Stratifying On-Shell Cluster Varieties

Jacob L. Bourjaily

Amplitudes 2022 Summer School
Charles University, Prague, Czech Republic

The Niels Bohr International Academy

PennState Eberly College of Science

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Organization and Outline

Amalgamating Diagrams from Three-Particle Amplitudes

Recall that on-shell diagrams built out of **three-point amplitudes** are always meaningful functions—even when the result is non-planar

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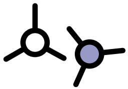

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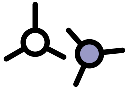

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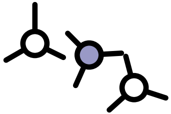

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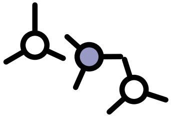

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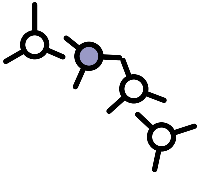

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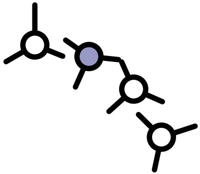

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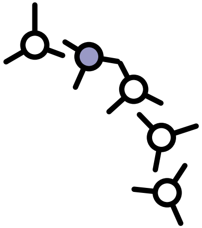

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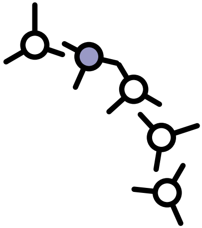

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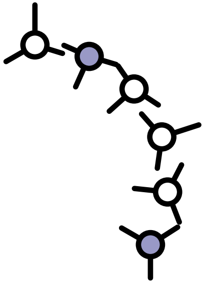

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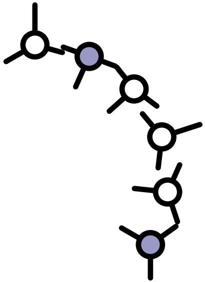

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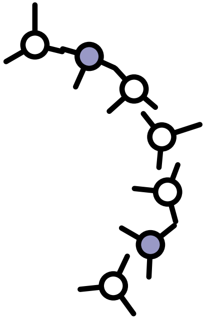

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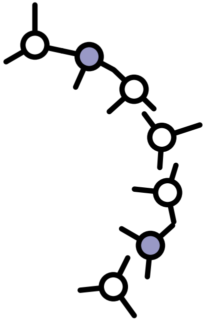

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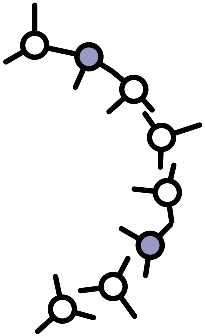

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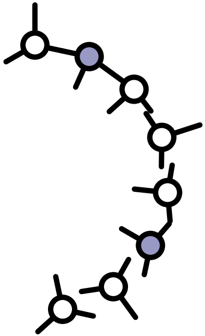

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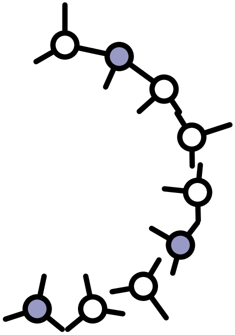

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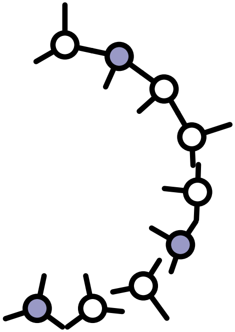

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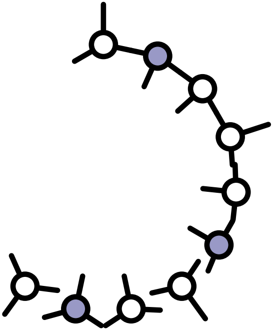

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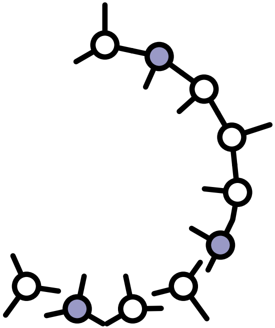

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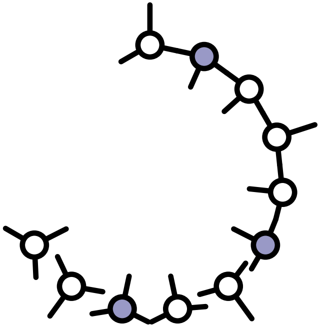

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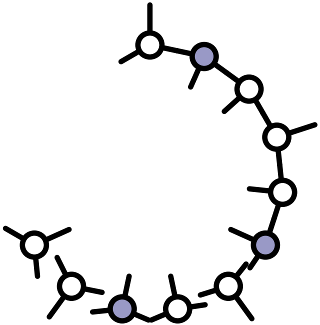

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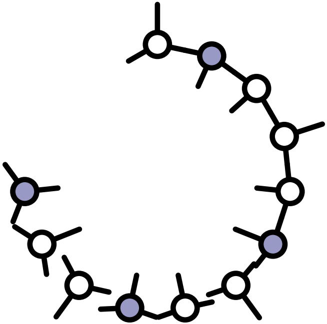

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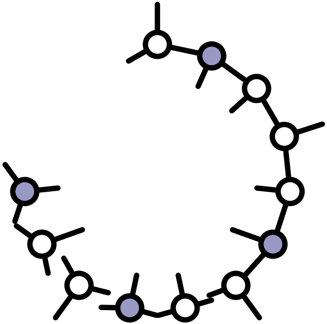

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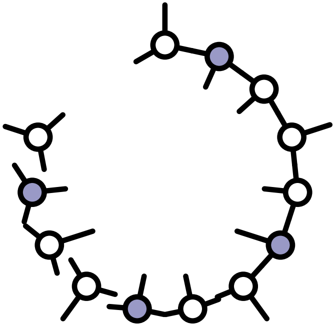

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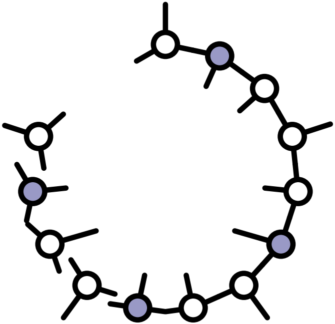

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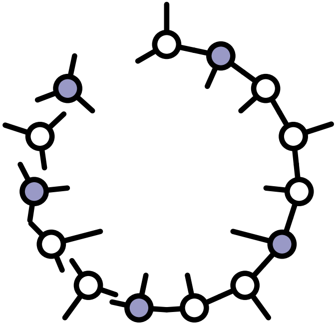

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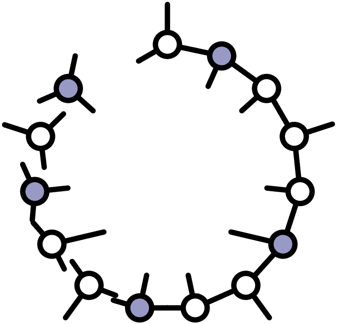

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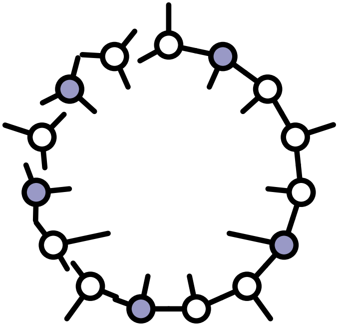

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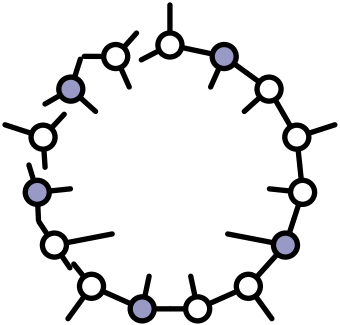

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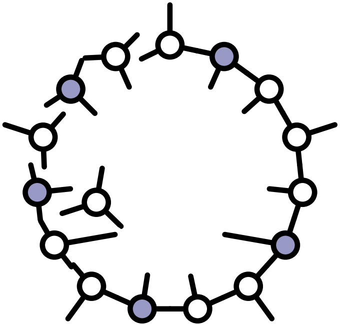

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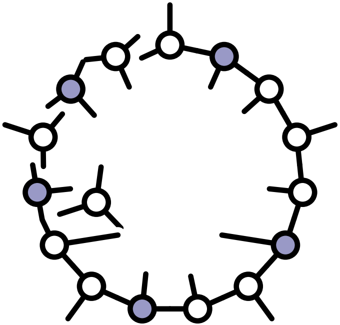

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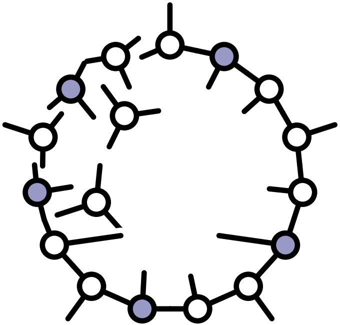

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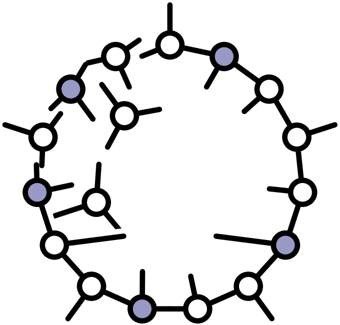

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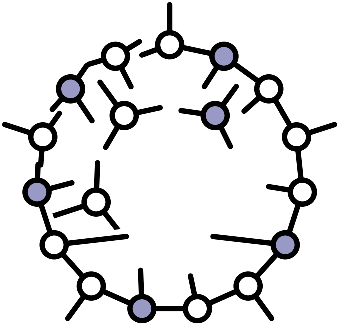

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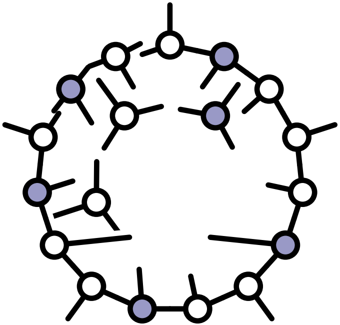

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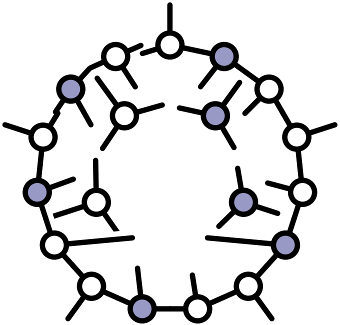

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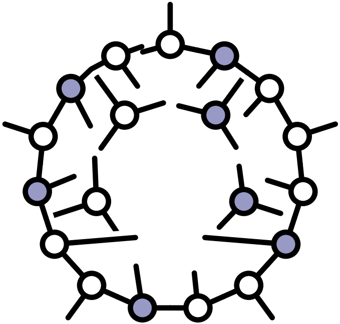

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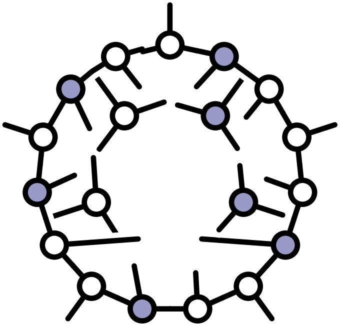

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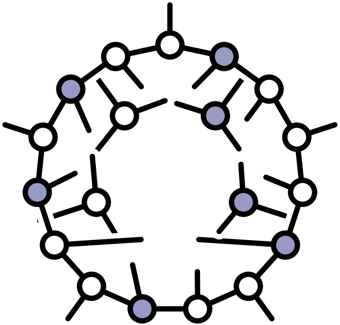

