Group graded algebras over G-graded G-algebras

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Introduction

Representation theory of finite groups

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Local-global conjectures

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Reduction theorems

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Character triples and relations between them [Späth 2017-2018]

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Group graded Morita equivalences [MM 2020-2021]

Notations and preliminaries

Assumptions and notations

- G is a finite group, $N \trianglelefteq G$;
- $\bar{G} := G/N$;
- $\mathcal{O}$ is an associative and commutative ring with unity $1 \neq 0$

Definition

An algebra $\mathcal{C}$ is a $\bar{G}$-graded $\bar{G}$-acted $\mathcal{O}$-algebra if

1. $\mathcal{C}$ is $\bar{G}$-graded, i.e. $\mathcal{C} = \oplus_{\bar{g}\in\bar{G}}\mathcal{C}_{\bar{g}}$ and $\mathcal{C}_{\bar{g}}\mathcal{C}_{\bar{h}} \subseteq \mathcal{C}_{\bar{g}\bar{h}}$, for all $\bar{g}, \bar{h} \in \bar{G}$;
2. $\bar{G}$ acts on $\mathcal{C}$ (always on the left in this presentation);
3. $\forall \bar{h} \in \bar{G}$, $\forall c \in \mathcal{C}_{\bar{h}}$ we have that ${}^{\bar{g}}c \in \mathcal{C}_{\bar{g}\bar{h}}$ for all $\bar{g} \in \bar{G}$.

Examples of $\bar{G}$-graded $\bar{G}$-acted $\mathcal{O}$-algebras

Example 1

- Let $A = \bigoplus_{\bar{g} \in \bar{G}} A_{\bar{g}}$ be a strongly $\bar{G}$-graded $\mathcal{O}$-algebra ($A_{\bar{g}} A_{\bar{h}} = A_{\bar{g}\bar{h}}$, for all $\bar{g}, \bar{h} \in \bar{G}$).
- Let $B := A_1$.
- Then, the centralizer $C_A(B)$ of B in A is a $\bar{G}$-graded $\bar{G}$-acted $\mathcal{O}$-algebra.

Example 2

- $\mathcal{C} := \mathcal{O}C_G(N)$ is a $\bar{G}$-graded $\bar{G}$-acted algebra:

$\bar{G}$-graded $\mathcal{O}$-algebras over $\mathcal{C}$

- Let $\mathcal{C}$ be a $\bar{G}$-graded $\bar{G}$-acted $\mathcal{O}$-algebra.

Definition

We say that A is a $\bar{G}$-graded $\mathcal{O}$-algebra over $\mathcal{C}$ if there is a $\bar{G}$-graded $\bar{G}$-acted algebra homomorphism

$$\zeta : \mathcal{C} \to C_A(B),$$

i.e. for any $\bar{h} \in \bar{G}$ and $c \in \mathcal{C}_{\bar{h}}$, we have $\zeta(c) \in C_A(B)_{\bar{h}}$, and for every $\bar{g} \in \bar{G}$, $\zeta({}^{\bar{g}}c) = {}^{\bar{g}}\zeta(c)$.

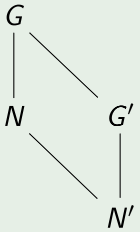
Example 1

If $b \in \mathrm{Z}(\mathcal{O}N)$ is a $\bar{G}$-invariant block idempotent, then $A := b\mathcal{O}G$ is a $\bar{G}$-graded crossed product ($hU(A) \cap A_{\bar{g}} \neq \varnothing$, for all $\bar{g} \in \bar{G}$) over $\mathcal{C} := \mathcal{O}C_G(N)$, with structural map induced by inclusion. Note that $B := A_1 = b\mathcal{O}N$.

Examples of $\bar{G}$-graded $\mathcal{O}$-algebras over $\mathcal{C}$

Example 2

- Let $G' \leq G$ such that $G = G'N$.
- Let $N' = G' \cap N$, hence $N' \trianglelefteq G'$.
- Therefore $\bar{G} := G/N \simeq G'/N'$.



- Let $b' \in Z(\mathcal{O}N')$ be a $\bar{G}$-invariant block idempotent.
- Let $A' := b'\mathcal{O}G'$, which is clearly a $\bar{G}$-graded crossed product, with $B' := A'_1 = b'\mathcal{O}N'$.
- If $C_G(N) \subseteq G'$, then A' is a $\bar{G}$-graded algebra over $\mathcal{C} := \mathcal{O}C_G(N)$, with structural map also induced by inclusion.

$\bar{G}$-graded bimodules over $\mathcal{C}$

- Let A and A' be two $\bar{G}$-graded crossed products over $\mathcal{C}$, with $B := A_1$ and $B' := A'_1$.

