Generalized Foulkes Module

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Representation Theory of S_n

Let $\lambda \vdash n$. A *Young diagram* of λ is a 2 dimensional diagram with n boxes put together such that j^{th} row has λ_j boxes. If $\lambda = (2,2)$, its Young diagram is

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$$

We can fill the boxes of a Young diagram with elements from $\{1, .., n\}$. Such a box is called a *Young tableau*. As an example

$$t_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad t_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}.$$

Representation Theory of S_n

Let C_t and R_t be the set of column and row stabilizers of tableau t and

$$k_t = \sum_{\sigma \in C_t, \nu \in R_t} sgn(\sigma)\sigma\nu.$$

Then the polytabloid $e_t = k_t t$ generates a KS_n module, the Specht module S^μ.

Specht Module S^μ

Let $char(K) = 0$. Then the set $\{S^\mu \mid \mu \vdash n\}$ forms the complete set of irreducible modules of KS_n.

Foulkes Module

Foulkes Module

$$F^a_{(b)} = \mathrm{Inf}_{S_b}^{S_a \wr S_b} 1 \uparrow^{S_{ab}} . \tag{1}$$

In other words, let P^{a^b} be the set of partitions of $\{1, \ldots, ab\}$ into b sets of size a each. Then the Foulkes module is the permutation module of S_{ab} acting on P^{a^b}.

Foulkes Module

Thrall, 1942

- $$F^2_{(b)} = \bigoplus_{\lambda \vdash b} S^{2\lambda}, \tag{2}$$
- $$F^b_{(2)} = \bigoplus_{\substack{\lambda \vdash b \\ \lambda \text{ has 2 parts}}} S^{2\lambda} \tag{3}$$

Generalized Foulkes Module

Generalized Foulkes Module

$$F_\nu^a = \mathrm{Inf}_{S_b}^{S_a \wr S_b} S^\nu \uparrow^{S_{ab}} . \quad (4)$$

- Let t be a ν tableau and X be an (a^b) ordered partition of the set $\{1, .., ab\}$. Then t_X is the ν shaped diagram with X_l as the $(i,j)^{th}$ entry where l is the $(i,j)^{th}$ entry of t.
- A set of basis elements of $\mathrm{Inf}_{S_b}^{S_a \wr Sb} S^\nu$ is $\{e_{t_X} \mid t \text{ is a standard } \nu \text{ tableau}\}$.
- The generalized Foulkes module F_ν^a is generated by e_{t_X}.

Generalized Foulkes Module

As an example, let $\nu = (2,2)$. For

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad t_X = \begin{array}{|c|c|}\hline X_1 & X_2 \\ \hline X_3 & X_4 \\ \hline \end{array}.$$

Generalized Foulkes module

- Paget and Wildon in 2019 gave the description of minimal constituents of the generalized Foulkes module.
- de Boeck in 2015 gave a description of certain irreducible constituents of the Foulkes module and the twisted Foulkes module $F^a_{(1^b)}$.

Generalized Foulkes Module

To get more insight into F_ν^a we restrict it to $S_b \times S_{n-b}$. It is decomposed as a sum of natural submodules. One such module is V_ν^a. It is generated by $e_{t_{1,Y}}$, where Y is a $(a-1)^b$ ordered partition of $\{b+1, .., ab\}$ and

$$t_{1,Y} = \begin{array}{|c|c|} \hline (1, Y_1) & (2, Y_2) \\ \hline (3, Y_3) & (4, Y_4) \\ \hline \end{array}.$$

$$V_{(b)}^a = \bigoplus_{\lambda \vdash b} S^\lambda \otimes F_\lambda^{(a-1)} \tag{5}$$

$$V_{(1^b)}^a = \bigoplus_{\lambda \vdash b} S^\lambda \otimes F_{\lambda^\perp}^{(a-1)} \tag{6}$$

Generalized Foulkes Module

Let $H = S_b \times S_b$ and $G = S_b \times (S_{a-1} \wr S_b)$. Then

$$V_\nu^a \cong \mathrm{Inf}_H^G V_\nu^2 \uparrow^{S_b \times S_{n-b}} \tag{7}$$

Thus, the study of V_ν^2 gives an interesting insight on the general case.

Generalized Foulkes Module

Let $\lambda, \mu, \nu \vdash b$.

Kronecker Coefficient

The Kronecker coeffiecient $K_\nu^{\lambda,\mu}$ is the multiplicity of S^ν in the S_b module $S^\lambda \otimes S^\mu$.

Main Theorem

The multiplicity of $S^\lambda \otimes S^\mu$ in V_ν^2 is equal to the Kronecker coefficient $k_\nu^{\lambda,\mu}$.

Interestingly, it is a NP hard problem to decide whether $K_\nu^{\lambda,\mu}$ is zero.

Generalized Foulkes Module

Let $\lambda + (1)^b = (\lambda_1 + 1, \ldots, \lambda_b + 1, \lambda_{b+1}, \ldots, \lambda_k\}$.

If λ has more then b parts then the multiplicity of S^λ in F^a_ν is 0.

Corollary

The multiplicity of $S^{\lambda+(1)^b}$ in $F^{a+1}_{\nu^\perp}$ is the same as multiplicity of S^λ in F^a_ν.

A generalized form of this corollary has been proved by de Boeck, Paget and Wildon. Though, their technique is quite different.

Questions

- Determine the complexity of calculating the multiplicity of S^λ in F^a_ν?
- Find a way to generalize F^a_ν into F^λ_ν with $\lambda \vdash a$. Can we find an analog of main theorem in F^λ_ν?

Thank You