Representation growth of semisimple profinite groups

Young Group theorists workshop: exploring new connections.

Les Diablerets, Switzerland

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Motivation

Universe of finite groups

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We divide them accordingly to the nature of some invariants

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A key result [Lubotzky, Mann, and Segal (1993)]

Complete characterization of finitely generated groups with polynomial subgroup growth in terms of algebraic properties.

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A profinite group is a Hausdorff, compact, and totally disconnected topological group.

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$$\rho : G \to \mathrm{GL}_n(\mathbb{C}).$$

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Let G be a finitely generated profinite group. A (complex) representation of G of dimension n is a continuous group homomorphism

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Representation rigidity

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For finitely generated profinite groups $\rightarrow$ Yes

Lemma [Bass, Lubotzky, Magid, and Mozes (2002)]

Let G be a finitely generated profinite group. Then G is rigid if and only if G is FAb, i.e. $H/[H, H]$ is finite for every open subgroup H of G.

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In the class of profinite groups, we consider the subclass of semisimple profinite groups

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$$\prod_{i\geq 1} \mathsf{PSL}_2(p^i)$$

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Rigidity for semisimple profinite groups

Recall:

Lemma [Bass, Lubotzky, Magid, and Mozes (2002)]

Let G be a finitely generated profinite group. Then G is rigid if and only if G is FAb, i.e. $H/[H, H]$ is finite for every open subgroup H of G.

Finitely generated semisimple profinite groups are always rigid.

Use the fact that finite non-abelian simple groups are perfect.

Representation zeta function

To better study the representation growth, we encode the informations that we have, in the representation zeta function

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^{-s}, \qquad s \in \mathbb{C}$$

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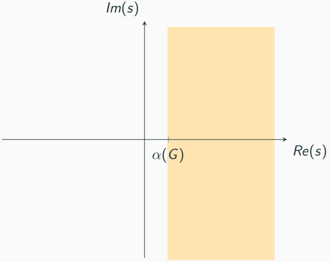

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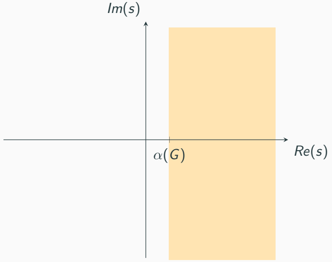

For $\mathrm{PSL}_2(q)$ and $q \equiv 1 \bmod 4$, we have

$$\zeta_{\mathrm{PSL}_2(q)}(s) = 1 + q^{-s} + \frac{q-5}{4}\cdot(q+1)^{-s} + 2\cdot\left(\frac{q+1}{2}\right)^{-s} + \frac{q-1}{4}\cdot(q-1)^{-s} \sim 1 + q^{1-s}$$

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$$\text{It converges} \quad \Leftrightarrow \quad \sum_{i\geq 1} p^{i(1-s)} \text{ converges}$$
$$\Leftrightarrow \quad \mathrm{Re}(s) > 1$$

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Theorem [Kassabov and Nikolov (2006)]

Let G be semisimple profinite group i.e. $G = \prod_{i \in \mathbb{N}} S_i$.
If G is finitely generated then

$$\operatorname{rk} S_i \rightarrow \infty \Leftrightarrow G \text{ is a profinite completion.}$$

Recall: $\operatorname{rk} S = n$ if $S = \operatorname{Alt}(n)$ and $\operatorname{rk} S = \operatorname{rk} L$ if S is a simple group of Lie type L.

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Theorem [Kassabov and Nikolov (2006)]

For any $a > 0$, there exists a group $G \in \mathcal{A}$ that has polynomial representation growth and such that $\alpha(G) = a$. Moreover, G is a profinite completion.

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Theorem [García-Rodríguez and Klopsch (2016)]

For any $a > 0$, there exists a group $G \in \mathcal{L}$ that has polynomial representation growth and such that $\alpha(G) = a$.

The groups constructed in the proof are not profinite completions.

New result

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Further interests

Study the spectrum of growth for d-generated semisimple profinite groups, where $d \in \mathbb{N}_{\geq 2}$.

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Study the interplay of groups of the class of semisimple profinite groups with polynomial representation growth and those that are positively finitely generated.

Thank you