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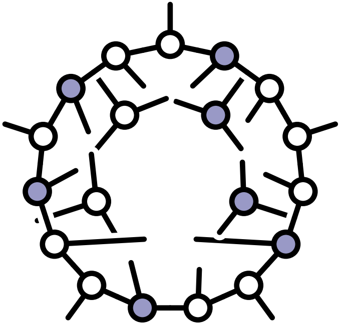

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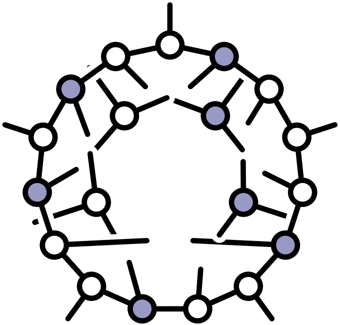

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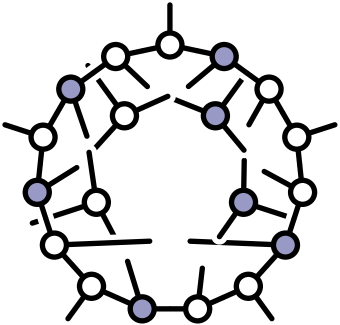

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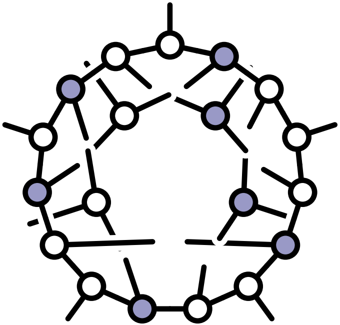

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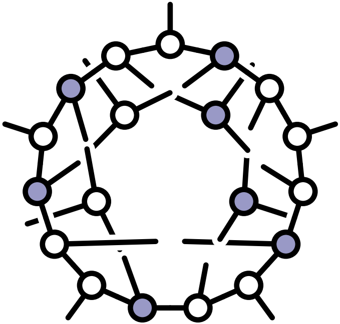

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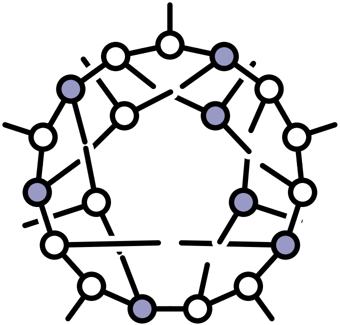

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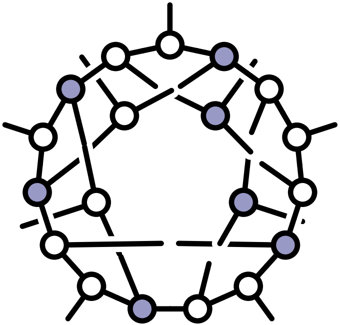

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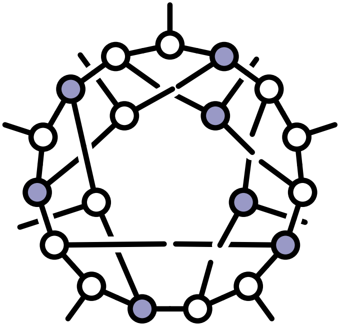

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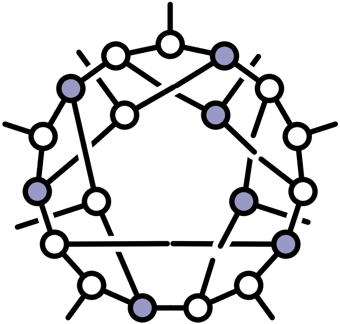

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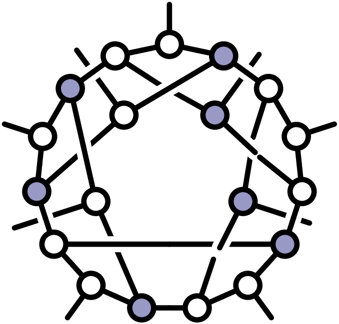

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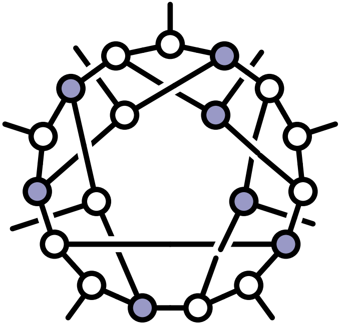

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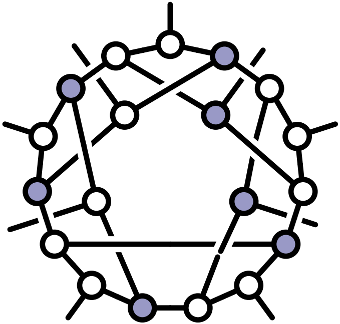

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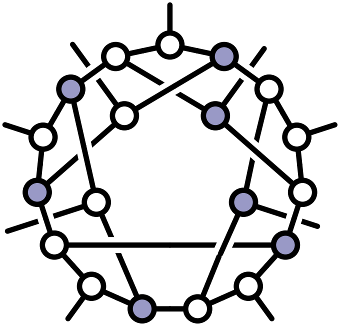

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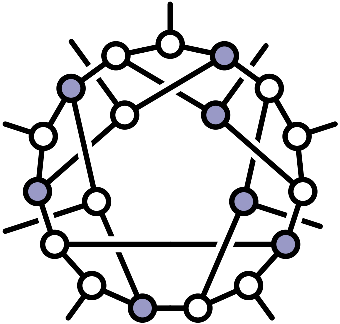

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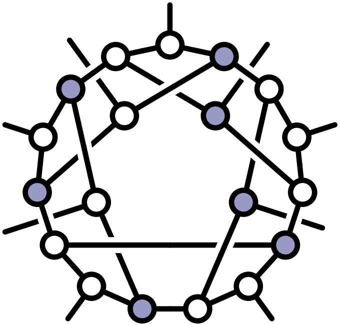

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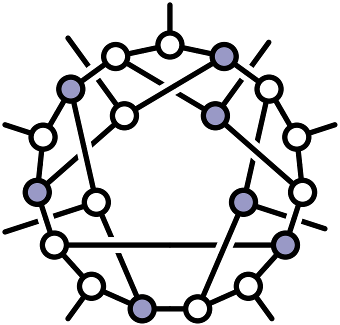

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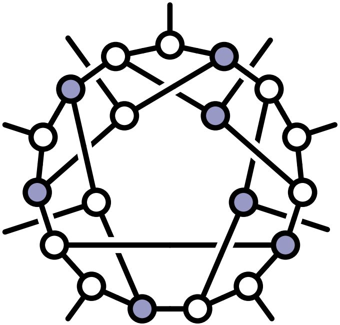

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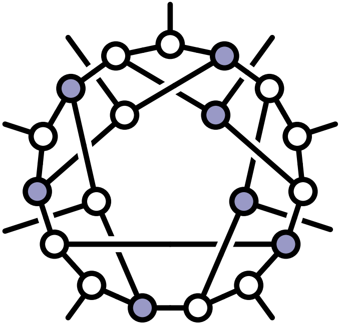

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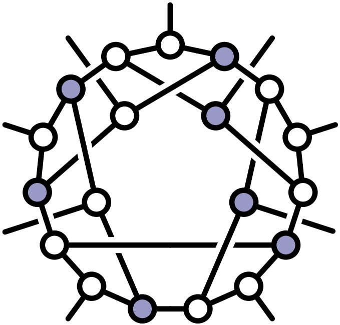

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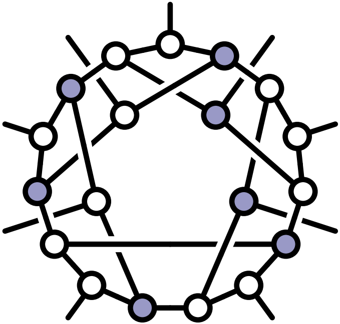

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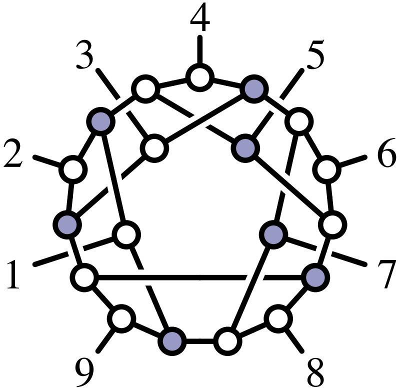

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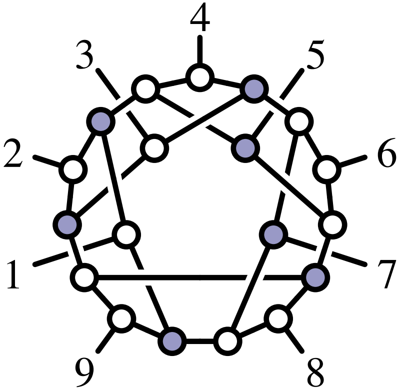

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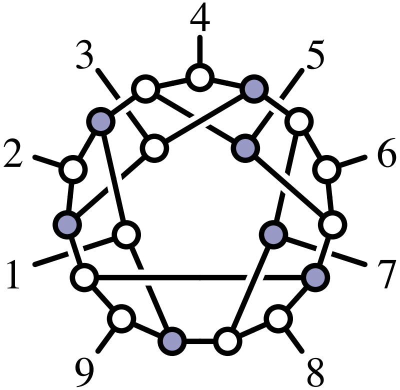

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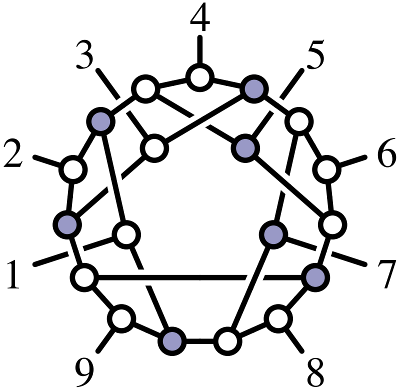

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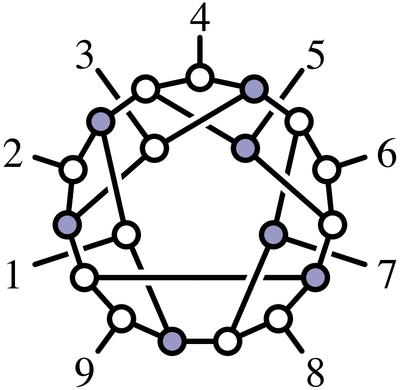

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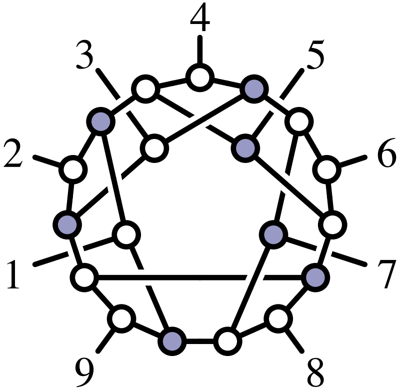

$$= \frac{(\langle 91\rangle\langle 23\rangle\langle 46\rangle - \langle 16\rangle\langle 34\rangle\langle 29\rangle)^2 \quad \delta^{2\times 4}(\lambda\cdot\widetilde{\eta})\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 67\rangle\langle 78\rangle\langle 81\rangle\langle 14\rangle\langle 42\rangle\langle 29\rangle\langle 96\rangle\langle 63\rangle\langle 39\rangle\langle 91\rangle}$$

