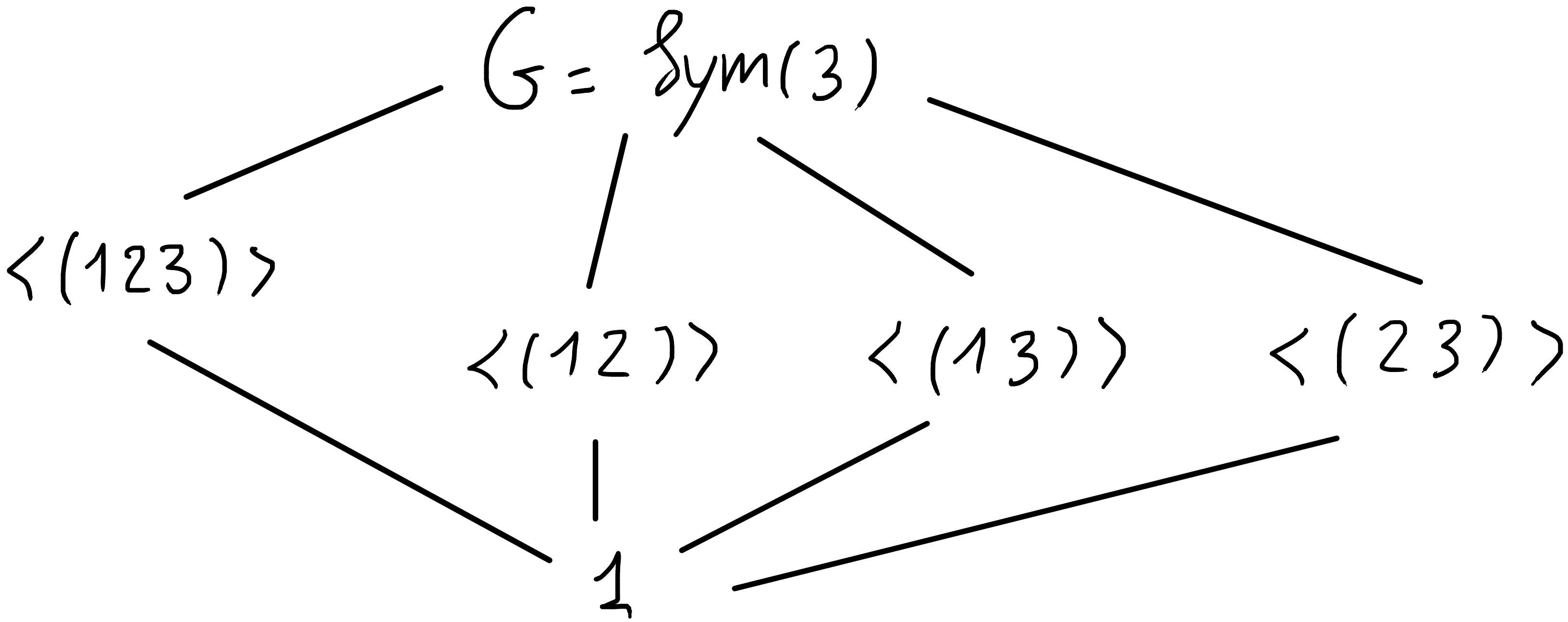
Applications

- Probability that k elements generate G: (Hall) $\boxed{P(G,k) = \sum_{H \leq G} \frac{\mu_G(H)}{|G:H|^k}}$
- Topology of simplicial complexes related to G (Bouc, Brown, ...)