Definition

1. We say that $\tilde{M}$ is a $\bar{G}$-graded (A, A')-bimodule over $\mathcal{C}$ if:
   1. $\tilde{M}$ is an (A, A')-bimodule;
   2. $\tilde{M}$ has a decomposition $\tilde{M} = \bigoplus_{\bar{g}\in\bar{G}} \tilde{M}_{\bar{g}}$ such that $A_{\bar{g}}\tilde{M}_{\bar{x}}A'_{\bar{h}} \subseteq \tilde{M}_{\bar{g}\bar{x}\bar{h}}$, for all $\bar{g}, \bar{x}, \bar{h} \in \bar{G}$;
   3. $\tilde{m}_{\bar{g}} \cdot c = {}^{\bar{g}}c \cdot \tilde{m}_{\bar{g}}$, for all $c \in \mathcal{C}$, $\tilde{m}_{\bar{g}} \in \tilde{M}_{\bar{g}}, \bar{g} \in \bar{G}$, where $c \cdot \tilde{m} = \zeta(c) \cdot \tilde{m}$ and $\tilde{m} \cdot c = \tilde{m} \cdot \zeta'(c)$, for all $c \in \mathcal{C}, \tilde{m} \in \tilde{M}$.
2. $\bar{G}$-graded (A, A')-bimodules over $\mathcal{C}$ form a category,

$$A\text{-Gr}/\mathcal{C}\text{-}A',$$

where the morphisms between $\bar{G}$-graded (A, A')-bimodules over $\mathcal{C}$ are just homomorphism between $\bar{G}$-graded (A, A')-bimodules.

$\Delta^{\mathcal{C}}$

- We regard A'^{op} as a $\bar{G}$-graded algebra with components $(A'^{\mathrm{op}})_{\bar{g}} = A'_{\bar{g}^{-1}}$, $\forall \bar{g} \in \bar{G}$.
- We consider the diagonal part of $A \otimes_{\mathcal{C}} A'^{op}$:

$$\Delta^{\mathcal{C}} := \Delta(A \otimes_{\mathcal{C}} A'^{op}) := \bigoplus_{\bar{g}\in\bar{G}} A_{\bar{g}} \otimes_{\mathcal{C}} A'_{\bar{g}^{-1}},$$

which is clearly well-defined.

Lemma-Example

- $\Delta^{\mathcal{C}}$ is an $\mathcal{O}$-algebra;
- $A \otimes_{\mathcal{C}} A'^{op}$ is a right $\Delta^{\mathcal{C}}$-module and a $\bar{G}$-graded (A, A')-bimodule over $\mathcal{C}$.

$\Delta^{\mathcal{C}}$

Theorem

We have three naturally isomorphic equivalences of categories, and their inverse is $(-)_1$:

$$\Delta^{\mathcal{C}}\text{-mod} \underset{(-)_1}{\overset{(A\otimes_{\mathcal{C}} A'^{\mathrm{op}})\otimes_{\Delta^{\mathcal{C}}} -,\ A\otimes_B -,\ -\otimes_{B'} A'}{\rightleftarrows}} A\text{-Gr}/\mathcal{C}\text{-}A'.$$

- Let M be a $\Delta^{\mathcal{C}}$-module, then $A \otimes_B M$, $M \otimes_{B'} A'$ and $(A \otimes_{\mathcal{C}} A'^{op}) \otimes_{\Delta^{\mathcal{C}}} M$ are isomorphic as $\bar{G}$-graded (A, A')-bimodules over $\mathcal{C}$. We shall denote them by $\widetilde{M}$.

$\bar{G}$-graded Morita equivalences over $\mathcal{C}$

- Let $\tilde{M}$ be a $\bar{G}$-graded (A, A')-bimodule over $\mathcal{C}$, then its A-dual $\tilde{M}^* = \mathrm{Hom}_A(\tilde{M}, A)$ of $\tilde{M}$ is a $\bar{G}$-graded (A', A)-bimodule over $\mathcal{C}$.

Definition

We say that $\tilde{M}$ induces a $\bar{G}$-graded Morita equivalence over $\mathcal{C}$ between A and A', if the following conditions hold:

1. $\tilde{M} \otimes_{A'} \tilde{M}^* \cong A$ as $\bar{G}$-graded (A, A)-bimodules over $\mathcal{C}$;
2. $\tilde{M}^* \otimes_A \tilde{M} \cong A'$ as $\bar{G}$-graded (A', A')-bimodules over $\mathcal{C}$.