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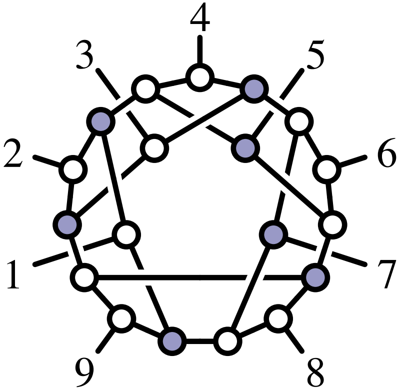

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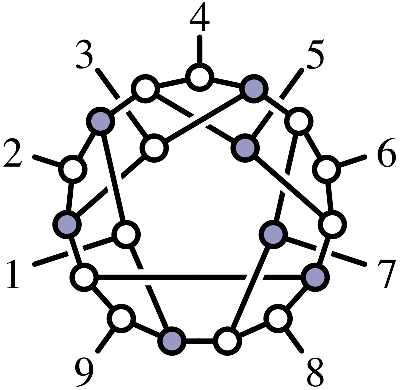

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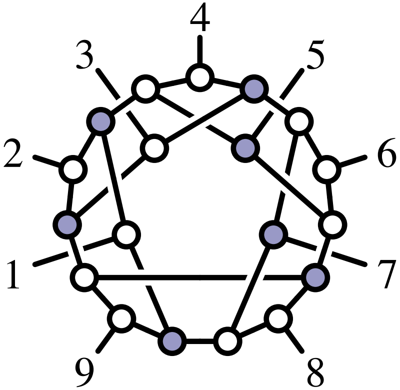

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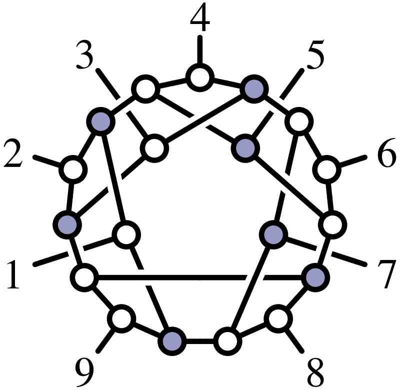

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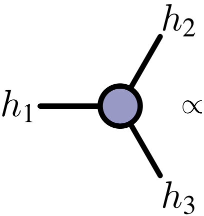
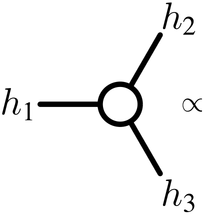
Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).



$$\propto \quad \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1}$$
$$h_1+h_2+h_3\leq 0$$



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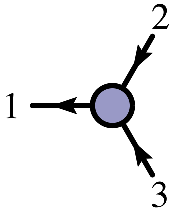

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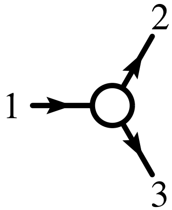

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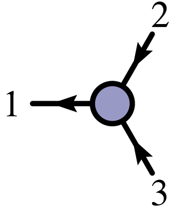

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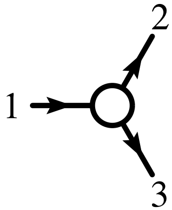

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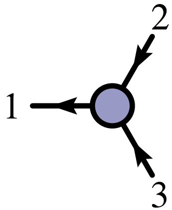

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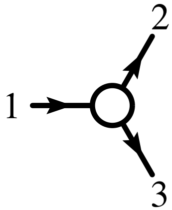

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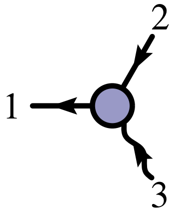

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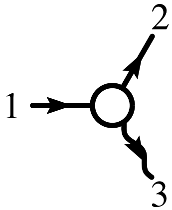

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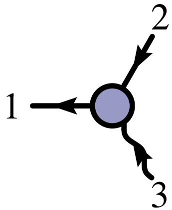

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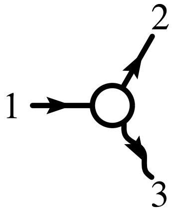

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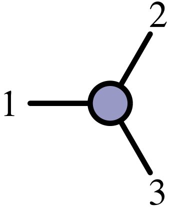

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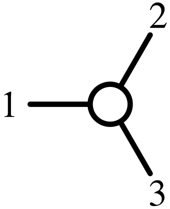

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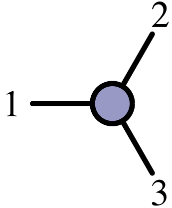

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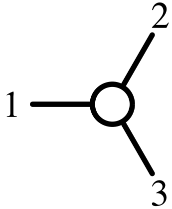

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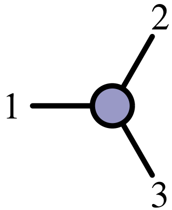

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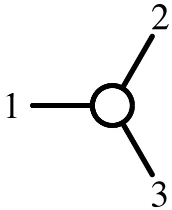

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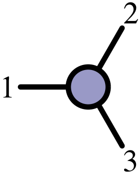

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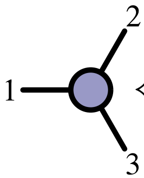

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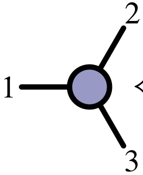

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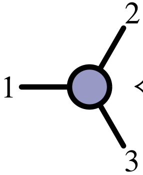

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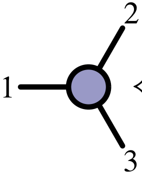
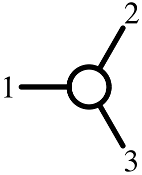


$$\Leftrightarrow B \equiv \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \end{pmatrix}$$



$$\mathcal{A}_3^{(2)} = \frac{\delta^{2\times4}\big(\lambda\cdot\widetilde{\eta}\big)}{\langle12\rangle\langle23\rangle\langle31\rangle}\,\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big) \equiv \int\!\frac{d^{2\times3}B}{\mathrm{vol}(GL_2)}\,\frac{\delta^{2\times4}\big(B\cdot\widetilde{\eta}\big)}{(12)(23)(31)}\;\delta^{2\times2}\big(B\cdot\widetilde{\lambda}\big)\;\delta^{1\times2}\big(\lambda\cdot B^{\perp}\big)$$

$$\mathcal{A}_3^{(1)} = \frac{\delta^{1\times4}\big(\widetilde{\lambda}^{\perp}\cdot\widetilde{\eta}\big)}{[12][23][31]}\,\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)$$

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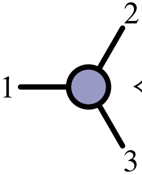

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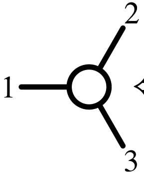

$$\Leftrightarrow W$$

$$\mathcal{A}_3^{(2)}=\frac{\delta^{2\times4}\big(\lambda\!\cdot\!\widetilde{\eta}\big)}{\langle12\rangle\langle23\rangle\langle31\rangle}\delta^{2\times2}\big(\lambda\!\cdot\!\widetilde{\lambda}\big)\equiv\int\!\frac{d^{2\times3}B}{\mathrm{vol}(GL_2)}\,\frac{\delta^{2\times4}\big(B\!\cdot\!\widetilde{\eta}\big)}{(12)(23)(31)}\,\delta^{2\times2}\big(B\!\cdot\!\widetilde{\lambda}\big)\,\delta^{1\times2}\big(\lambda\!\cdot\!B^{\perp}\big)$$

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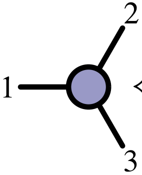

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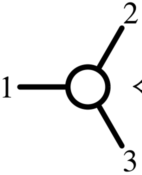

$$\Leftrightarrow W\equiv\begin{pmatrix}w_1^1 & w_2^1 & w_3^1\end{pmatrix}$$

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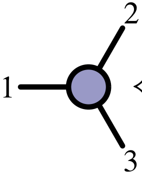

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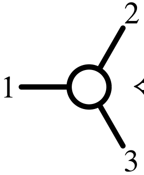

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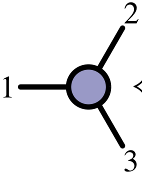

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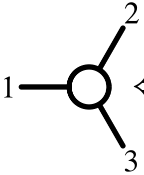

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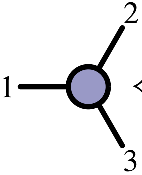

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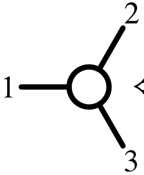

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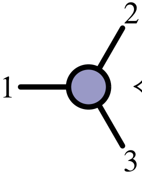

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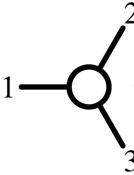

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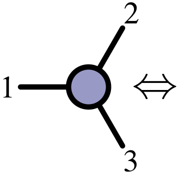

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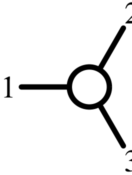

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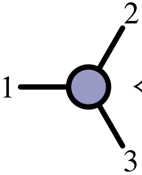

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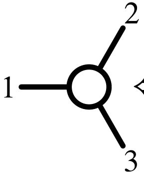

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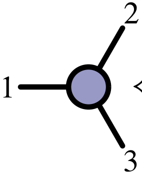

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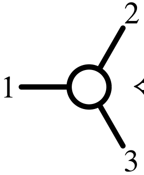

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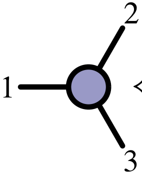
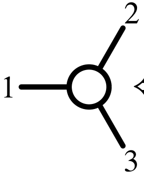


$$\Leftrightarrow B \equiv \begin{pmatrix} b_1^1 & 1 & 0 \\ b_1^2 & 0 & 1 \end{pmatrix}$$



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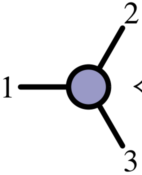

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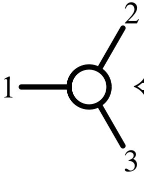

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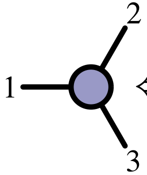

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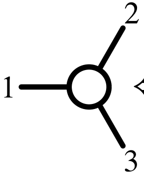

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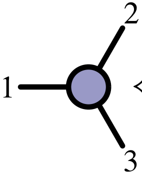

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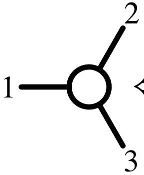

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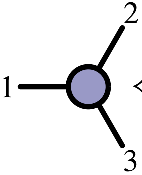

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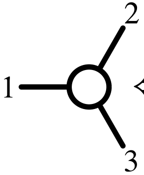

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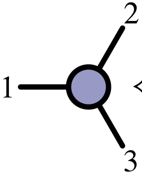

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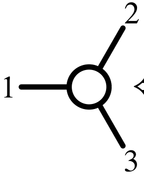

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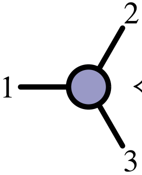

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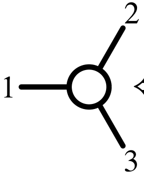

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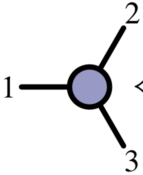

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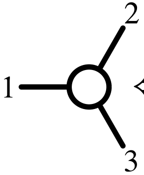

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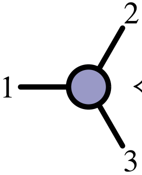

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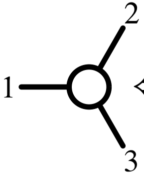

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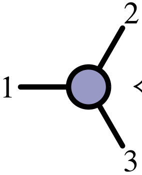

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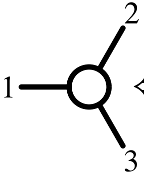

