Finite Subgroups of Transformation Groups

SwissMAP Young Group theorists workshop: exploring new connections

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Outline

1. Geometric motivation

2. Structure results of class 2 nilpotent groups

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2. Structure results of class 2 nilpotent groups

A classical theorem

Question (Old for warm-up)

What are the (isomorphism classes of) *finite* subgroups of $\mathrm{GL}_2(\mathbb{C})$?

Answer (F. Klein): Cyclic central extensions of

- family 1: cyclic groups, C_n ($n \in \mathbb{N}_+$) or [index 1]
- family 2: dihedral groups, D_{2n} ($n \in \mathbb{N}_+$) or [index 2]
- 3 other groups: A_4, A_5, S_5 [index $\leq 5!$]

Full list is not easy. Note: these are *'almost' abelian*: $\exists$ normal abelian subgroup, index $\leq 5!$.

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Theorem (C. Jordan, 1877)

$\forall n \in \mathbb{N}_+ \quad \forall$ *finite* $F \leq \mathrm{GL}_n(\mathbb{C})$ *is 'almost abelian', i.e.* $\mathrm{GL}_n(\mathbb{C})$ *is Jordan, i.e.*
$\exists J_n \in \mathbb{N}_+ \quad \forall F \quad \exists$ *abelian* $A \lhd F$ *with* $|F : A| \leq J_n$.

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$\mathrm{GL}_n(\mathbb{C}) \rightsquigarrow$ "transformation group" G:

- $\Omega \in$ {compact manifolds, algebraic varieties}
- faithful action via homeomorphisms/diffeomorphisms, resp. birational automorphisms

Question (Q1)

Are all finite subgroups of *every* transformation group are 'almost' abelian?

- Past two decades: positive answer for many Ω's (Mundet, Zimmermann, Serre, Popov, Prokhorov, Shramov etc).
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Break – a natural local question ;)

Figure: Landscape at Le Lechere this afternoon

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How *close*(=près) is G to being *Jordan*?

Recent development

Are finite subgroups of *every* transformation group G 'almost' [family of groups to be found]?

Yes!

Theorem

- *birational automorphisms: nilpotent of class ≤ 2 [Guld, '20]*
- *homeomorphisms: nilpotent [Csikós – Pyber – E. Szabó, '22]*

True for some smaller family of groups?

Theorem (Sz's thesis, 2022)

- *birational automorphisms: the theorem is optimal*
- *homeomorphisms: lower bound: nilpotent of class ≤ 2 (conjecture: sharp)*

Idea: action of *all* class 2 groups in both cases.

Tools: *common group theory core* for both cases (+ cohomologies, sheaves, number theory)

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Central product decomposition

Let G be a finite(ly generated) nilpotent group of class ≤ 2 (i.e. $G' \subseteq \mathrm{Z}(G)$).

Previous counterexamples to **Q1**

- work for $G' \subseteq \mathrm{Z}(G)$ cyclic
- can be extended to *all* finite 2-generated class ≤ 2 groups
- can be naturally combined using central products (on external tensor product of bundles)

This and the structural description of special p-groups lead to an independent rediscovery of

Theorem ($\approx$ Cheng, '82)

1. $\exists$ **subdirect product** $G \leq \prod_{i=1}^{d(\mathrm{Z}(G))} H_i$ *where* H_i' *cyclic. (Done by induction.)*
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(Recall central product: $H = K_1 \curlyvee \ldots \curlyvee K_n$ if $H = K_1 \ldots K_n$ and $[K_i, K_j] = 1$ for $i \neq j$.)

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- $\log_p \#(\text{groups of order } p^n \text{ of class 2}) = \frac{2}{27}n^3 - \frac{12}{27}n^2$ [Higman '60]
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(Recall central product: $H = K_1 \curlyvee \ldots \curlyvee K_n$ if $H = K_1 \ldots K_n$ and $[K_i, K_j] = 1$ for $i \neq j$.)

- $\exists$ classification of these building blocks $E_{i,j}$, cf. [Ahmad–Magidin–Morse, '12] finite p case.
- $\log_p \#(\text{groups of order } p^n \text{ of class 2}) = \frac{2}{27}n^3 - \frac{12}{27}n^2$ [Higman '60]
  $\log_p \#(\text{groups of order } p^n) = \frac{2}{27}n^3 + \mathcal{O}(n^{8/3})$ [Sims, '65]

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Our proof (of $\exists H = Z \curlyvee E_1 \curlyvee \ldots \curlyvee E_t$): works for all orders (prime or not, odd or even, finite or infinite) and gives bounds to the ranks (needed for the Motivation)

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Embedding to concrete Heisenberg groups

Corollary of Theorem: one can construct a certain action of G on invertible sheaves.
Mumford ('66) classified finite subgroups of these automorphisms using matrix groups giving

Corollary (cf. Magidin (98) for more general, less concrete statement)

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How to get the realising actions of G from Motivation?

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Summary

Finite class ≤ 2 nilpotent groups

1. is basically the largest family that can act simultaneously on a variety birationally.
2. admit a structural decomposition using 2-generated groups via central/subdirect products.
3. embed to concrete unitriangular matrix groups (facilitating computations, applications).

Thank you for your attention!

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[3] Attila Guld. *Finite subgroups of the birational automorphism group are 'almost' nilpotent of class at most two*. 2020. arXiv: 2004.11715 [math.AG].