



# Useful things to know about accelerators - part I

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**Science and  
Technology  
Facilities Council**



# Accelerators - A Window on Nature

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- Particle accelerators provide the source for most high energy physics experiments
  - Provide high luminosity, high energy beams for colliders
  - Provide high brightness beams for secondary particle production
  - Also key technology for life sciences, engineering, chemistry
- How do they work?
  - How can we get to high energy?
  - How can we keep the beam in the accelerator?
  - How can we get to high luminosity?
- What are the main HEP facilities in the world today?
- What might HEP facilities look like in the future?



# Accelerator Components

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- Most accelerators share similar components
- Main components of an accelerator
  - Bending - dipoles
  - Focussing - quadrupoles
  - Acceleration - RF cavities
- Also
  - Vacuum
  - Diagnostics
  - Targets for secondary particle production
- First Lecture: Derive basic theory of accelerator physics
- Second Lecture: Discuss accelerator equipment and techniques

# Lorentz force law

- Fundamental equation for particles moving through fields

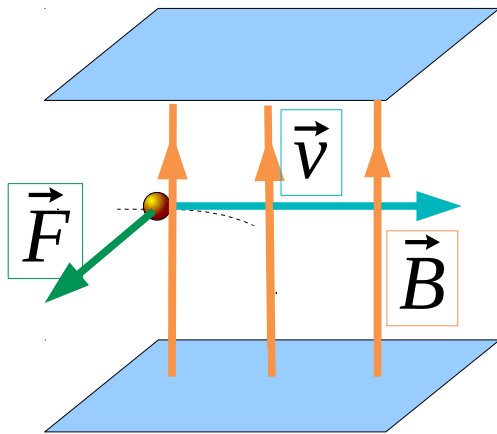
The diagram shows the Lorentz force law equation  $\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$  with five colored boxes above it: 'Force' (green), 'Charge' (cyan), 'Velocity' (red), 'Magnetic Field' (purple), and 'Electric Field' (orange). Arrows of corresponding colors point from each box to its respective variable in the equation. The 'Force' box points to  $\vec{F}$ , 'Charge' to  $q$ , 'Velocity' to  $\vec{v}$ , 'Magnetic Field' to  $\vec{B}$ , and 'Electric Field' to  $\vec{E}$ . The equation is labeled '(eq. 1)' to the right.

$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E} \quad (\text{eq. 1})$$

- **Magnetic force** is perpendicular to velocity
  - Magnetic field conserves energy
- **Electric force** is weaker by factor velocity
  - Magnets are better for bending and focussing

# Magnetic Rigidity and Bending

- Simplest magnet - “dipole”
  - Uniform magnetic field perpendicular to beam direction



Lorentz force (eq. 1) + centripetal motion:

$$qvB = \frac{pv}{\rho}$$

Radius

Rearranging:

$$B\rho = \frac{p}{q}$$

Magnetic Rigidity

- Constant force  $\rightarrow$  constant curvature  $\rightarrow$  circular motion
- Magnetic rigidity parameterises momentum
- Charge-to-mass ratio important when accelerating multiple particle species

# Worked example - LHC

- If we wanted to accelerate, say, 7 TeV particles, what bending radius is required?
- Maximum dipole field around 8.3 T