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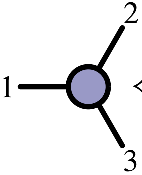

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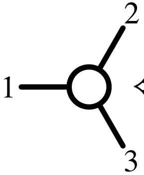

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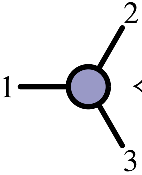

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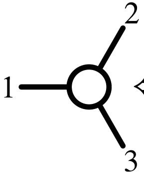

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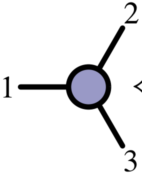

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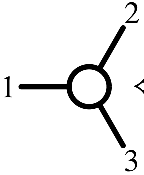

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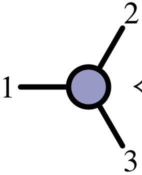

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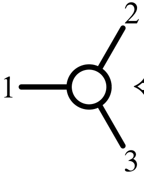

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Constructing the Correspondence: Amalgamations & Bridges

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Direct/Outer Products

$$(f_1,f_2)\mapsto f_1\times f_2$$
$$(C_1,C_2)\mapsto C_1\oplus C_2\subset G(k_1+k_2,n_1+n_2)$$
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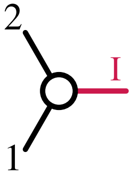

Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

$$(f_1,f_2) \mapsto f_1 \times f_2$$
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$$\begin{array}{ccc} 1 & 2 & \mathrm{I} \\ \end{array}$$
$$\begin{pmatrix} 1 & w_2 & w_{\mathrm{I}} \end{pmatrix}$$

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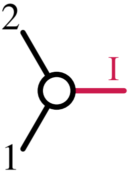

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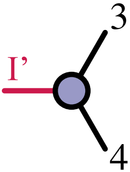

$$\begin{array}{c} \begin{array}{ccc} \mathrm{I'} & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & b_4^1 \\ 0 & 1 & b_4^2 \end{pmatrix} \end{array}$$

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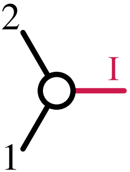

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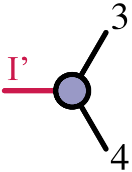

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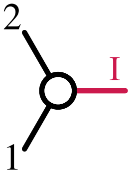

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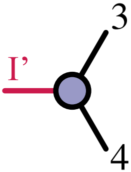

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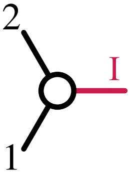
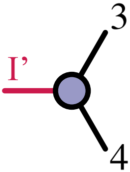


$$\begin{array}{c} \begin{array}{ccc} 1 & 2 & \mathrm{I} \end{array} \\ \begin{pmatrix} 1 & w_2 & w_{\mathrm{I}} \end{pmatrix} \end{array}$$



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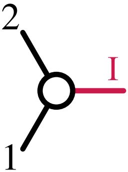

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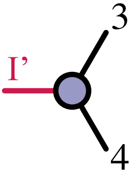

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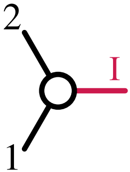

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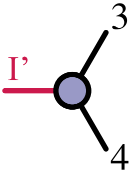

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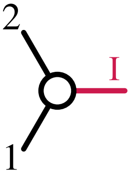
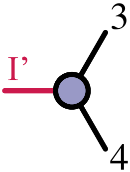


$$\begin{array}{ccc}1&2&\mathrm{I}\\\hline\end{array}\qquad\left(\begin{array}{ccc}1&w_2&w_{\mathrm{I}}\end{array}\right)$$



$$\begin{array}{ccc}\mathrm{I'}&3&4\\\hline\end{array}\qquad\left(\begin{array}{ccc}1&0&b_4^1\\0&1&b_4^2\end{array}\right)$$

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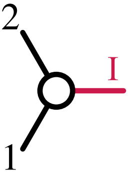
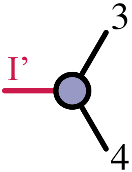


$$\begin{array}{ccc}1 & 2 & \mathrm{I}\\ \hline \end{array}$$
$$\begin{pmatrix}1 & w_2 & w_{\mathrm{I}}\end{pmatrix}$$



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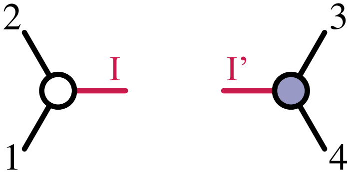

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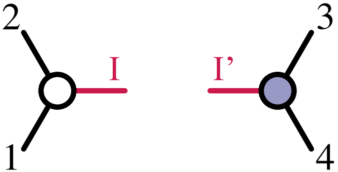

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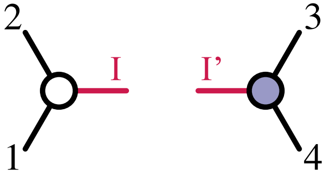

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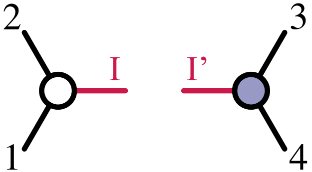

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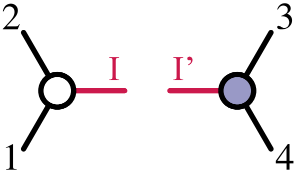

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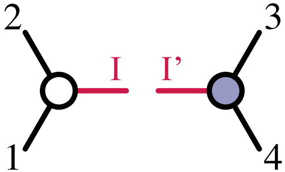

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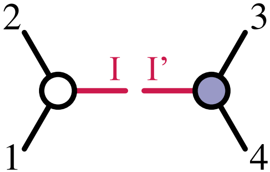

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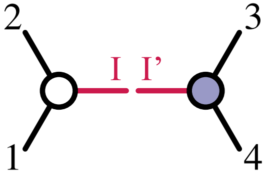

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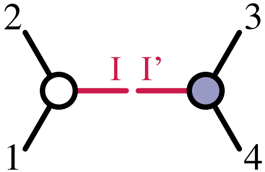

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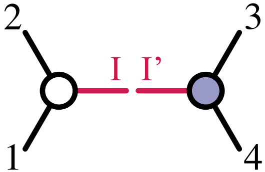

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \mathrm{I} & \mathrm{I'} & 3 & 4 \end{array} \\ \left(\begin{array}{cccccc} 1 & w_2 & w_{\mathrm{I}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & b_4^1 \\ 0 & 0 & 0 & 0 & 1 & b_4^2 \end{array}\right) \end{array}$$

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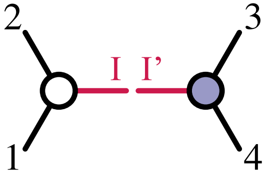

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1&2&\mathrm{I}&\mathrm{I'}&3&4\end{array}\\ \left(\begin{array}{cccccc}1&w_2&w_{\mathrm{I}}&0&0&0\\0&0&0&1&0&b_4^1\\0&0&0&0&1&b_4^2\end{array}\right)\end{array}$$

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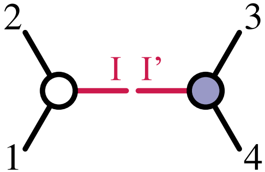

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1 & 2 & \mathrm{I} & \mathrm{I'} & 3 & 4\end{array}\\ \left(\begin{array}{cccccc}1 & w_2 & w_{\mathrm{I}} & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & b_4^1\\ 0 & 0 & 0 & 0 & 1 & b_4^2\end{array}\right)\end{array}$$

Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

$$(f_1, f_2) \mapsto f_1 \times f_2$$
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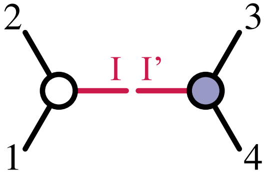

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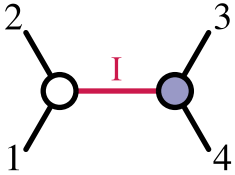

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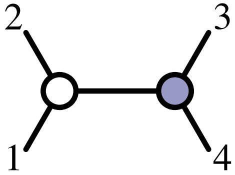

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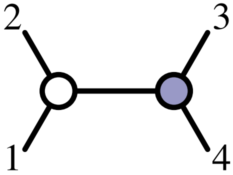

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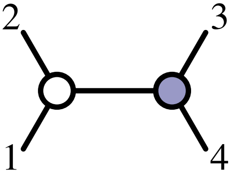

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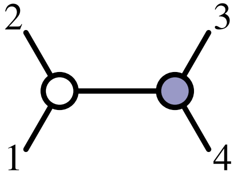

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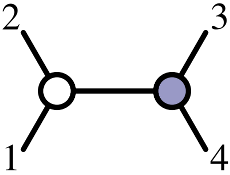

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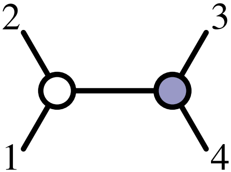

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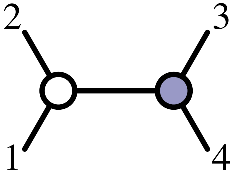

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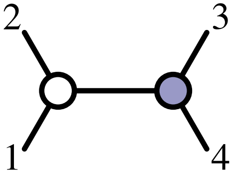

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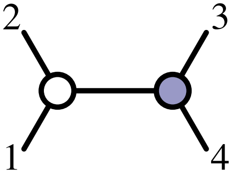

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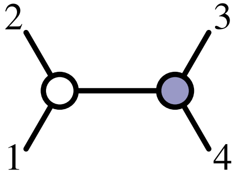

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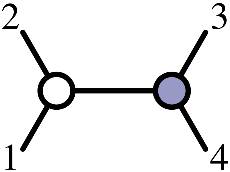

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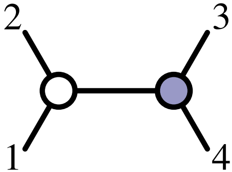

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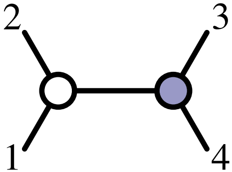

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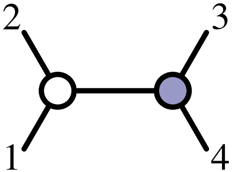

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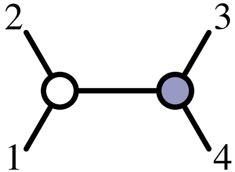

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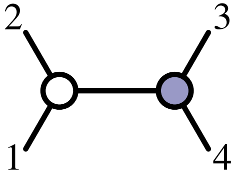

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Constructing the Correspondence: Amalgamations & Bridges

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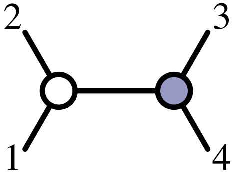

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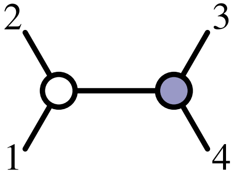

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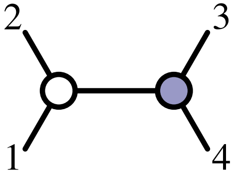

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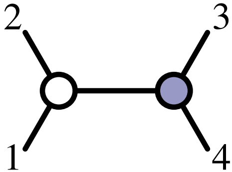

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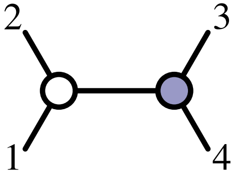

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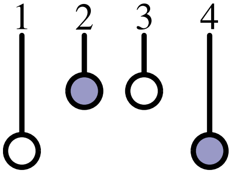

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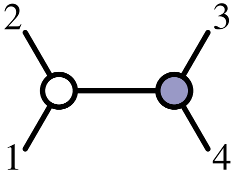