Example



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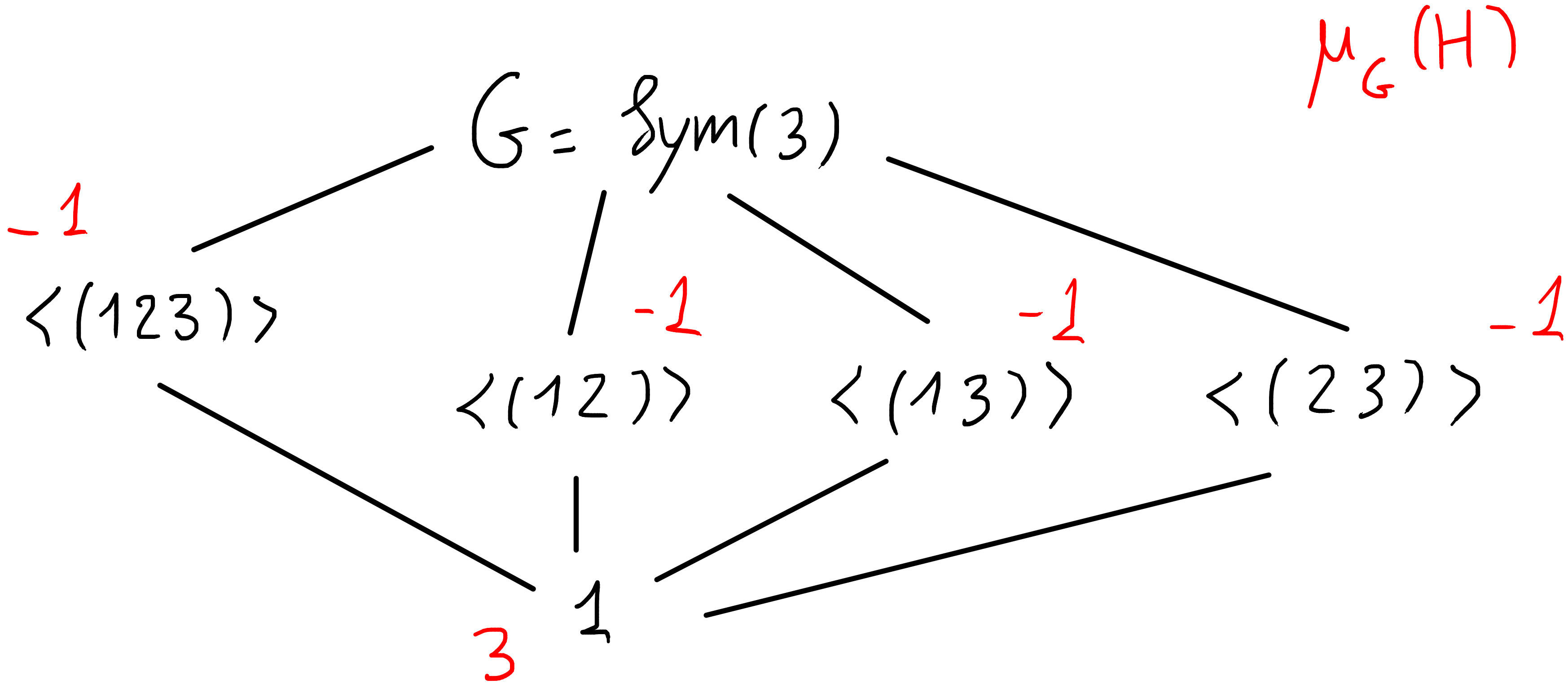

(Shareshian)

Let p be an odd prime, $G = \mathrm{Sym}(p)$. Then

$$\mu_G(1) = \frac{p!}{2}$$

Main Property

Proposition

Let H be a proper subgroup of G ($H < G$) such that

$$\mu_G(H) \neq 0.$$

Then $\exists$ maximal subgroups $M_1, \dots, M_n$ of G so that

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PROOF. Let $M_1, \dots, M_t$ be all the maximal subgroups of G that contain H. Let $J = M_1 \cap \dots \cap M_t$.

Show that:
$$H < J \implies \mu_G(H) = 0.$$

J = intersection of all maximal subgroups containing H.

Let H be properly contained in J.

Then

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$$= -\sum_{J\leq K\leq G} \mu_G(K) - \sum_{H<L,\ J\not\leq L} \mu_G(L)$$

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$$-\Big(\mu_G(J) + \underbrace{\sum_{J<K\le G} \mu_G(K)}_{-\mu_G(J)}\Big) = 0$$

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- $M \leq_{max} G$ such that $L \subseteq M \Rightarrow M \in \{M_1, \dots, M_t\}$
- L is not an intersection of some maximal subgroups in $\{M_1, \dots, M_t\}$

$\rightsquigarrow$ By an inductive argument we have $\mu_G(L) = 0 \ \forall L$.

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$$= 0$$

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$$G = \langle x, y \mid x^5 = y^4 = 1,\ x^y = x^2 \rangle \cong C_5 \rtimes C_4$$