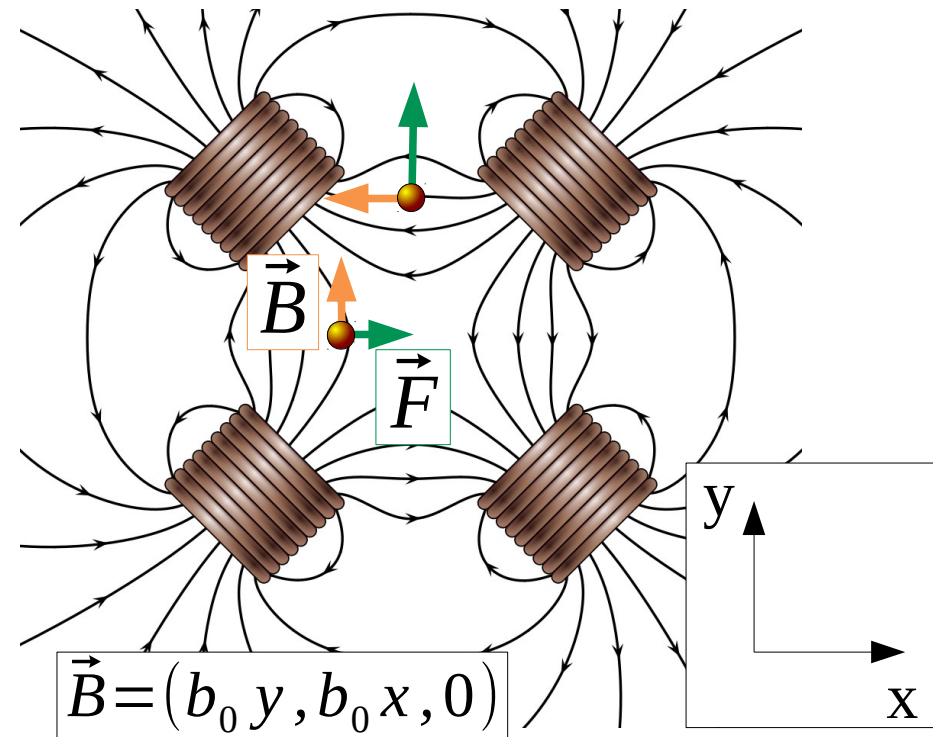
$$B\rho = \frac{p}{q}$$

$$\rho = \frac{p}{qB} = \frac{7}{0.3 \times 8.3} = 2.8 \text{ km}$$

- Nb: LHC radius  $\sim 4.1$  km
  - Need space for detectors, etc

# Quadrupole magnets

- If we only had bending magnets, particles would soon be lost from the accelerator
- Need to keep the particles in the accelerator using focussing elements
  - Usually use quadrupoles
- Field stronger away from beam centre
  - Like a spring or pendulum
  - Simple harmonic motion
- Overall focussing by alternating the gradient



# Quadrupole field - horizontal (1)

- For a particle moving near to the z-axis

$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$$

$$\vec{B} = (b_0 y, b_0 x, 0)$$

- Considering only  $p_x$  for now

$$\frac{dp_x}{dt} = q \frac{dz}{dt} B_y$$

- Use the chain rule

$$\frac{dp_x}{dt} = \frac{dp_x}{dz} \frac{dz}{dt}$$

- Combining these equations:

$$\frac{dp_x}{dz} = q b_0 x$$



# Quadrupole field - horizontal (2)

$$\frac{dp_x}{dz} = q b_0 x \quad \text{😊}$$

- Definition of x-component of momentum

$$p_x = m \gamma v_x = m \gamma \frac{dz}{dt} \frac{dx}{dz} = p_z \frac{dx}{dz}$$

- Substitute this definition into 😊 gives

$$p_z \frac{d^2 x}{dz^2} = q b_0 x$$

- Rearrange and wrap up constant terms in focussing strength  $k$

$$\frac{d^2 x}{dz^2} - k x = 0$$

# Quadrupole field - vertical

- Lorentz force law with quadrupole field definition

$$\frac{dp_y}{dt} = -q b_0 v_z y$$

- Use chain rule and eliminate  $v_z$

$$p_z \frac{d^2 y}{dz^2} = -q b_0 y$$

- Rearrange and wrap up constant terms in defocussing strength  $k$

$$\frac{d^2 y}{dz^2} + k y = 0$$

# Solutions

- Motion is governed by

$$\frac{d^2 x}{dz^2} - k x = 0 \qquad \frac{d^2 y}{dz^2} + k y = 0$$

- This is simple harmonic motion - solutions are of form

$$x = x_0 \cos(\sqrt{k} z) + \frac{dx_0}{dz} \frac{1}{\sqrt{k}} \sin(\sqrt{k} z)$$

- Taking derivative

$$\frac{dx}{dz} = -x_0 \sqrt{k} \sin(\sqrt{k} z) + \frac{dx_0}{dz} \cos(\sqrt{k} z)$$

For y

$$y = y_0 \cosh(\sqrt{k} z) + \frac{dy_0}{dz} \frac{1}{\sqrt{k}} \sinh(\sqrt{k} z)$$

$$\frac{dy}{dz} = y_0 \sqrt{k} \sinh(\sqrt{k} z) + \frac{dy_0}{dz} \cosh(\sqrt{k} z)$$

# Transfer Matrix

- Just thinking about  $x$ , the particles move according to

$$x = x_0 \cos(\sqrt{k} z) + \frac{dx_0}{dz} \sin(\sqrt{k} z)$$

$$\frac{dx}{dz} = -x_0 \sqrt{k} \sin(\sqrt{k} z) + \frac{dx_0}{dz} \sqrt{k} \cos(\sqrt{k} z)$$

- We can rewrite this as a matrix

$$\begin{pmatrix} x \\ \frac{dx}{dz} \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k} z) & \frac{1}{\sqrt{k}} \sin(\sqrt{k} z) \\ -\sqrt{k} \sin(\sqrt{k} z) & \cos(\sqrt{k} z) \end{pmatrix} \begin{pmatrix} x_0 \\ \frac{dx_0}{dz} \end{pmatrix}$$

- This matrix is known as the quadrupole's **transfer matrix**

$$\underline{u}_1 = \mathbf{M}_{01} \underline{u}_0$$



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# Questions



# Questions

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- Exercise – what is the transfer matrix for a drift space, that is a region with no fields at all?
  - What is the force acting on the particle?
  - What is  $x(z)$  in terms of  $dx_0/dz$  and  $x_0$
  - What is  $dx/dz$  in terms of  $dx_0/dz$
  - Now write that as a matrix

# Questions

- Exercise – what is the transfer matrix for a drift space?