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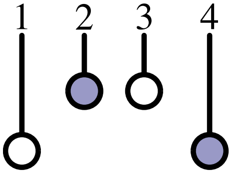

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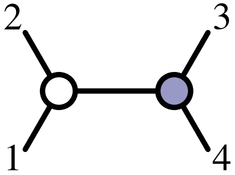

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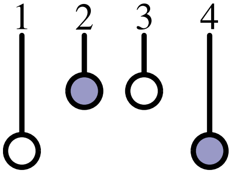

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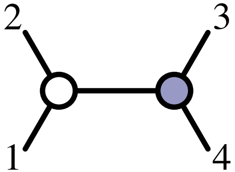

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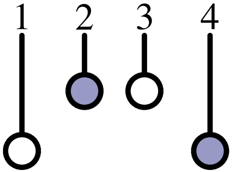

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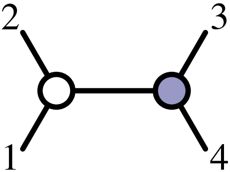

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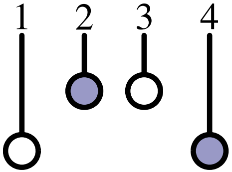

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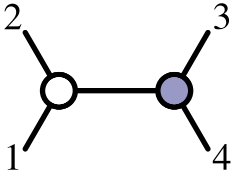

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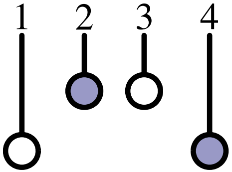

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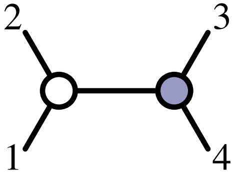

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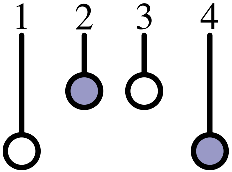

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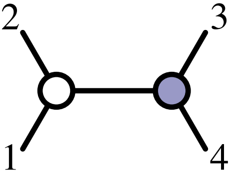

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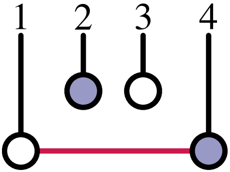

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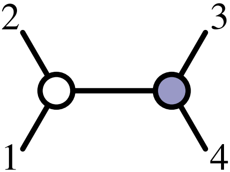

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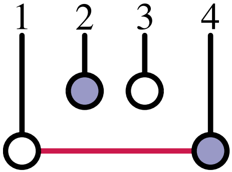

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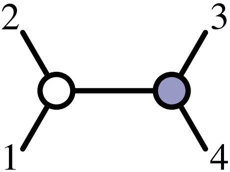

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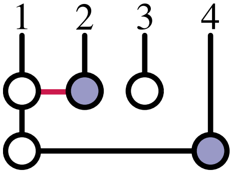

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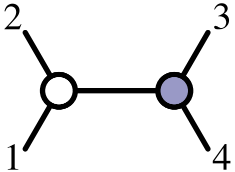

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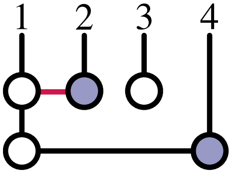

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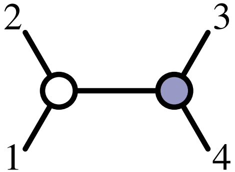

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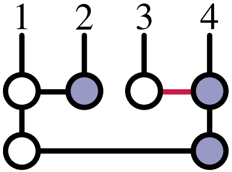

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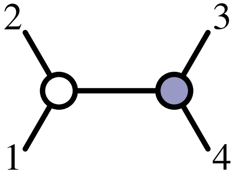

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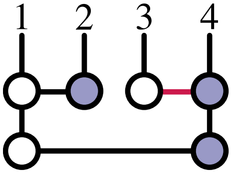

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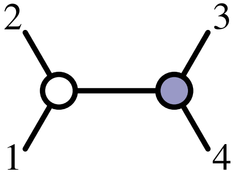

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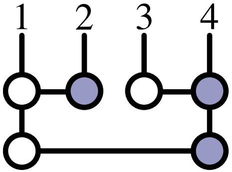

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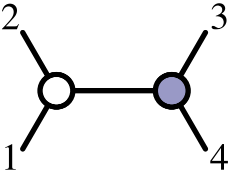

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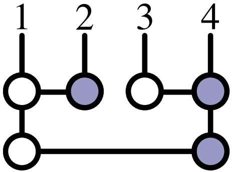

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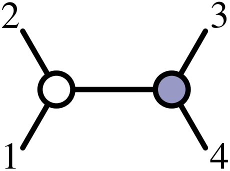

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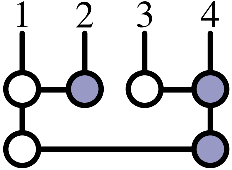

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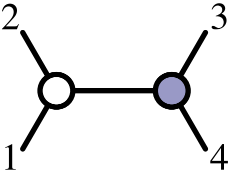

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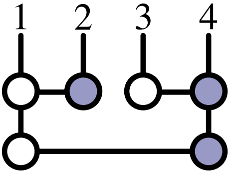

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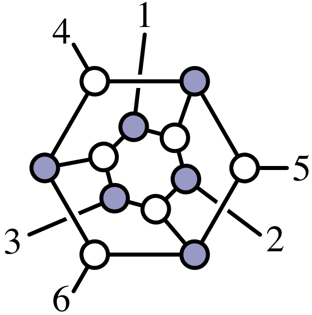

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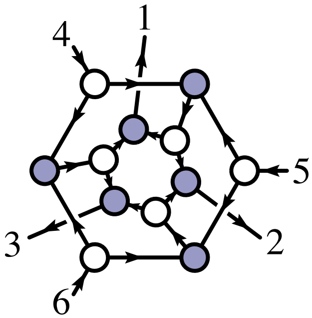

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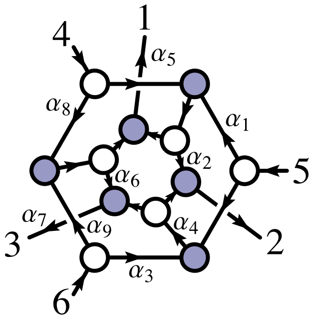

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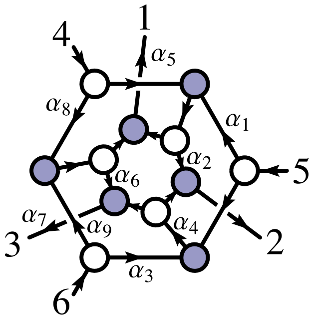

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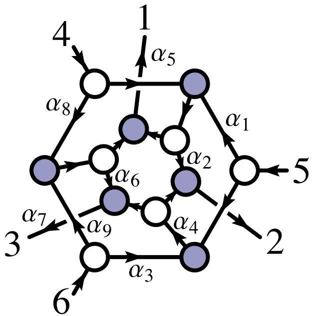

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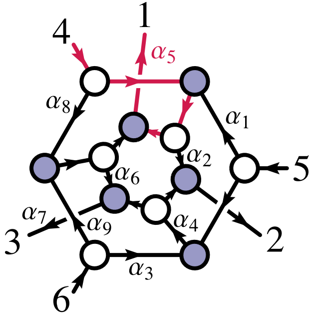

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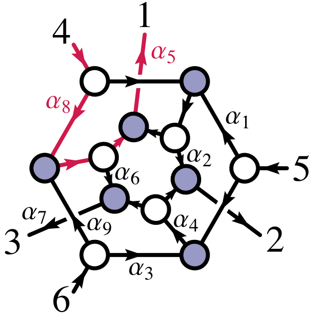

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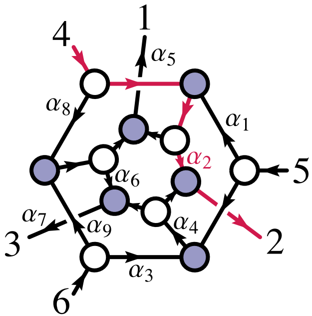

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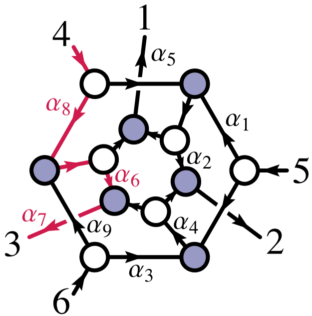

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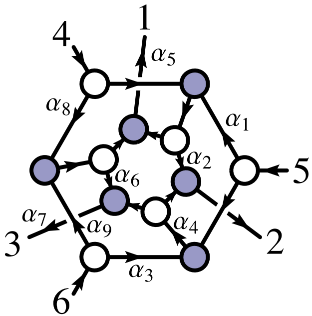

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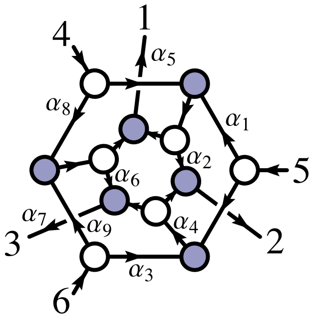

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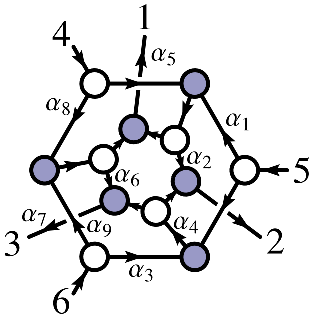

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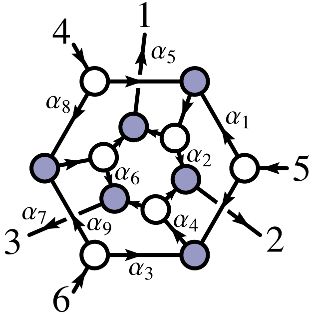

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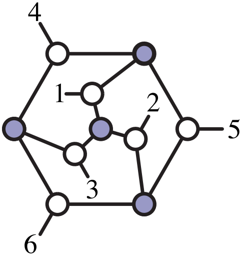

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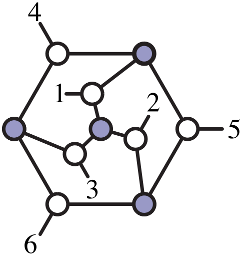

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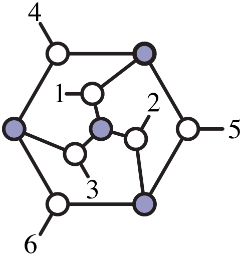

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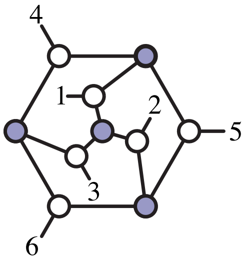

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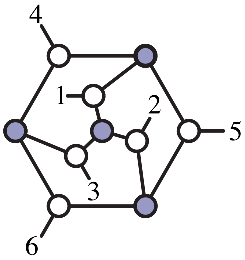

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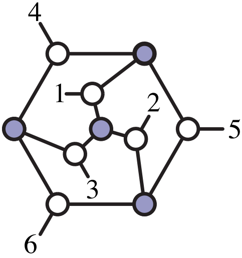

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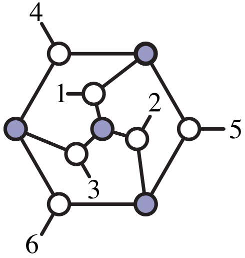

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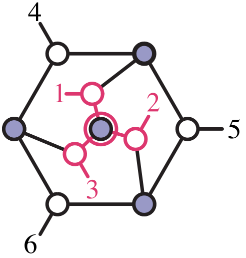