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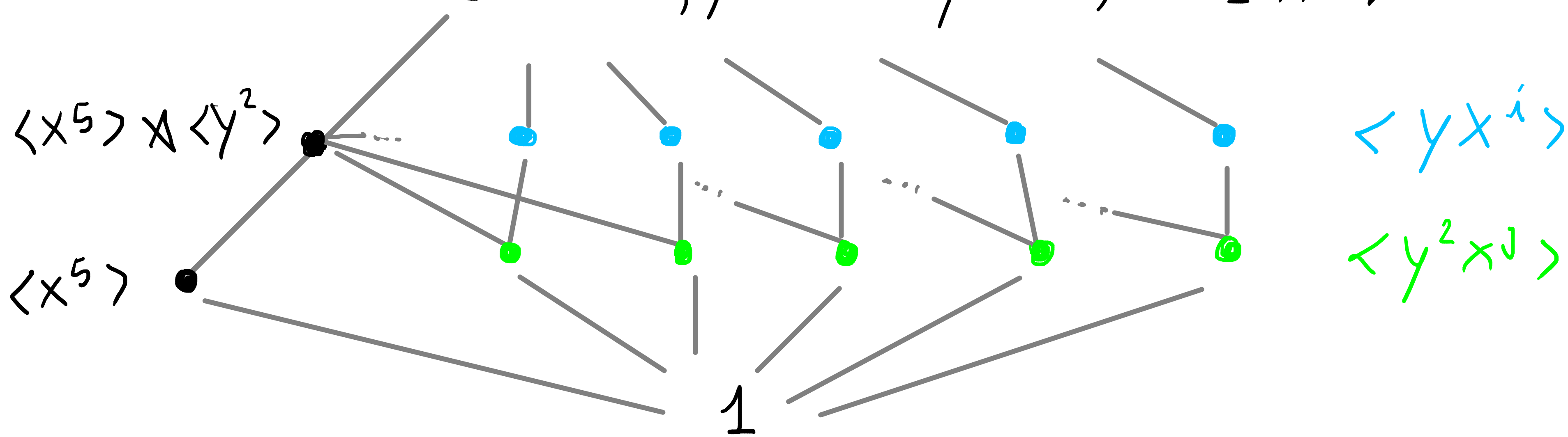

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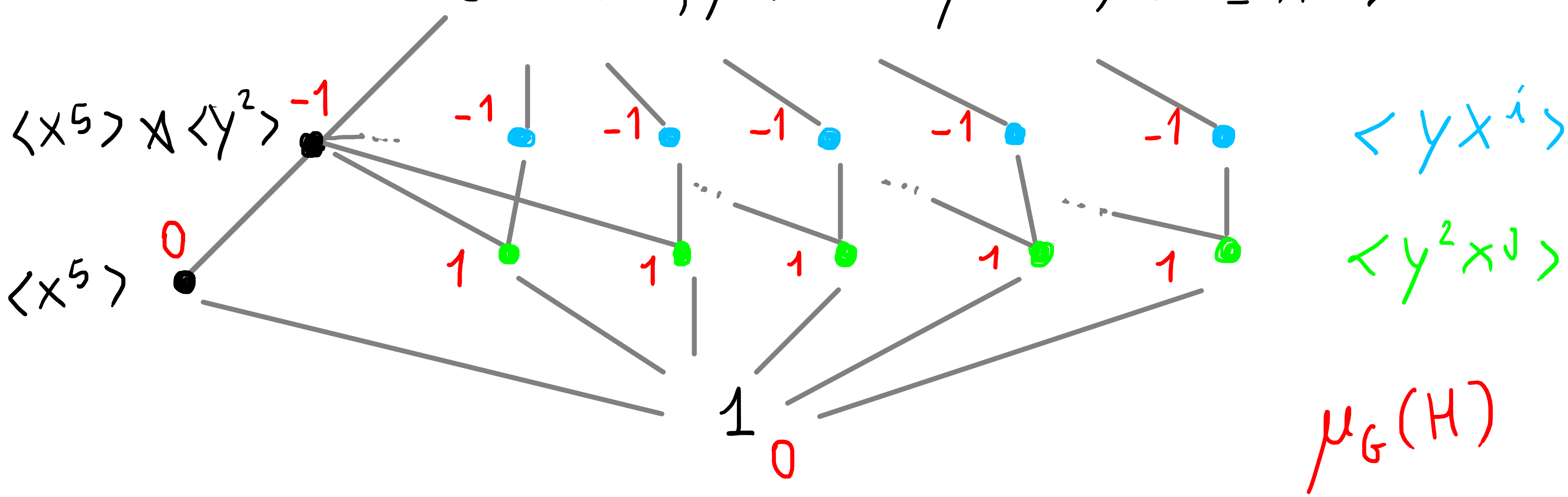