- What is the force acting on the particle?

- No force

- What is  $x(z)$  in terms of  $dx_0/dz$  and  $x_0$

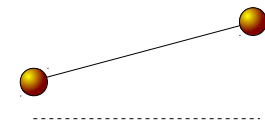
$$x = x_0 + \frac{dx_0}{dz} z$$

- What is  $dx/dz$  in terms of  $dx_0/dz$

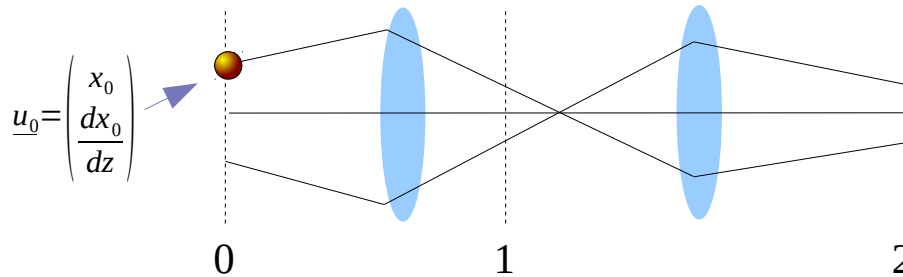
$$\frac{dx}{dz} = \frac{dx_0}{dz}$$

- Now write that as a matrix

$$\begin{pmatrix} x \\ \frac{dx}{dz} \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ \frac{dx_0}{dz} \end{pmatrix}$$



# Transfer Lines



- Transfer matrix defines transport through a region
- Transfer matrices can be combined by multiplication
- Say we have transfer matrices like:

$$\underline{u}_1 = \mathbf{M}_{01} \underline{u}_0$$

$$\underline{u}_2 = \mathbf{M}_{12} \underline{u}_1$$

- Then

$$\underline{u}_2 = \mathbf{M}_{12} \mathbf{M}_{01} \underline{u}_0$$

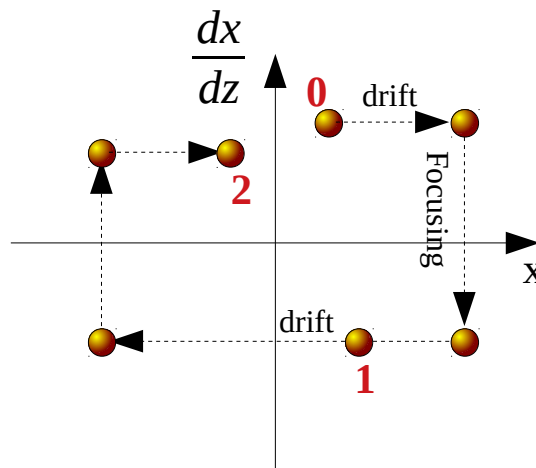
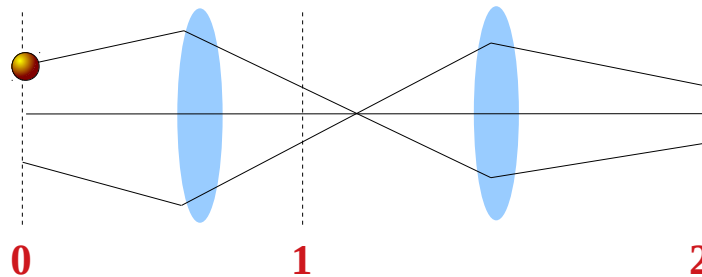
- i.e. we can define a combined transfer matrix like

$$\mathbf{M}_{02} = \mathbf{M}_{12} \mathbf{M}_{01}$$

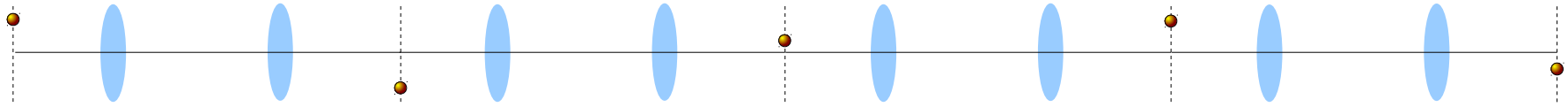


# Phase space

- Another instructive way to look at beam optics is by considering the phase space



# Periodic Lattices



- Following  $n$  identical cells or turns in a ring with one-turn matrix  $\mathbf{M}$

$$\underline{u}_n = \mathbf{M}^n \underline{u}_0$$

- Rewrite

$$\mathbf{M} = \mathbf{I} \cos \mu + \mathbf{J} \sin \mu$$

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \quad \text{with} \quad \gamma\beta - \alpha^2 = 1 \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