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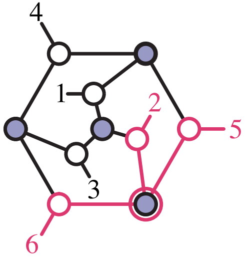

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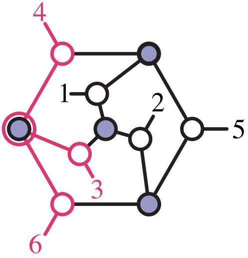

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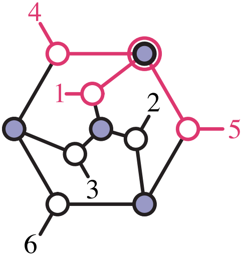

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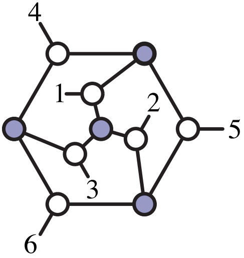

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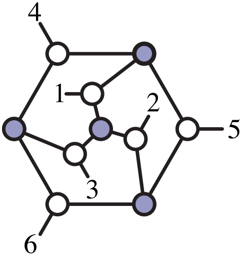

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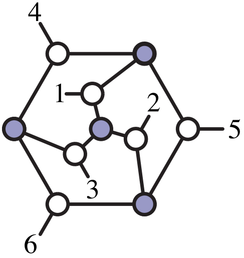

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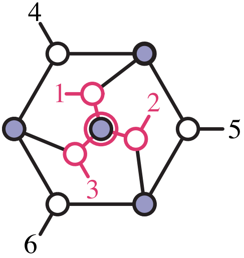

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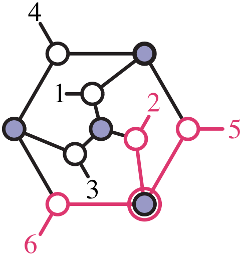

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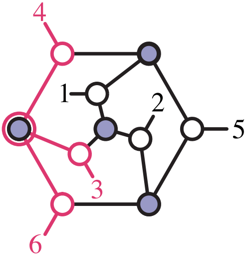

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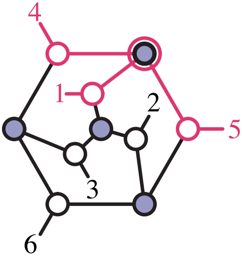

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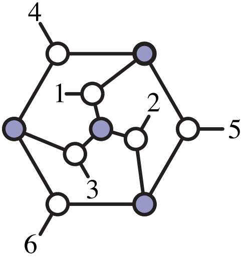

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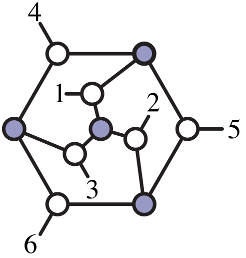

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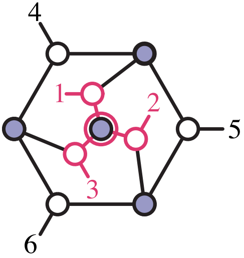

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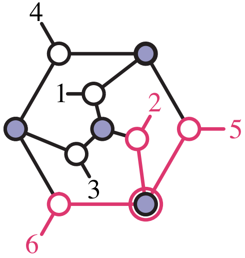

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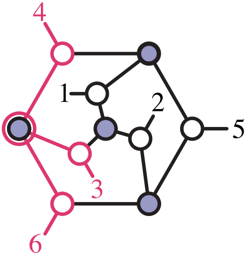

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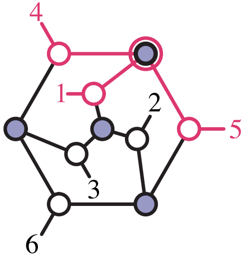

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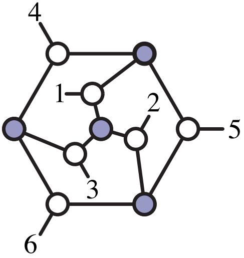

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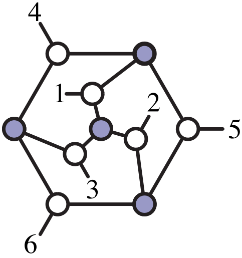

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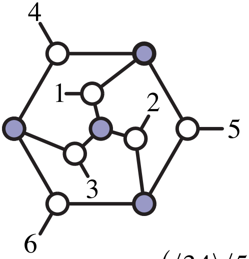

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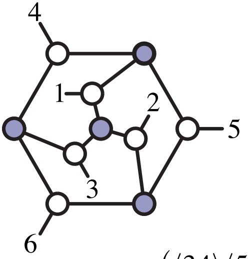

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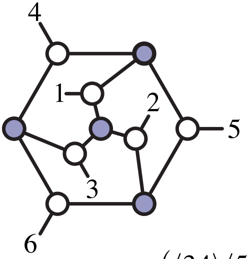

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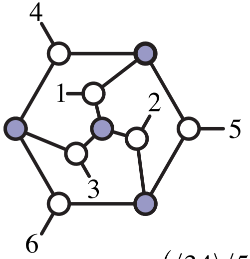

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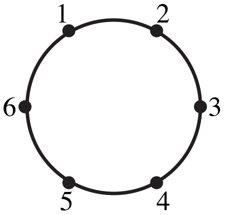

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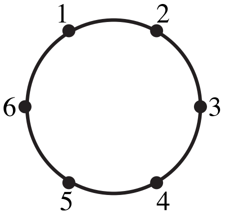

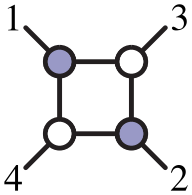

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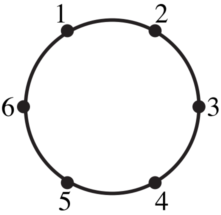

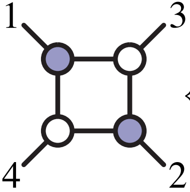

$$\left\{\begin{matrix}(1\,3\,4)\\(3\,2\,4)\end{matrix}\right\}$$

Extended ‘Positivity’ and Parke-Taylor Completeness

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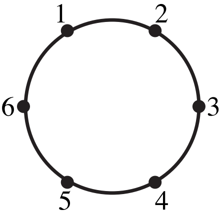

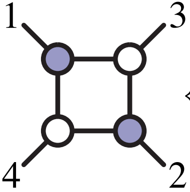

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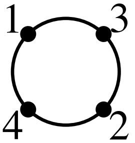

$\mathrm{PT}(1,3,2,4)$

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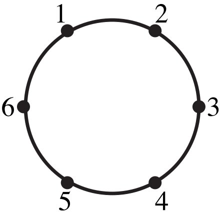

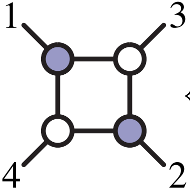

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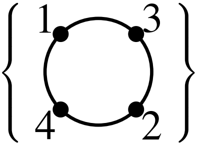

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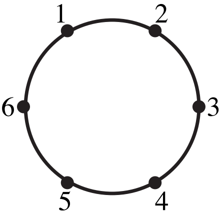
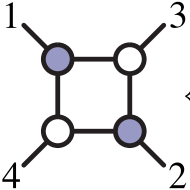
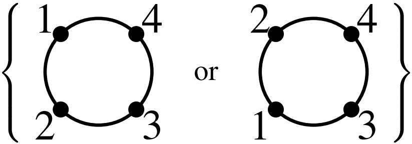




$$\left\{\begin{matrix}(1\,{\color{red}4\,3})\\(3\,2\,4)\end{matrix}\right\} \Leftrightarrow$$



$\mathrm{PT}(1,4,3,2)$, $\mathrm{PT}(1,2,4,3)$

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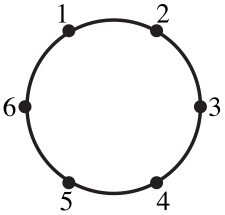

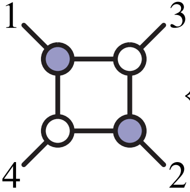

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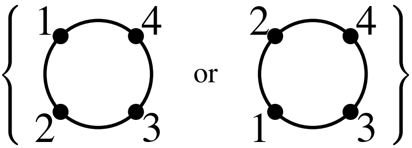

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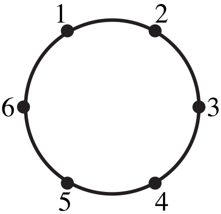

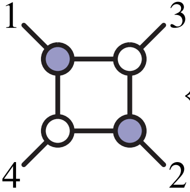

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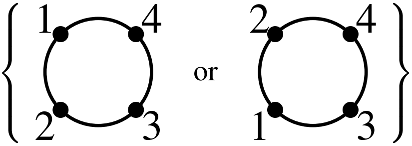

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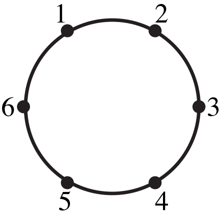

$$\boxed{\widetilde{f}_{\Gamma} = \sum_{\{\sigma\in(\mathfrak{S}_n/\mathbb{Z}_n)|\forall\tau\in T:\,\sigma_{\tau_1}<\sigma_{\tau_2}<\sigma_{\tau_3}\}} \mathrm{PT}(\sigma_1,\ldots,\sigma_n),}$$

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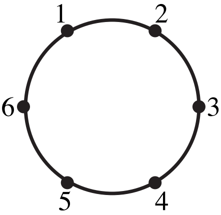

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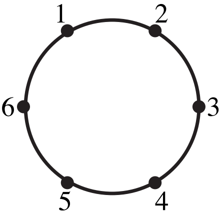

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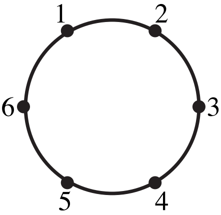

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Geometry of Kleiss-Kuijf Relations and $U(1)$-Decoupling

This gives a geometric interpretation of the $U(1)$-decoupling and KK-relations:

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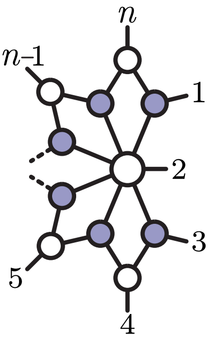

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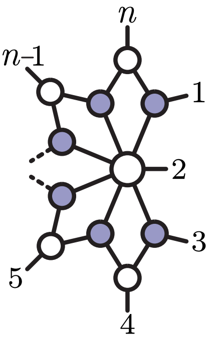

$$\left\{\begin{matrix} (2 & n & 1) \\ (2 & 3 & 4) \\ (2 & 4 & 5) \\ & \vdots & \\ (2 & n-1 & n) \end{matrix}\right\}$$

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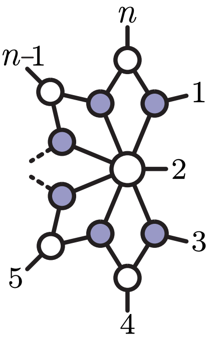

$$\frac{\left\{\begin{matrix}(2 & n & 1)\\(2 & 3 & 4)\\(2 & 4 & 5)\\ & \vdots & \\(2 & n{-}1 & n)\end{matrix}\right\}}{\mathrm{PT}(1,2,\ldots,n)}$$

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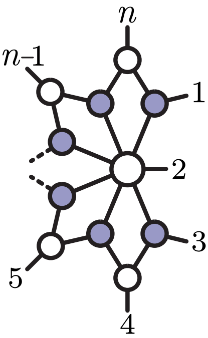