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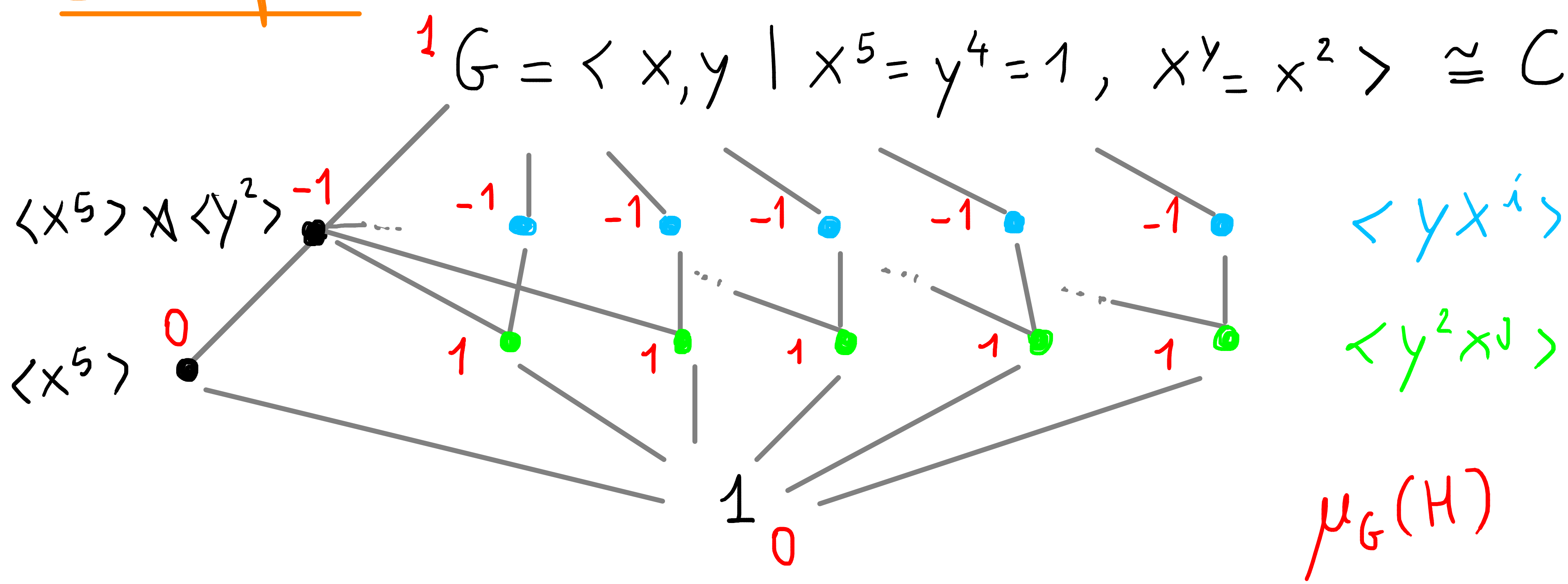

(Shareshian)

Let p be an odd prime, $p \not\equiv 1, 2, 4, 7, 8, 13 \mod 15$.

Let $G = PSL(3, p^a)$, $a \in \mathbb{N}$. Then $\mu_G(1) = 0$.

Subgroups $H \leq G$ such that $\mu_G(H) \neq 0$

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Let G be solvable, $H < G$, $\mu_G(H) \neq 0$.

Then $\exists$ $M_1, \dots, M_n$ maximal subgroups of G such that

$$H = M_1 \cap \dots \cap M_n \quad \& \quad |G:H| = |G:M_1| \cdot \ldots \cdot |G:M_n| .$$

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(Colombo-Lucchini)

Let $G \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\}$. Then the probability of finding $H \leq G$ such that $\mu_G(H) \neq 0$ tends to 0 as $n \to +\infty$.

Subgroups $H \leq G$ such that $\mu_G(H) \neq 0$

Let
$$b_m(G) = \#\{H \leq G \mid |G:H| = m \ \& \ \mu_G(H) \neq 0\}$$

<u>Conjecture</u> (Mann-Lucchini)

There exists an absolute constant α such that for every finite almost-simple group G
$$b_m(G) \leq m^{\alpha} \quad \forall\, m \in \mathbb{N}.$$

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Such a constant exists $\forall\, G \in \{\mathrm{Alt}(n), \mathrm{Sym}(n) \mid n \geq 5\}$

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... What about finite classical groups?

Closure operator

Let P be a poset : $c : P \longrightarrow P$ such that

1) $x \leq c(x) \qquad \forall x \in P$

2) $x \leq y \Rightarrow c(x) \leq c(y) \qquad \forall x, y \in P$

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In a (projective) linear group

Let $G \in \{GL(V), PGL(V)\}$, $V \cong \mathbb{F}_q^n$.

Let K be an irreducible subgroup of G.

For every $H \leq K$, the set of H-invariant subspaces of V is

$$\mathcal{S}(V,H) := \{ W \leq V \mid Wh = W \ \forall h \in H \}$$

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the closure of H in K

(H is closed if $H = \overline{H}$)

$V \cong \mathbb{F}_q^n$, $\quad G \in \{GL(V), PGL(V)\}$, $\quad H \leq G$.