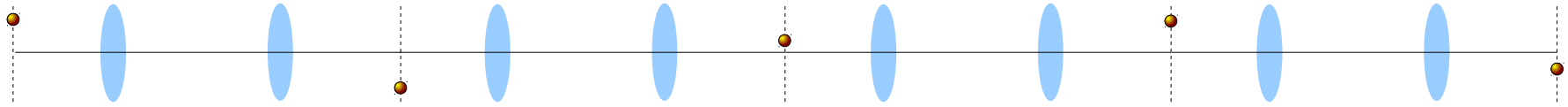
- So

$$\mathbf{J}^2 = -\mathbf{I}$$

- And

$$\mathbf{M}^n = \mathbf{I} \cos(n\mu) + \mathbf{J} \sin(n\mu)$$

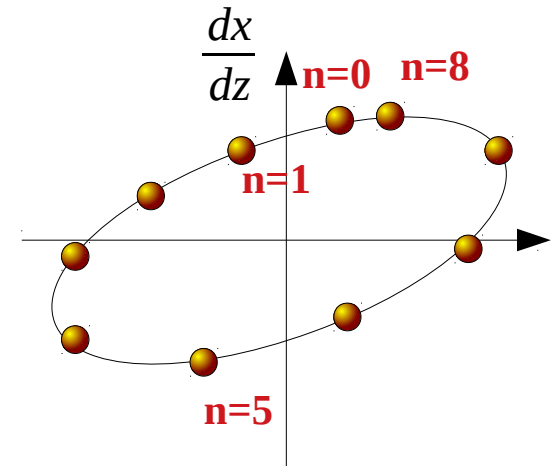
# Periodic Lattices



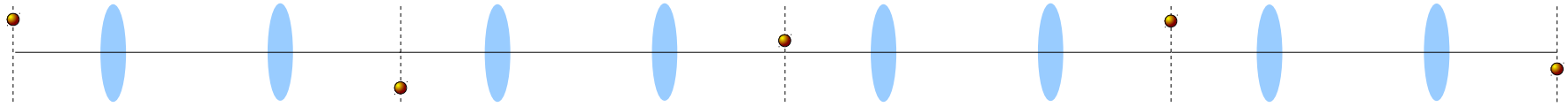
- What does this mean?

$$\mathbf{M}^n = \mathbf{I} \cos(n\mu) + \mathbf{J} \sin(n\mu)$$

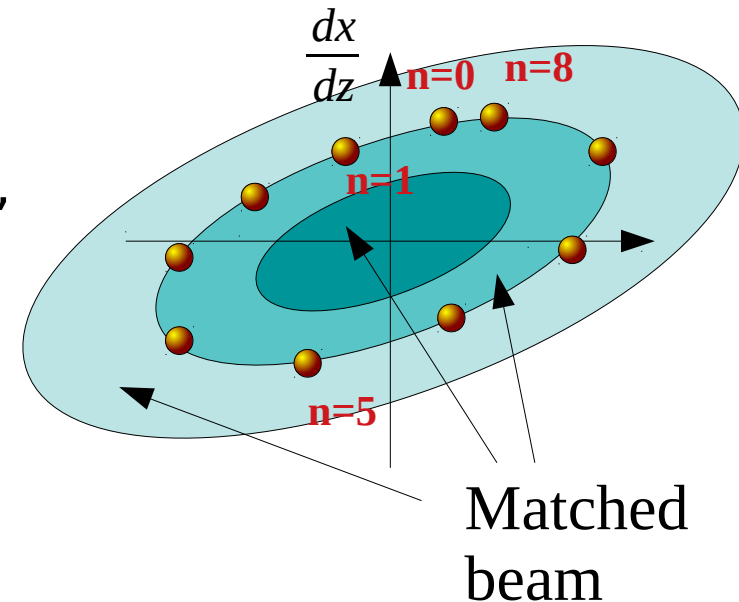
- Particles move around an ellipse in phase space if  $\text{Trace}(\mathbf{M}) < 2$
- $\mu$  is the “phase advance”
- $\alpha$ ,  $\beta$  and  $\gamma$  are “Twiss parameters”
  - Tell us the alignment of the ellipse
- Each particle sits on ellipse area  $\varepsilon$  - the particle’s amplitude