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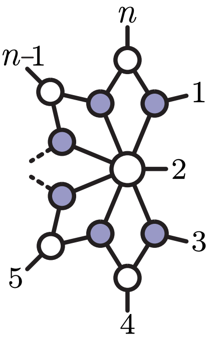

$$\underbrace{\left\{\begin{matrix}(2 & n & 1)\\(2 & 3 & 4)\\(2 & 4 & 5)\\ & \vdots & \\(2 & n-1 & n)\end{matrix}\right\}}_{\mathrm{PT}(1,2,\ldots,n)} \qquad \underbrace{\left\{\begin{matrix}(2 & 1 & n)\\(2 & 3 & 4)\\(2 & 4 & 5)\\ & \vdots & \\(2 & n-1 & n)\end{matrix}\right\}}_{\mathrm{PT}(1,n,2,\ldots,n-1)+\ldots+\mathrm{PT}(1,3,\ldots,n,2)}$$

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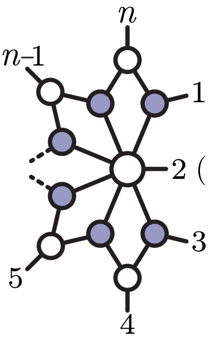

$$(-1)\frac{\left\{\begin{matrix}(2 & n & 1)\\ (2 & 3 & 4)\\ (2 & 4 & 5)\\ & \vdots & \\ (2 & n{-}1 & n)\end{matrix}\right\}}{\mathrm{PT}(1,2,\ldots,n)} = \frac{\left\{\begin{matrix}(2 & 1 & n)\\ (2 & 3 & 4)\\ (2 & 4 & 5)\\ & \vdots & \\ (2 & n{-}1 & n)\end{matrix}\right\}}{\mathrm{PT}(1,n,2,\ldots,n{-}1)+\ldots+\mathrm{PT}(1,3,\ldots,n,2)}$$

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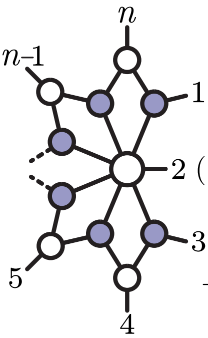

$$(-1)\underline{\left\{\begin{matrix}(2 & n & 1)\\(2 & 3 & 4)\\(2 & 4 & 5)\\ & \vdots & \\(2 & n-1 & n)\end{matrix}\right\}} = \underline{\left\{\begin{matrix}(2 & 1 & n)\\(2 & 3 & 4)\\(2 & 4 & 5)\\ & \vdots & \\(2 & n-1 & n)\end{matrix}\right\}}$$

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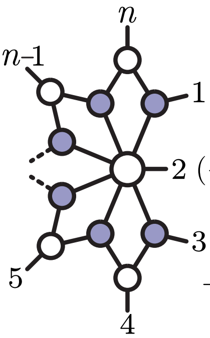

$$(-1)\left\{\begin{matrix}(2 & n & 1)\\(2 & 3 & 4)\\(2 & 4 & 5)\\ & \vdots & \\(2 & n-1 & n)\end{matrix}\right\} = \left\{\begin{matrix}(2 & {\color{red}1} & {\color{red}n})\\(2 & 3 & 4)\\(2 & 4 & 5)\\ & \vdots & \\(2 & n-1 & n)\end{matrix}\right\}$$

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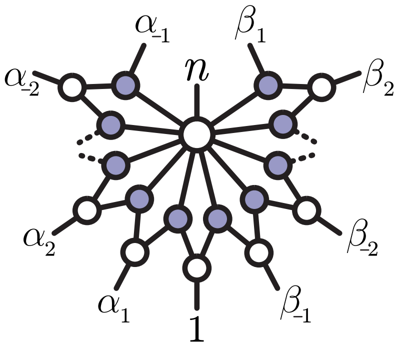

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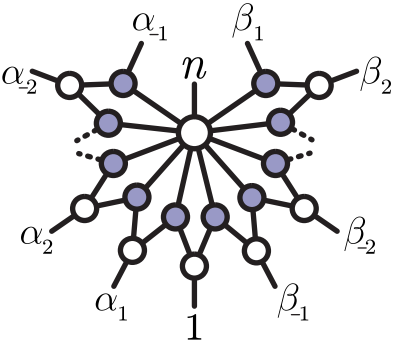

$$\left\{\begin{array}{c}(\,1\;\;\alpha_1\;\;n\,)\\(\alpha_1\;\alpha_2\;\;n\,)\\\vdots\\(\alpha_{-2}\;\alpha_{-1}\;\;n\,)\\(\,n\;\;\beta_1\;\beta_2)\\\vdots\\(\,n\;\;\beta_{-2}\;\beta_{-1})\\(\,n\;\;\beta_{-1}\;\;1\,)\end{array}\right\}$$

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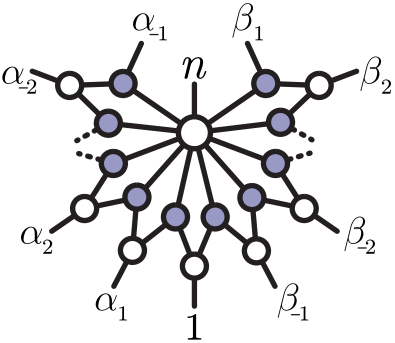

$$\left\{\begin{array}{c}(1\;\alpha_1\;n)\\(\alpha_1\;\alpha_2\;n)\\\vdots\\(\alpha_{-2}\;\alpha_{-1}\;n)\\(n\;\beta_1\;\beta_2)\\\vdots\\(n\;\beta_{-2}\;\beta_{-1})\\(n\;\beta_{-1}\;1)\end{array}\right\} \qquad \left\{\begin{array}{c}(1\;\alpha_1\;n)\\(\alpha_1\;\alpha_2\;n)\\\vdots\\(\alpha_{-2}\;\alpha_{-1}\;n)\\(n\;\beta_2\;\beta_1)\\\vdots\\(n\;\beta_{-1}\;\beta_{-2})\\(n\;1\;\beta_{-1})\end{array}\right\}$$

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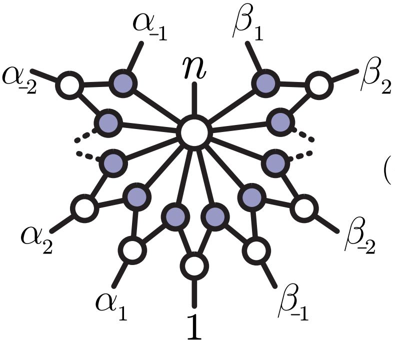

$$(-1)^{n_\beta}\left\{\begin{matrix}(1\ \ \alpha_1\ \ n)\\(\alpha_1\ \alpha_2\ \ n)\\ \vdots\\(\alpha_{-2}\ \alpha_{-1}\ \ n)\\(n\ \ \beta_1\ \beta_2)\\ \vdots\\(n\ \ \beta_{-2}\ \beta_{-1})\\(n\ \ \beta_{-1}\ \ 1)\end{matrix}\right\} = \left\{\begin{matrix}(1\ \ \alpha_1\ \ n)\\(\alpha_1\ \alpha_2\ \ n)\\ \vdots\\(\alpha_{-2}\ \alpha_{-1}\ \ n)\\(n\ \ \beta_2\ \beta_1)\\ \vdots\\(n\ \ \beta_{-1}\ \beta_{-2})\\(n\ \ 1\ \ \beta_{-1})\end{matrix}\right\}$$

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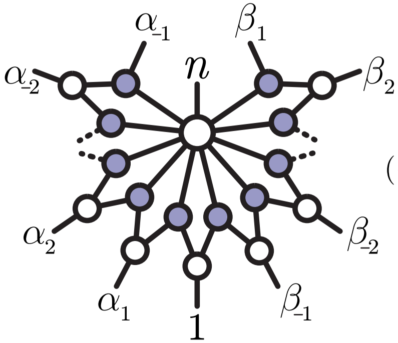

$$(-1)^{n_\beta}\begin{Bmatrix}(1\ \alpha_1\ n)\\(\alpha_1\,\alpha_2\ n)\\\vdots\\(\alpha_{-2}\,\alpha_{-1}\ n)\\(n\ \beta_1\,\beta_2)\\\vdots\\(n\ \beta_{-2}\,\beta_{-1})\\(n\ \beta_{-1}\ 1)\end{Bmatrix}=\begin{Bmatrix}(1\ \alpha_1\ n)\\(\alpha_1\,\alpha_2\ n)\\\vdots\\(\alpha_{-2}\,\alpha_{-1}\ n)\\(n\ \beta_2\,\beta_1)\\\vdots\\(n\ \beta_{-1}\,\beta_{-2})\\(n\ 1\ \beta_{-1})\end{Bmatrix}$$

$$(-1)^{n_\beta}\times\mathrm{PT}(1,\alpha_1,\ldots,\alpha_{-1},n,\beta_1,\ldots,\beta_{-1})=\sum_{\sigma\in(\{\alpha_1,\ldots,\alpha_{-1}\}\sqcup\{\beta_{-1},\ldots,\beta_1\})}\mathrm{PT}(1,\sigma_1,\ldots,\sigma_{n-2},n).$$

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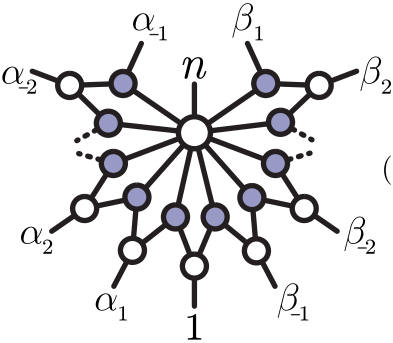

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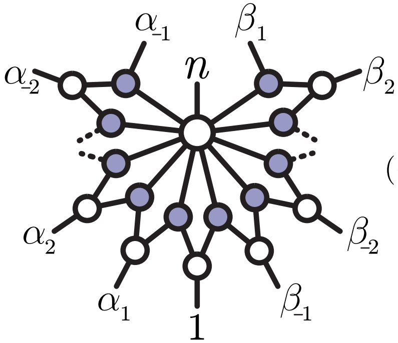

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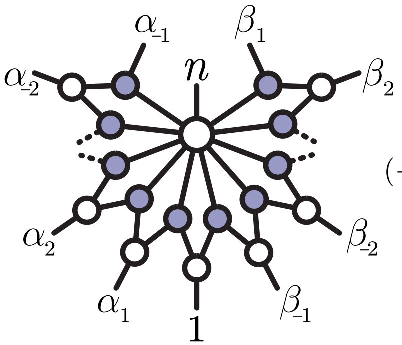

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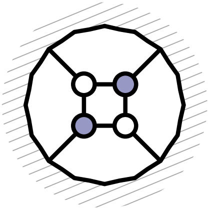

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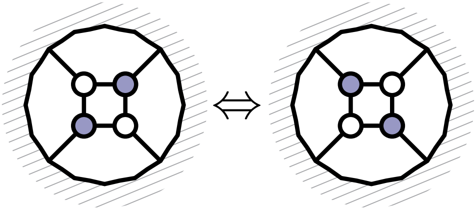

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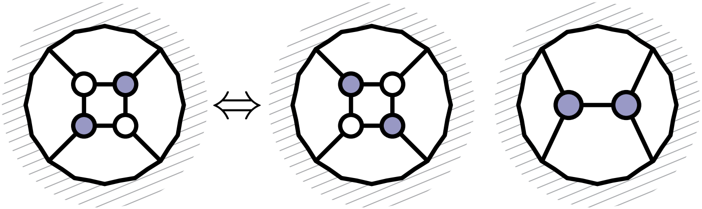

