The Gyrogroups and the G-Graph of some Gyrogroups

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Contents

Gyrogroup

Gyrogroup was discovered by Abraham Ungar in his study of the parametrization of the Lorentz transformation group. A gyrogroup can be considered as a generalization of a group with a binary operation, where the associative property is replaced by the left gyroassociative and the left loop properties.

The first example of a gyrogroup is given by Ungar in 1988. He applied the Einstein's velocity addition to define a gyrogroup on unit sphere. It is the groupoid $(V_c, \oplus_E)$ with their composition law given by Einstein's addition. The Vectorial version of this addition in the c-ball of Euclidian space, $\mathbb{R}^3_c = \{X \in \mathbb{R}^3 : \|X\| < c\}$, and $\oplus$ is given by the equation:

$$X \oplus Y = \frac{X + Y}{1 + \frac{\langle X, Y\rangle}{c^2}} - \frac{1}{c^2}\frac{\gamma x}{1 + \gamma x}\frac{\langle X, X\rangle Y - \langle X, Y\rangle X}{1 + \frac{\langle X, Y\rangle}{c^2}}$$

Where $\gamma x = \frac{1}{1 - \frac{\langle X, X\rangle}{c^2}}$. For $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$, the usual inner product of these vectors are $\langle X, Y\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$.

Definition

A non-empty set G together with a binary operation $\oplus$ on G is called a Gyrogroup if it satisfies the following axioms:

G1) $\exists e \in G$ s.t. $e \oplus a = a, \forall a \in G$. $(e = 0)$

G2) $\forall a \in G$, $\exists$ an element $b \in G$, s.t. $a \oplus b = e$. $(b = \ominus a)$

G3) *For each* $a, b \in G$, *there is an automorphism* $gyr[a, b] \in Aut(G, \oplus)$ *s.t.*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b](c), \forall c \in G$$

G4) *For all* $a, b \in G$, $gyr[a \oplus b, b] = gyr[a, b]$.

$\gamma : G \times G \longrightarrow Aut(G)$
$(a, b) \leftrightsquigarrow gyr[a, b](c) = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)).$

The gyrogroup G is called gyrocommutative if and only if for all $a, b \in G, a \oplus b = gyr[a, b](b \oplus a)$.

Subgyrogroup:
Let $\emptyset \neq H \subseteq G$, if H is a gyrogroup under the operation inherited from G and $\forall a, b \in H$, $gyr[a, b](H) = H$ then $H \leq_* G$.

L-Subgyrogroup:
If $\forall a \in G, h \in H$, $gyr[a, h](H) = H$ then $H \leq_L G$ and $\{g \oplus H | g \in G\}$ partition G and $|H| \mid |G|$, $|G| = |G : H||H|$.

Symmetric:

A subset S of a gyrogroup $(G, \oplus)$ is said to be symmetric if $\forall s \in S$, $\ominus s \in S$.

left generating set:

The left generating set $(S\rangle$ is defined as:

$$(S\rangle = \{s_n \oplus (\cdots \oplus (s_3 \oplus (s_2 \oplus s_1))\cdots)|s_1, s_2, \cdots, s_n \in S\}$$

if $(S\rangle = G$, then G is Left-generated by S.

G-graph [Bretto 2005]:

Let G be a finite group with the non-empty subset $S = \{s_1, s_2, \cdots, s_k\}$, $k \geq 1$. For all $s \in S$, the right cosets $\langle s \rangle x$, $x \in G$ partition G. Let $g_s : G \to G$, $g_s(x) = sx$ of S_G and for $x \in G$ consider the following disjoint cycles that are used in the disjoint decomposition of g_s: $(s)x = (x, sx, s^2x, \cdots, s^{o(s)-1}x)$

- $V(\Phi(G, S)) = \sqcup_{s \in S} V_s$ with $V_s = \{(s)x, x \in T_s\}$.
- $(s)x - (t)y \in E(\Phi(G, S))$, when $|\langle s \rangle x \cap \langle t \rangle y| = d \geq 1$, and $(s)x - (t)y$ is a d-edge.

G-gyrographs of G_8

Example

Let $G = G_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with $A = (1\ 6)(2\ 5)$ be a gyrogroup.

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	5	6	0	7	1	2	4
4	4	2	1	7	0	6	5	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

gyro	0	1	2	3	4	5	6	7
0	I	I	I	I	I	I	I	I
1	I	I	A	A	A	A	I	I
2	I	A	I	A	A	I	A	I
3	I	A	A	I	I	A	A	I
4	I	A	A	I	I	A	A	I
5	I	A	I	A	A	I	A	I
6	I	I	A	A	A	A	I	I
7	I	I	I	I	I	I	I	I

$V(\Phi(G_8, S = \{1, 2\})) = V_1 \cup V_2$
$= \{(1) \oplus x = (x, 1 \oplus x) | x \in G_8\} \cup \{(2) \oplus y = (y, 2 \oplus y) | y \in G_8\}$
$= \{(0, 1), (2, 3), (4, 5), (6, 7)\} \cup \{(0, 2), (1, 3), (4, 6), (5, 7)\}$.
$\Phi(G_8, S)$ is not connected and contains two cycles of C_4.