$\bar{G}$-graded Morita equivalences over $\mathcal{C}$

Theorem

Let ${}_BM_{B'}$ and ${}_{B'}M^*{}_B := \mathrm{Hom}_B(M, B)$ (the B-dual of M) be two bimodules that induce a Morita equivalence between B and B':

$$B \underset{M\otimes_{B'}-}{\overset{M^*\otimes_B -}{\rightleftarrows}} B'$$

If M extends to a Δ^C-module, then we have the following:

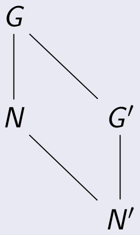
1. M^* becomes a $\Delta(A' \otimes_C A^{op})$-module;
2. $\tilde{M} := (A \otimes_C A'^{op}) \otimes_{\Delta^C} M$ and $\tilde{M}^* := (A' \otimes_C A^{op}) \otimes_{\Delta(A' \otimes_C A^{op})} M^*$ are $\bar{G}$-graded bimodules over C and they induce a $\bar{G}$-graded Morita equivalence over C between A and A':

$$A \overset{\sim}{\rightleftarrows} A'.$$

Main framework

Assumptions and notations

- $G' \leq G$, such that $G = G'N$
- $N' = G' \cap N$, hence $N' \trianglelefteq G'$
- Therefore, $\bar{G} := G/N \simeq G'/N'$.



- $b \in Z(\mathcal{O}N)$ and $b' \in Z(\mathcal{O}N')$ are $\bar{G}$-invariant block idempotents
- $A := b\mathcal{O}G$ and $A' := b'\mathcal{O}G'$, which are $\bar{G}$-graded crossed products
- $B := A_1 = b\mathcal{O}N$ and $B' := A'_1 = b'\mathcal{O}N'$
- Assume $C_G(N) \subseteq G'$, hence A and A' are $\bar{G}$-graded crossed products over $\mathcal{C} := \mathcal{O}C_G(N)$

Proposition

Assume that:

1. $C_G(N) \subseteq G'$;
2. M induces a Morita equivalence between B and B';
3. $zm = mz$, for all $m \in M$ and $z \in Z(N)$.

Then there is a $\bar{C}_G(N) := NC_G(N)/N$-graded Morita equivalence between C and C' over $\mathcal{C} := \mathcal{O}C_G(N)$

$$\begin{array}{ccc}
A := b\mathcal{O}G & & A' := b'\mathcal{O}G' \\
\Big| & & \Big| \\
C := b\mathcal{O}NC_G(N) & \overset{}{\underset{\sim}{\rule{4em}{0.4pt}}} & C' := b'\mathcal{O}N'C_G(N) \\
\Big| & & \Big| \\
B := b\mathcal{O}N & \overset{{}_B M_{B'}}{\underset{\sim}{\rule{4em}{0.4pt}}} & B' := b'\mathcal{O}N',
\end{array}$$

induced by the $\bar{C}_G(N)$-graded (C, C')-bimodule over $\mathcal{C}$
$C \otimes_B M \simeq M \otimes_{B'} C' \simeq (C \otimes_{\mathcal{C}} C'^{\mathrm{op}}) \otimes_{\Delta(C \otimes_{\mathcal{C}} C'^{\mathrm{op}})} M.$

The butterfly theorem for group graded Morita equivalences over $\mathcal{C}$

The butterfly theorem for group graded Morita equivalences over $\mathcal{C}$

Let $\hat{G}$ be another finite group with normal subgroup N, such that the block b is also $\hat{G}$-invariant. Assume that:

1. $C_G(N) \subseteq G'$,
2. $\tilde{M}$ induces a $\bar{G}$-graded Morita equivalence over C between A and A';
3. the conjugation maps $\varepsilon : G \to \mathrm{Aut}(N)$ and $\hat{\varepsilon} : \hat{G} \to \mathrm{Aut}(N)$ satisfy $\varepsilon(G) = \hat{\varepsilon}(\hat{G})$.

Denote $\hat{G}' = \hat{\varepsilon}^{-1}(\varepsilon(G'))$. Then there is a $\hat{G}/N$-graded Morita equivalence over $\hat{C} := \mathcal{O}C_{\hat{G}}(N)$ between $\hat{A} := b\mathcal{O}\hat{G}$ and $\hat{A}' := b'\mathcal{O}\hat{G}'$.

The butterfly theorem for group graded Morita equivalences over $\mathcal{C}$

The butterfly theorem for group graded Morita equivalences over $\mathcal{C}$

$$
\begin{array}{ccccccc}
\hat{A} := b\mathcal{O}\hat{G} & & A := b\mathcal{O}G & \overset{\tilde{M}}{\underset{\sim}{\text{———}}} & A' := b'\mathcal{O}G' & & \hat{A}' := b'\mathcal{O}\hat{G}' \\
\Big| & & \Big| & & \Big| & & \Big| \\
b\mathcal{O}NC_{\hat{G}}(N) & & b\mathcal{O}NC_G(N) & \underset{\sim}{\text{———}} & b'\mathcal{O}N'C_G(N) & & b'\mathcal{O}N'C_{\hat{G}}(N) \\
& \diagdown & \Big| & & \Big| & \diagup & \\
& & B := \mathcal{O}Nb & \overset{M}{\underset{\sim}{\text{———}}} & B' := \mathcal{O}N'b'. & &
\end{array}
$$

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