On algebraic groups, their Lie algebras, and nilpotent pieces

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- $\mathrm{Sp}_4(\mathbf{k}) := \{A \in \mathrm{GL}_4(\mathbf{k}) \mid A^{tr} J_4 A = J_4\}$, where $J_4 := \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & -1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{pmatrix}$.

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> - Borel subgroups are closed, connected, solvable subgroups of G.

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- $B := \mathrm{Sp}_4(\mathbf{k}) \cap \left\{ \left(\begin{smallmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{smallmatrix} \right) \right\}$ is a Borel subgroup.

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$$\alpha : T \longrightarrow \mathbf{k}^\times, \quad \begin{pmatrix} t_1 & . & . & . \\ . & t_2 & . & . \\ . & . & t_2^{-1} & . \\ . & . & . & t_1^{-1} \end{pmatrix} \longmapsto t_1 t_2^{-1},$$

$$\beta : T \longrightarrow \mathbf{k}^\times, \quad \begin{pmatrix} t_1 & . & . & . \\ . & t_2 & . & . \\ . & . & t_2^{-1} & . \\ . & . & . & t_1^{-1} \end{pmatrix} \longmapsto t_2^2.$$

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$\mathrm{Lie}(\mathrm{Sp}_4(\mathbf{k})) = \{A \in M_4(\mathbf{k}) \mid J_4 A + A^{tr} J_4 = 0\}$

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Example (Adjoint map for $\mathrm{GL}_n(\mathbf{k})$)

We have the map

$$\mathrm{Ad} : \mathrm{GL}_n(\mathbf{k}) \mapsto \mathrm{GL}(\mathfrak{gl}_n(\mathbf{k})),\ A \mapsto (B \mapsto ABA^{-1}).$$

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- $\mathfrak{g}_\alpha$ – **root subspace**, a one-dimensional subspace (the *weight space*) for each root $\alpha \in \Phi$; $\mathfrak{g} = \mathrm{Lie}(T) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$,

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$$\mathfrak{g}_\alpha = \left\{ \begin{pmatrix} \cdot & a & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -a \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| \, a \in \mathbf{k} \right\}, \qquad \mathfrak{g}_\beta = \left\{ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| \, a \in \mathbf{k} \right\},$$

$$\mathfrak{g}_{\alpha+\beta} = \left\{ \begin{pmatrix} \cdot & \cdot & a & \cdot \\ \cdot & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| \, a \in \mathbf{k} \right\}, \qquad \mathfrak{g}_{2\alpha+\beta} = \left\{ \begin{pmatrix} \cdot & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| \, a \in \mathbf{k} \right\},$$

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- $e_\alpha \in \mathfrak{g}$ such that $\langle e_\alpha \rangle = \mathfrak{g}_\alpha$,
- $U_\alpha \subseteq G$ – the **root subgroups**; $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$, $G = \langle T, U_\alpha \mid \alpha \in \Phi \rangle$

Bad Characteristic

Bad Characteristic

For certain primes p, we call p a **bad prime**, depending on the type of the root system:

Root system type	prime
$B_n (n \geqslant 3),\ C_n (n \geqslant 2),\ D_n (n \geqslant 4)$	2
$G_2,\ F_4,\ E_6,\ E_7$	2,3
E_8	2,3,5

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Maybe there is a way to get a picture that is very similar to the orbits in good characteristic and holds in both good and bad characteristic!

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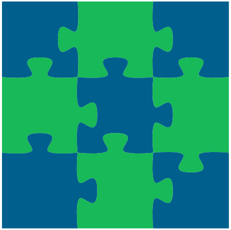

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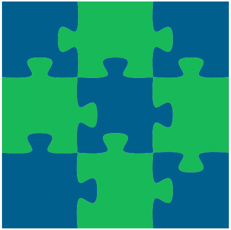

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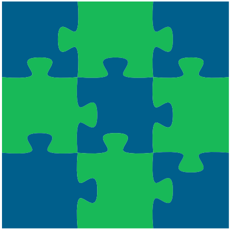

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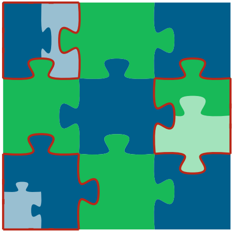

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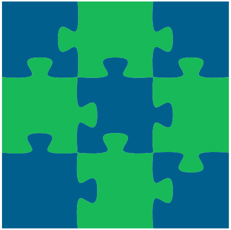

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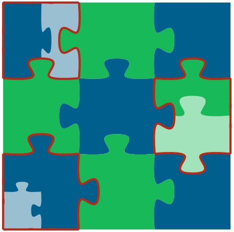

In bad characteristic

Nilpotent orbits can be linked to maps $\Delta := \{\delta : \Phi \to \mathbb{Z}\}$. In good characteristic a map $\delta \in \Delta$ uniquely determines a nilpotent orbit.

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Conjecture

$\mathcal{N}_{\mathfrak{g}}^{\delta} = \bigcup_{\substack{\delta' \in \Delta \\ \delta' = \delta}} \mathcal{O}_{\delta'}$

Nilpotent Pieces

Theorem (V.)

Let G be a simple algebraic group of type G_2, F_4, and E_6 and $\mathcal{O}_\delta$ denote the nilpotents orbit with map $\delta \in \Delta$. Then the nilpotent piece with respect to map δ and the group G is given by

$$\mathcal{N}_{\mathfrak{g}}^{\delta} = \bigcup_{\substack{\delta' \in \Delta \\ \delta' = \delta}} \mathcal{O}_{\delta'},$$

In particular, the nilpotent pieces $\mathcal{N}_{\mathfrak{g}}^{\delta}$ form a partition of $\mathcal{N}_{\mathfrak{g}}$ and are in bijection with the nilpotent orbits in good characteristic.