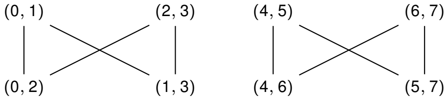

Figure 1.

Note that $G_8 = \langle S \rangle$. $3 = 1 \oplus 2$, $4 = 1 \oplus ((1 \oplus 2) \oplus 1)$, $5 = (1 \oplus 2) \oplus 1$, $6 = (1 \oplus 2) \oplus 2$ and $7 = 2 \oplus ((1 \oplus 2) \oplus 1)$.
But $(S\rangle \neq G_8$.

Example

$V(\Phi(G_8, S = \{1,3\})) = V_1 \cup V_3$
$= \{(1) \oplus x = (x, 1 \oplus x) | x \in G_8\} \cup \{(3) \oplus y = (y, 3 \oplus y) | y \in G_8\}$
$= \{(0,1), (2,3), (4,5), (6,7)\} \cup \{(0,3), (1,5), (2,6), (4,7)\}$.
$\Phi(G_8, S)$ is a bipartite connected G-gyrograph isomorphic to cycle C_8. Here $G_8 = (S\rangle$.

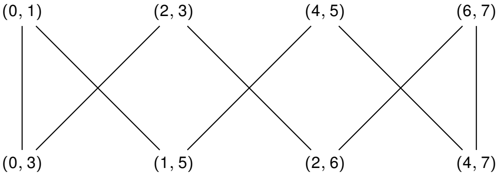

Figure 2.

Example

Now consider the gyrogroup G_8 with the left generating set $S = \{1, 2, 3\}$:
$V(\Phi(G_8, S)) = V_1 \cup V_2 \cup V_3 =$
$\{(0, 1), (2, 3), (4, 5), (6, 7)\} \cup \{(0, 2), (1, 3), (4, 6), (5, 7)\} \cup \{(0, 3), (1, 5), (2, 6), (4, 7)\}$
Each vertex has two numbers in common with the vertices in other levels. Then $\Phi(G_8, S) \cong K_{4,4,4}$ is a 3-partite, 4-regular connected G-gyrograph and $G_8 = (S\rangle$.

The G-gyrograph $\Phi(G_8, S)$ is connected if and only if $G_8 = (S\rangle \subset \langle S\rangle$.

There are some papers about the Cayley graph of gyrogroups. In the paper by Bussaban, for example $Cay(G_8, \{1, 2\})$ is not connected, but $Cay(G_8, \{1, 3\}) \cong C_8$ and $Cay(G_8, \{1, 2, 3\})$ are connected. Also it's proved that

Theorem

For a gyrogroup G with a nonempty symmetric subset S, $Cay(G, S)$ is connected if and only if $\langle S\rangle = G$.

L. Bussaban, A. Kaewkhao and S. Suantai,
Cayley graphs of gyrogroups,
Quasigroups and Related Systems, 27 (2019) 25-32.

The 2-gyrogroup $G(n), n \geq 3$

A class of 2-gyrogroups constructed by Ashrafi, et.al., whose every proper subgyrogroup is either a cyclic or a dihedral group. For an integer $n \geq 3$ let $G(n) = P(n) \cup H(n)$, where

$$P(n) = \{0, 1, 2, \ldots, 2^{n-1} - 1\},$$
$$H(n) = \{2^{n-1}, 2^{n-1} + 1, \cdots, 2^n - 1\}.$$

A. R. Ashrafi, S. Mahdavi, M. A. Salahshour and A. A. Ungar,
Construction of 2-Gyrogroups in Which Every Proper Subgyrogroup Is Either a Cyclic or a Dihedral Group,
MDPI (2021)

Let $m = 2^{n-1}$, for all $i, j \in G(n)$,

$$i \oplus j = \begin{cases} t & (i,j) \in P(n) \times P(n) \\ t + m & (i,j) \in P(n) \times H(n) \\ s + m & (i,j) \in H(n) \times P(n) \\ k & (i,j) \in H(n) \times H(n) \end{cases}$$

where $t, s, k \in P(n)$,

$$\begin{cases} t \equiv i + j & (modm) \\ s \equiv i + (\frac{m}{2} - 1)j & (modm) \\ k \equiv (\frac{m}{2} + 1)i + (\frac{m}{2} - 1)j & (modm) \end{cases}$$

$$A : G(n) \longrightarrow G(n),$$
$$x \rightsquigarrow \begin{cases} x & x \in P(n) \\ r+m & x \in H(n) \end{cases} \quad r \in P(n), r \equiv x + \frac{m}{2}, mod\ m.$$

$$gyr : G(n) \times G(n) \longrightarrow Aut(G(n), \oplus)$$
$$gyr(a,b) = gyr[a,b] = \begin{cases} A & (a,b) \in M \\ I & O.W. \end{cases}$$

The subgyrogroups of $G(n)$ are

1) $G(n)$,
2) $B \leq P(n)$,
3) $\{0, j\}, j \in H(n)$
4) $\exists r, s \in \mathbb{Z}$, $1 \leq r \leq n-2$, $0 \leq s \leq 2^r - 1$, $\langle 2^r \rangle \cup \langle 2^r \rangle + (m+s)$.