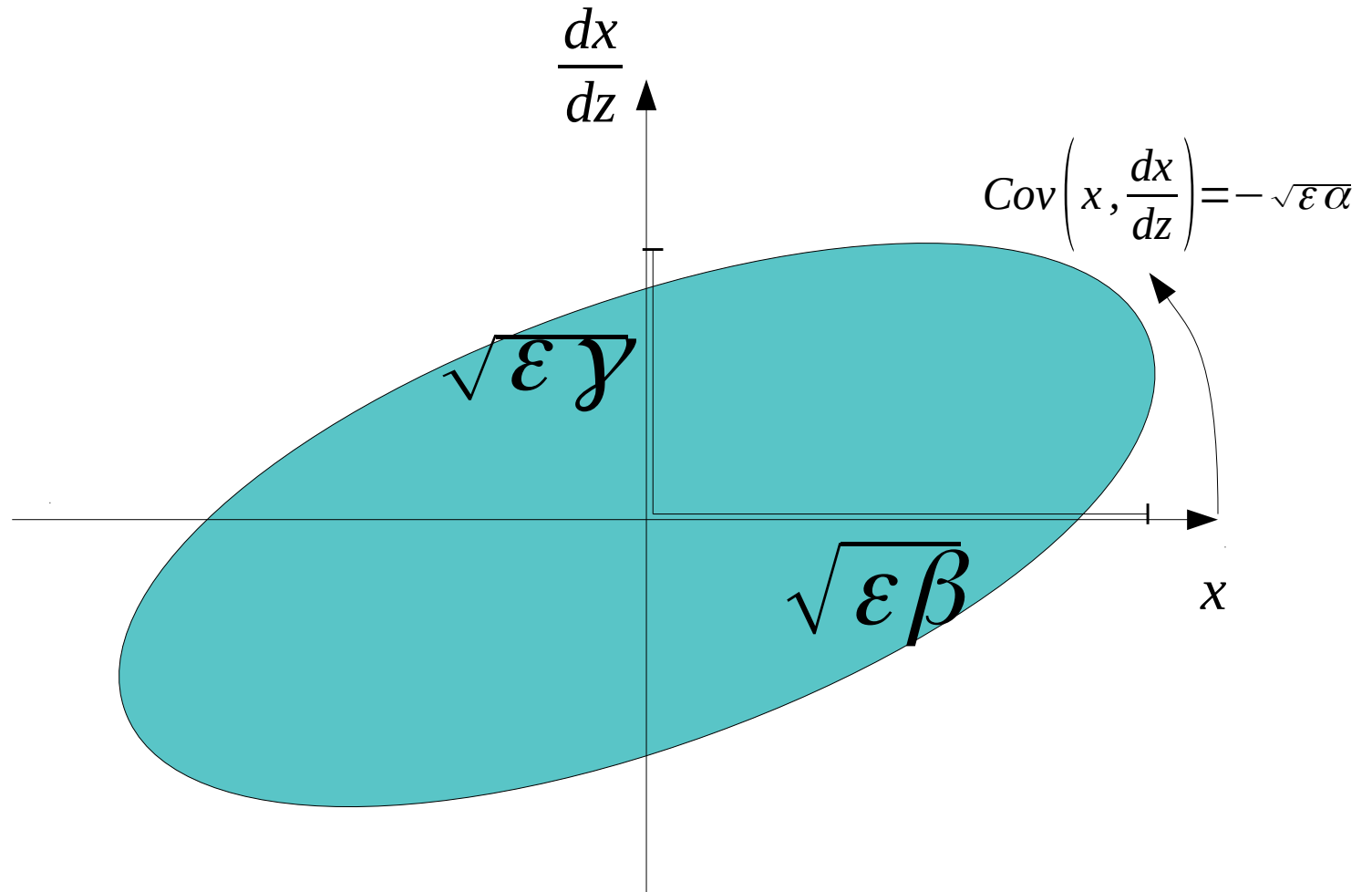
# Periodic Lattices and beams



- Beam is composed of many particles
  - Particles occupy a region in phase space
- “Emittance” is area occupied by the entire beam
- Sometimes classify “RMS emittance”
  - Area occupied by ellipse 1 RMS distance from beam centre
- Low emittance is crucial for
  - High luminosity
  - Low losses



