Average number of zeros of characters of finite groups

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Overview

Characters

$\mathrm{Irr}(G)$-set of irreducible characters of G.

$\chi(1)$ - degree of a character. If $\chi(1) = 1$, then χ is called a linear character.

χ is invariant on conjugacy classes.

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Fundamental question

Every finite group has an associated character table.

How does the information of a character table of a particular group affect its structure?

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Average character degree

Definition. Let $T(G) = \Sigma_{\chi \in \mathrm{Irr}(G)} \chi(1)$, $k(G)$ the number of conjugacy classes. Then $k(G) = |\mathrm{Irr}(G)|$. Define the average character degree of G by

$$\mathrm{acd}(G) := \frac{T(G)}{|\mathrm{Irr}(G)|}.$$

Examples: $\mathrm{acd}(G) = 1$ when G is abelian, $\mathrm{acd}(A_5) = \frac{16}{5}$, $\mathrm{acd}(A_4) = \frac{3}{2}$.

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Average character degree

How does the average character degree of a finite group affect its structure?

Theorem.(Magaard and Tong-Viet, 2011) If $\mathrm{acd}(G) < 2$, then G is solvable.

Conjecture 1.(Magaard and Tong-Viet, 2011) If $\mathrm{acd}(G) \leqslant 3$, then G is solvable.

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Theorem.(Isaacs, Loukaki and Moretó, 2013)
Conjecture 1 holds.

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Conjecture 2.(Isaacs, Loukaki and Moretó, 2013)
If $\mathrm{acd}(G) < \frac{16}{5}$, then G is solvable.

Theorem.(Moretó and Nguyen, 2014)
Conjecture 2 holds.

The bounds are optimal.
$\mathrm{acd}(A_5) = \frac{16}{5}$, $\mathrm{acd}(A_4) = \frac{3}{2}$ and $\mathrm{acd}(S_3) = \frac{4}{3}$.

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Zeros of Characters

Let $\chi \in \mathrm{Irr}(G)$ and $g \in G$. If $\chi(g) = 0$, then we say χ vanishes on g.

Theorem. (Burnside, 1904) Let $\chi \in \mathrm{Irr}(G)$ be non-linear. Then there exists $g \in G$ such that $\chi(g) = 0$.

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Average number of zeros of characters

Definition. Let $\mathrm{nz}(G)$ denote the number of zeros in the character table of G. Define the average number of zeros of characters of G by

$$\mathrm{anz}(G) := \frac{\mathrm{nz}(G)}{|\mathrm{Irr}(G)|}.$$

Character Table for A_5

Example. Character Table for A_5

$\chi \setminus \mathcal{C}$	$1A$	$2A$	$3A$	$5A$	$5B$
χ_1	1	1	1	1	1
χ_2	3	-1	0	A	$*A$
χ_3	3	-1	0	$*A$	A
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

Where $A = \dfrac{(\sqrt{5}+1)}{2}$ and $^*A = \dfrac{(-\sqrt{5}+1)}{2}$.
From the table, $\mathrm{anz}(A_5) = 1$.

Character Table for S_5

Example. Character Table for S_5

$\chi \setminus \mathcal{C}$	$1A$	$2A$	$2B$	$3A$	$4A$	$5A$	$6A$
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	1	-1
χ_3	4	-2	0	1	0	-1	1
χ_4	4	2	0	1	0	-1	1
χ_5	5	-1	1	-1	1	0	-1
χ_6	5	1	1	-1	-1	0	1
χ_7	6	0	-2	0	0	1	0

$\mathrm{anz}(S_5) = \frac{10}{7}$

Average number of zeros of characters

How does the average number of zeros of characters of a finite group affect its structure?

Average number of zeros of characters

Theorem. (M, 2021)
If $\mathrm{anz}(G) < 1$, then G is solvable. The bound is optimal since $\mathrm{anz}(A_5) = 1$.

Proof.

- Reduced the problem to almost simple case.
- If N is a minimal non-abelian normal subgroup of G, then there exists a non-linear character $\chi \in \mathrm{Irr}(G)$ such that χ_N is irreducible and χ vanishes on two conjugacy classes of G.
- (M, 2020) Classification of almost simple groups with an irreducible character that vanishes on one conjugacy class.

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Theorem. (M, 2021) If $\mathrm{anz}(G) < \frac{1}{2}$, then G is supersolvable. The bound is optimal since $\mathrm{anz}(A_4) = \frac{1}{2}$.

Theorem. (M, 2021) G is abelian if and only if $\mathrm{anz}(G) < \frac{1}{3}$. The bound is optimal since $\mathrm{anz}(S_3) = \frac{1}{3}$.

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Theorem. (Moretó, 2022) Classified all non-abelian finite groups such that $\mathrm{anz}(G) < 1$.

Proof.

- Used a different approach to show solvablity of the groups.
- Used the classification in which every non-linear irreducible character vanishes on at most two conjugacy classes. (Chillag, 1999 and Moretó-Sangroniz, 2004).
- only non-solvable groups with property above are A_5 and $\mathrm{PSL}_2(7)$.

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Theorem. (Moretó, 2022) Conjecture 3 holds. In fact there are only four non-abelian groups of odd order such that $\mathrm{anz}(G) < \frac{16}{11}$.

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Question. (Moretó, 2022) Is it true that there exists a real valued function f such that for every solvable group G the Fitting height of G, $h(G) \leq f(\mathrm{anz}(G))$?

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