Let
$$\mathcal{J}(G,H) := \{ K \leq G \mid H \leq K \text{ \& } K \text{ irreducible} \}$$

For every $K \in \mathcal{J}(G,H)$ we define
$$\psi(H,K) := \{ E \subseteq \mathcal{S}(V,H) \setminus \{0,V\} \mid \bigcap_{W \in E} \mathrm{stab}_K(W) \neq H \}$$

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$(>)$

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Let
$$\mathcal{Y}(G,H) := \{ K \leq G \mid H \leq K \ \& \ K \text{ irreducible} \}$$

For every $K \in \mathcal{Y}(G,H)$ we define
$$\psi(H,K) := \Big\{ E \subseteq \mathcal{S}(V,H) \setminus \{0,V\} \;\Big|\; \bigcap_{W \in E} \mathrm{stab}_K(W) \neq H \Big\}$$

Theorem

$$\mu_G(H) = \sum_{K \in \mathcal{Y}(G,H)} \mu_G(K) \cdot g(H,K)$$

where
$$g(H,K) = \sum_{E \in \psi(H,K)} (-1)^{|E|}.$$

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(a similar result holds in general for finite classical groups)

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Let $K \in \mathcal{I}(G,H)$ (i.e., $H \leq K \leq G$ and K irreducible)

If H is not closed in K, then $H < \overline{H} = \bigcap_{W \in \mathcal{S}(V,H)} \mathrm{stab}_K(W)$

$\Rightarrow$ $g(H,K) = 0$.

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Proposition

Assume that $\mu_G(H) \neq 0$. Then there exist

- an irreducible subgroup $K \leq G$ with $\mu_G(K) \neq 0$

&

- a closed subgroup C in G

so that

$$H = K \cap C.$$

At least one term in $\sum_{K \in \mathcal{J}(G,H)} \mu_G(K)\, g(H,K)$ must be $\neq 0$ ...

Subgroups $H \leq G$ such that $\mu_G(H) \neq 0$

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Conjecture (Mann-Lucchini)

There exists an absolute constant α such that for every finite almost-simple group G

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Theorem

Such a constant exists $\forall G \in \{PGL(n,q), PSL(n,q)\}$ if

- $\#\{K \leq G \mid K \text{ irreducible}, \mu_G(K) \neq 0, |G:K| = m\} \leq m^{\alpha_1} \quad \forall m \in \mathbb{N}$

&

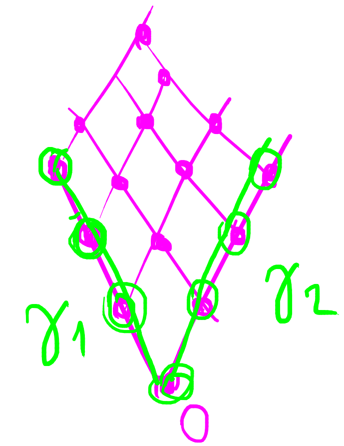
- $\#\{C \leq G \mid C \text{ closed in } G, |G:C| = m\} \leq m^{\alpha_2} \quad \forall m \in \mathbb{N}$

with α_1, α_2 independent of n, q.

Let $V \cong \mathbb{F}_q^n$ and $G \in \{GL(V), SL(V), PGL(V), PSL(V)\}$

$\xi \in G$ cyclic matrix $(char_\xi = min_\xi)$. Then

if $H = \langle \xi \rangle \Rightarrow \mathcal{S}(V,H) \cong \prod_i \gamma_i$, γ_i chains



Theorem

Let $\Gamma_m(G)$ be the set of closed subgroups H in G such that $|G:H| = m$ and $\mathcal{S}(V,H)$ is isomorphic to a product of chains. Then there exists an absolute constant c, independent of n and q, such that

$$|\Gamma_m(G)| \leq m^c \quad \forall m \in \mathbb{N}.$$

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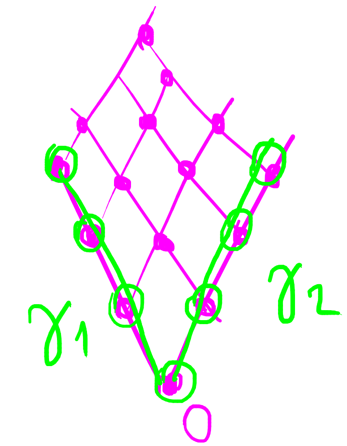

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¿ Number of closed subgroups containing a cyclic matrix ?

($\rightsquigarrow \mathcal{S}(V,H)$ distributive)