# Beam ellipse





# Emittance Growth

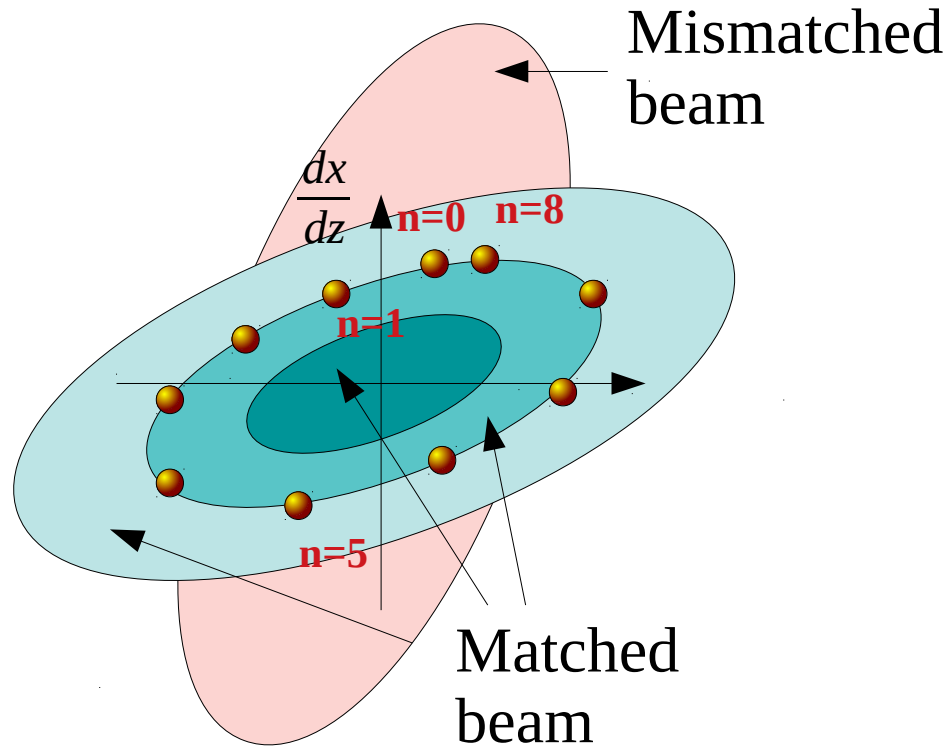
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- Ideally emittance is conserved, but this is not always the case
- Long list of effects that can cause emittance growth
  - Beam mismatch
  - Scattering off residual gas
  - Scattering off particles in the same beam
  - Scattering off particles in other beams (e.g. in collider)
  - Space charge
  - Resonances:

$$\nu_x = \frac{\mu_x}{2\pi}$$

$$j \nu_x + k \nu_y = N$$

# Mismatch





# Emittance Reduction (Cooling)

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- Several techniques to reduce emittance
  - Synchrotron radiation cooling
  - Stochastic cooling
  - Laser cooling
  - Electron cooling
  - Ionisation cooling
- Fundamental principle is to remove “heat” from the beam using a neighbouring heat sink
  - Comoving electron beam → electron cooling
  - Comoving laser → laser cooling
  - Emission of synchrotron radiation
    - Photon emission caused by (principally) electrons bending in magnetic field





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# Questions



# Questions

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- What is behaviour of particles in phase space if
  - $\text{Trace}(M) < 2$
  - $\text{Trace}(M) = 2$
  - $\text{Trace}(M) > 2$



# Questions

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- What is behaviour of particles in phase space if
  - $\text{Trace}(M) < 2$ 
    - Motion is an ellipse
  - $\text{Trace}(M) = 2$ 
    - $x \rightarrow +/- x$
  - $\text{Trace}(M) > 2$ 
    - Motion is a hyperbola