Also
$\langle 1 \rangle \cong \mathbb{Z}_m$, $\langle 2 \rangle \cong \mathbb{Z}_{\frac{m}{2}}, \cdots, \langle 2^{n-2} \rangle \cong \mathbb{Z}_4$.
$\langle 2, m \rangle \cong \langle 2, m+1 \rangle \cong D_m$,
$\langle 4, m \rangle \cong \langle 4, m+1 \rangle \cong \langle 4, m+2 \rangle \cong \langle 4, m+3 \rangle \cong D_{\frac{m}{2}}, \cdots$
$\langle m \rangle \cong \langle m+1 \rangle \cong \cdots \cong \langle 2m-1 \rangle \cong \mathbb{Z}_2$.

Example

The gyroaddition table of $(G(3) = \{0,1,2,3,4,5,6,7\}, \oplus)$ with $A = (4\ 6)(5\ 7)$:

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	5	6	7	0	1	2	3
5	5	6	7	4	3	0	1	2
6	6	7	4	5	2	3	0	1
7	7	4	5	6	1	2	3	0

gyr	0	1	2	3	4	5	6	7
0	I	I	I	I	I	I	I	I
1	I	I	I	I	A	A	A	A
2	I	I	I	I	I	I	I	I
3	I	I	I	I	A	A	A	A
4	I	A	I	A	I	A	I	A
5	I	A	I	A	A	I	A	I
6	I	A	I	A	I	A	I	A
7	I	A	I	A	A	I	A	I

G-gyrograph of $(G(n), P(n))$

Example

$\Phi(G(3), P(3))$:
Since $|P(3)| = 4$, then the graph is 4-partite and $\langle 1 \rangle \cong \mathbb{Z}_m = \mathbb{Z}_4$, then $\langle 1 \rangle \leq_L G(3)$ that means $|V_1| = |G(3) : \langle 1 \rangle| = 2$.

$$V_1 = \{(1) \oplus x = (x, 1 \oplus x, 2 \oplus x, 3 \oplus x)\} = \{(1, 2, 3, 0), (4, 5, 6, 7)\}.$$

Also $|V_2| = 4$ because $\langle 2 \rangle \cong \mathbb{Z}_{\frac{m}{2}} \cong \mathbb{Z}_2 = \{0, 1\}$ is an L-subgyrogroup of $G(3)$, then

$$V_2 = \{(2) \oplus x = (x, 2 \oplus x)\} = \{(0, 2), (1, 3), (4, 6), (5, 7)\}.$$

Since $\langle 3 \rangle$ is an L-subgyrogroup of index 4 in $G(3)$, then $|V(3)| = \frac{8}{4} = 2$.

$$V_3 = \{(3) \oplus x = (x, 3 \oplus x, 3 \oplus 3 \oplus x, 3 \oplus 3 \oplus 3 \oplus x)\}$$

$$= \{(x, 3 \oplus x, 2 \oplus x, 1 \oplus x)\} = \{(0, 3, 2, 1), (4, 7, 6, 5)\}.$$

Finally $V_0 = \{(0), (1), (2), (3), (4), (5), (6), (7)\}$.
We see that $\Phi(G(3), P(3))$ is a connected 4-partite graph with $deg(v_1) = 7$, $deg(v_2) = 4$, $deg(v_3) = 7$ and $deg(v_0) = 3$.

G-gyrograph of $(G(n), H(n))$

Example

$\Phi(G(3), H(3))$ is a $m = 4$-partite connected graph with the vertex set: $V = V_4 \cup V_5 \cup V_6 \cup V_7$. Since for every $j \in H(3)$, $\langle j \rangle \cong \mathbb{Z}_2$ is the L-subgyrogroup of $G(3)$, then
$|V_4| = |V_5| = |V_6| = |V_7| = |G(3) : V_j| = 4$.
$V = \{(4) \oplus x\} \cup \{(5) \oplus y\} \cup \{(6) \oplus z\} \cup \{(7) \oplus t\}$
$= \{(0,4),(1,5),(2,6),(3,7)\} \cup \{(0,5),(1,6),(2,7),(4,3)\}$
$\cup\{(0,6),(1,7),(2,4),(3,5)\} \cup \{(0,7),(1,4),(2,5),(3,6)\}$
This graph is $2(4-1) = 6$-regular.

$\Phi(G(n), H(n))$

Theorem

The G-gyrograph $\Phi(G(n), H(n))$ is connected and Hamiltonian.

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Thanks for your attention