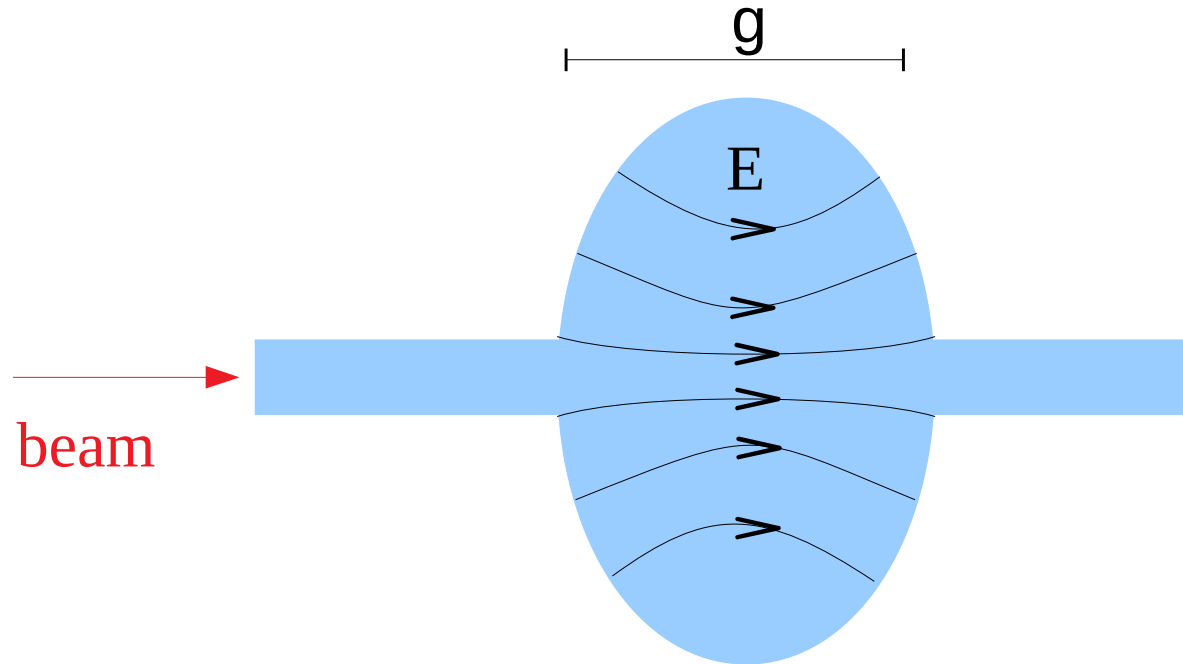


# Longitudinal Dynamics

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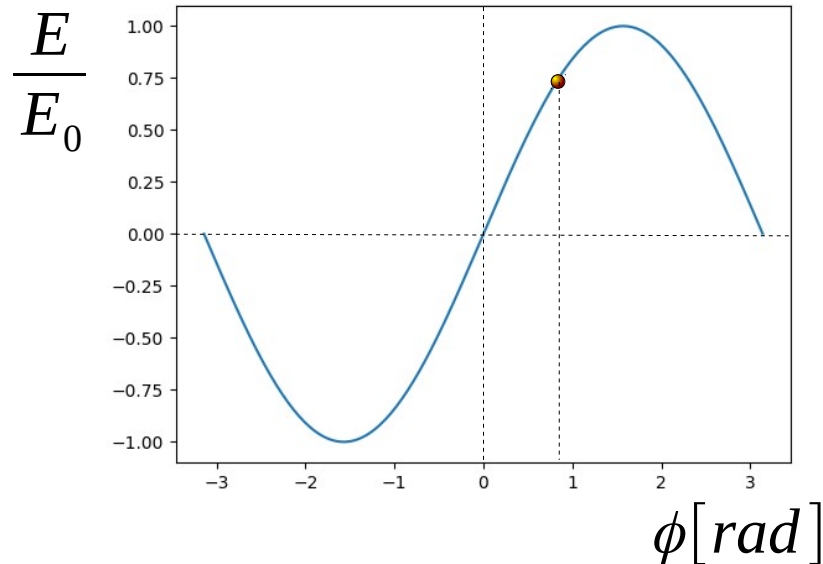
- So much for transverse motion (i.e.  $x$  and  $y$  planes)
- What about energy and acceleration?
- Electrostatic acceleration limited by breakdown potential
  - Change in energy is given by voltage differential
  - High voltage differentials cause breakdown (sparks)
  - Practically limits electrostatic acceleration to few MeV
- To accelerate beyond MeV require oscillating electric field
- RF Cavities

# RF cavity field



- RF cavity holds a resonating EM wave
- Recall Lorentz force law
$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$$
- Force is in direction of motion - energy changes!

# RF cavity field



- In RF cavity

$$\vec{E} = E_0 \sin(\omega t + \phi)$$

- Energy change of synchronous particle crossing at  $\phi_s$

$$\delta W = qTg E_0 \sin(\phi_s)$$

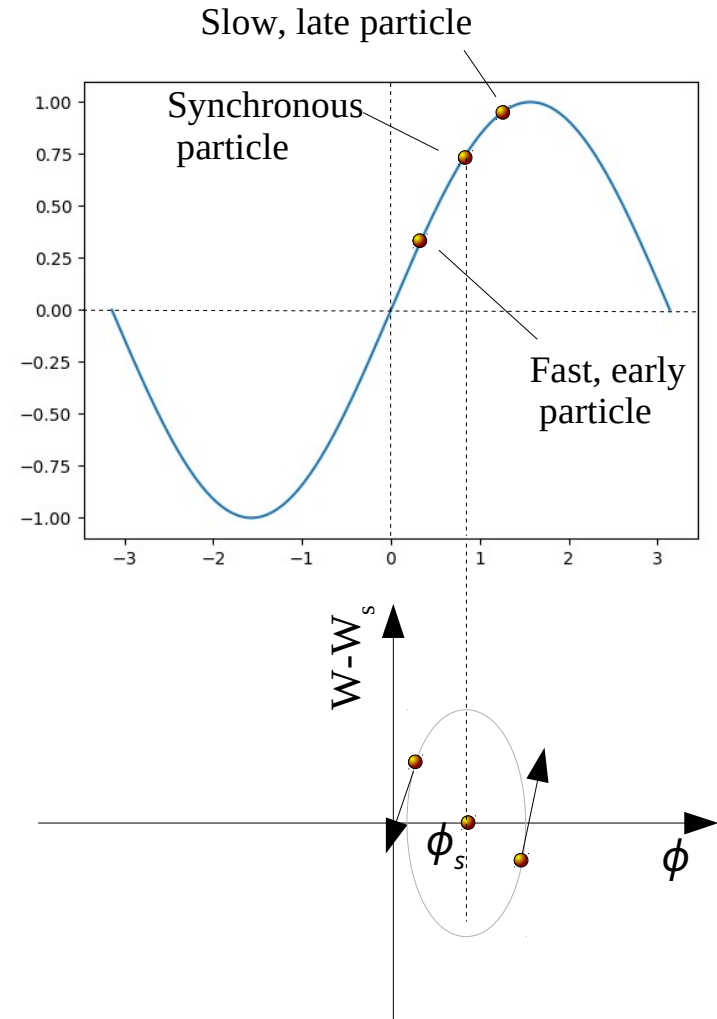
- T is factor to allow for phase to vary a bit during crossing
- g is the gap length

# Phase stability

- Particle crossing at phase  $\phi$  relative to synchronous particle

$$\delta W = q T g E_0 \sin(\phi + \phi_s)$$

- Particle arriving early
  - Fast
  - t negative
  - Gets smaller energy kick
  - Ends up relatively slower
- Particle arriving late
  - Slow
  - t positive
  - Gets bigger energy kick
  - Ends up relatively faster
- Phase stability!





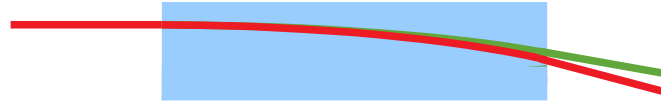
# Dealing with momentum spread

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- Momentum spread introduces a few effects
  - Dispersion
  - Chromaticity
  - Momentum compaction
- Dispersion:
  - Off-momentum particles follow a different trajectory
- Momentum compaction (rings):
  - Different path length yields different time of flight
- Chromaticity:
  - Off-momentum particles get a different focussing strength



# Dispersion



- Recall the definition of magnetic rigidity

$$B\rho = \frac{p}{q}$$

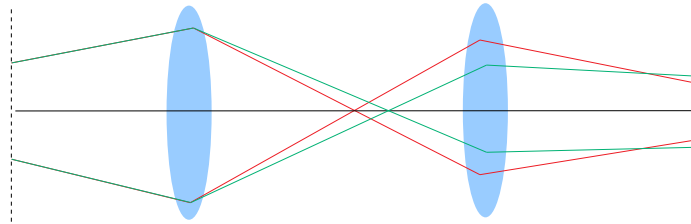
- Particles having different momentum ( $p$ ) get different radius of curvature
  - Introduce dispersion  $D$

$$D = p \frac{dx}{dp}$$

- Which is another optical function that we must make periodic

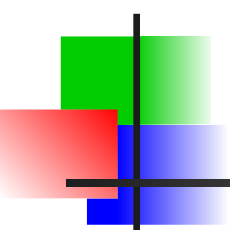
# Chromaticity

- Chromaticity arises because quadrupoles focus differently for different momenta



$$k = q \frac{b_0}{p}$$

- This often limits the degree of focussing at a collision point
  - Limits luminosity
- Can deliberately enhance/reduce chromaticity by
  - Introduce a dispersion
  - Using a magnet with variable focussing strength across the aperture - “sextupole”



# Questions



# Review

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- Dipoles are used to bend a beam - rigidity is  $B\rho = \frac{p}{q}$
- Quadrupoles are used to focus a beam:  $k = q \frac{b_0}{p}$
- Beam in each of  $x$  and  $y$  can be characterised by 3 Twiss parameters and an emittance
- Lattices can be characterised by a phase advance
- RF cavities are used to accelerate the beam
- Introducing momentum spread, one can also define a dispersion (and its derivative with respect to  $z$ )

# Finally... luminosity

- Luminosity defines the number of interactions in a collider per unit time for a given cross section
- Luminosity will increase if
  - Beam is narrower
  - Current is higher

Number of particles in each bunch

Revolution frequency

Number of bunches

$$\tilde{L} = \frac{N_1 N_2 f N_b}{4 \pi \sigma_x \sigma_y}$$

Width of Each bunch

The diagram illustrates the luminosity formula  $\tilde{L} = \frac{N_1 N_2 f N_b}{4 \pi \sigma_x \sigma_y}$ . Callouts identify the variables:  $N_1$  and  $N_2$  are grouped as 'Number of particles in each bunch';  $f$  is 'Revolution frequency';  $N_b$  is 'Number of bunches'; and  $\sigma_x$  and  $\sigma_y$  are grouped as 'Width of Each bunch'.



# What dictates luminosity?

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$$\tilde{L} = \frac{N_1 N_2 f N_b}{4 \pi \sigma_x \sigma_y}$$

- Typically
  - Number of particles → space charge
  - Revolution frequency → ring circumference
  - Number of bunches → RF frequency
  - Beam width →  $\sqrt{\varepsilon \beta}$ 
    - Emittance (cooling?)
    - Twiss beta (final focus and chromaticity)



# Next lecture...

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- Accelerator equipment
- Types of accelerator
- Current facilities
- Future facilities