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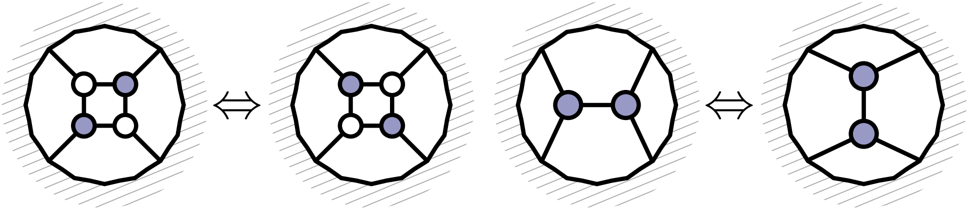

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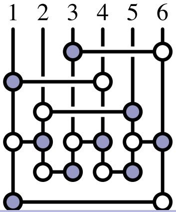

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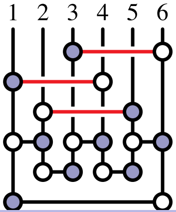

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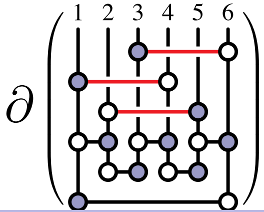

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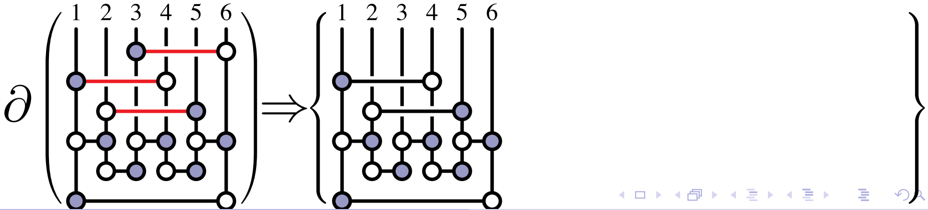

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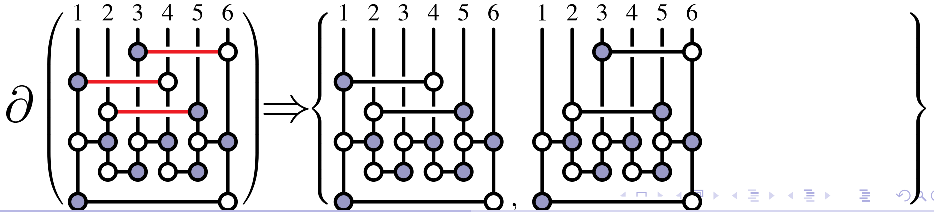

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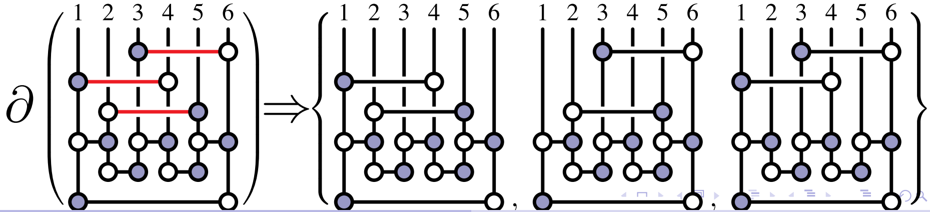

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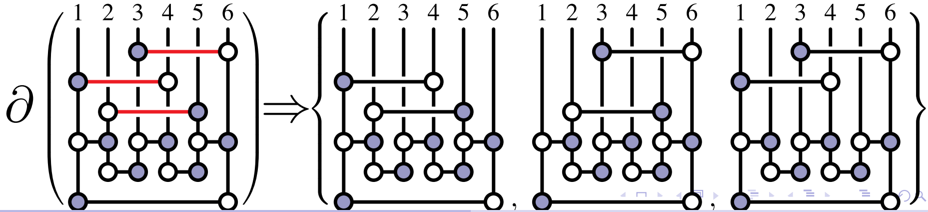

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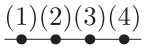

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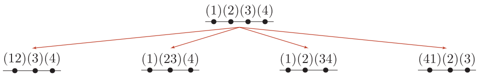

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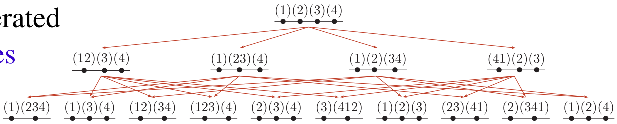

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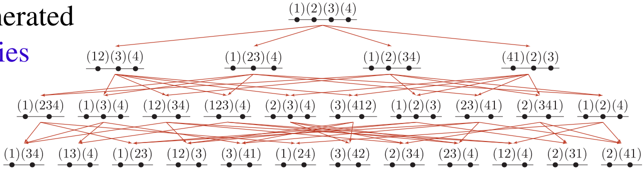

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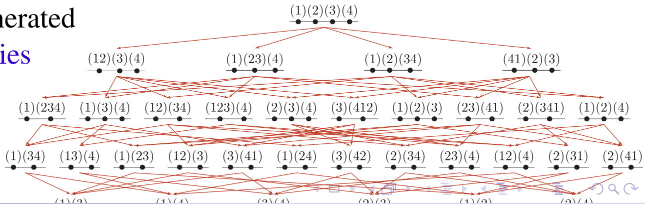

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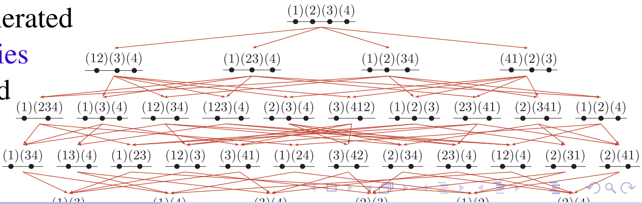

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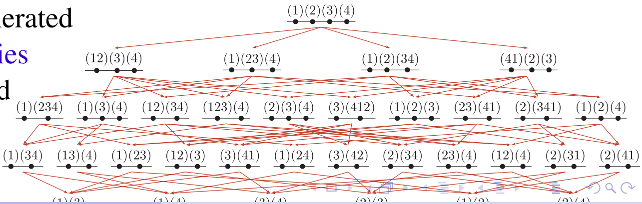

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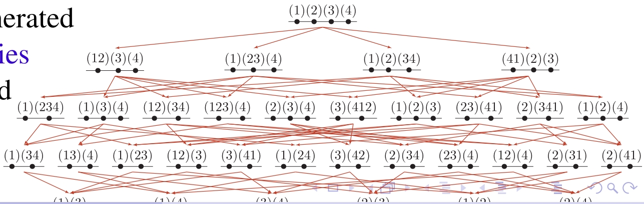

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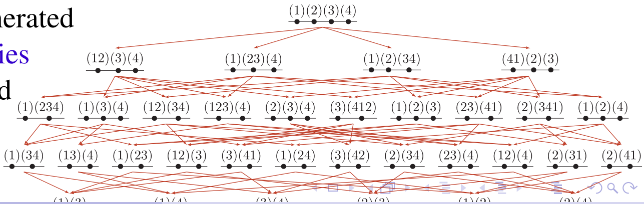

Summary of the Classification of On-Shell Varieties of $G(3,6)$

Classification of On-Shell Varieties for 6-Point NMHV $(k=3)$

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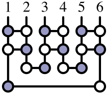

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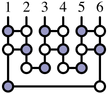

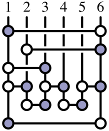

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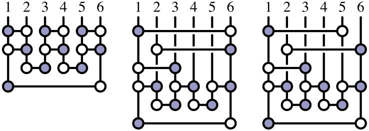

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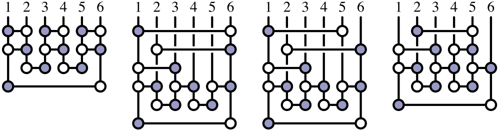

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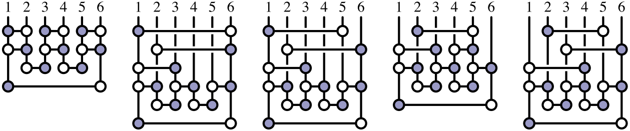

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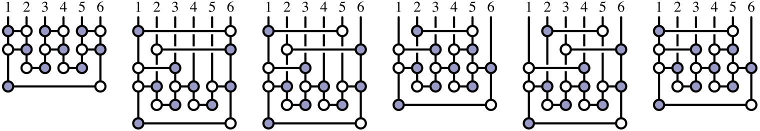

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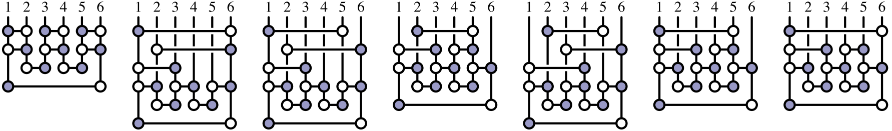

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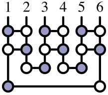

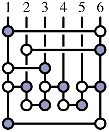

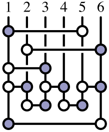

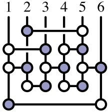

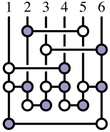

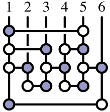

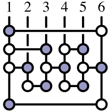

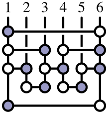

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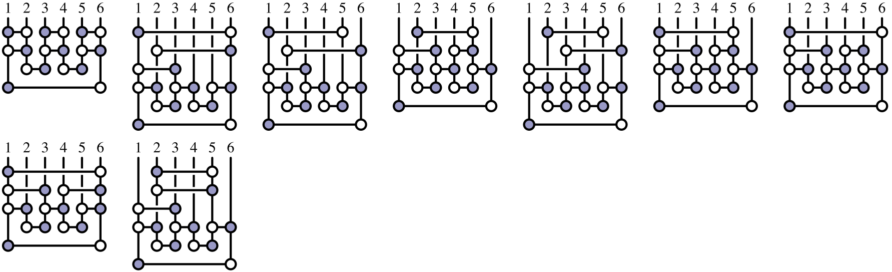

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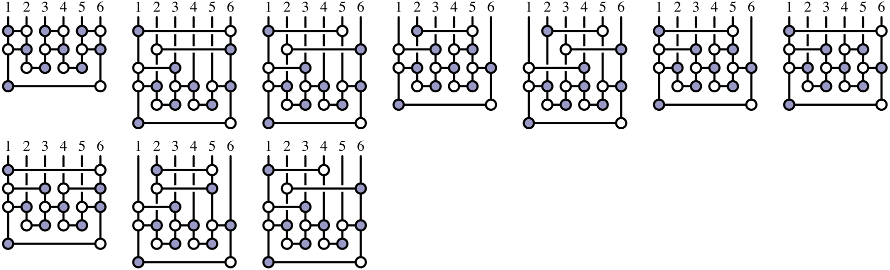

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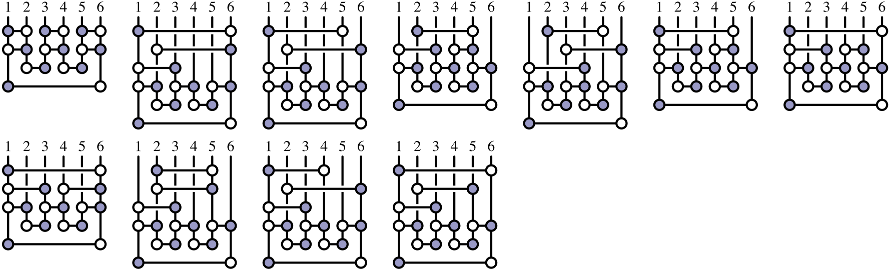

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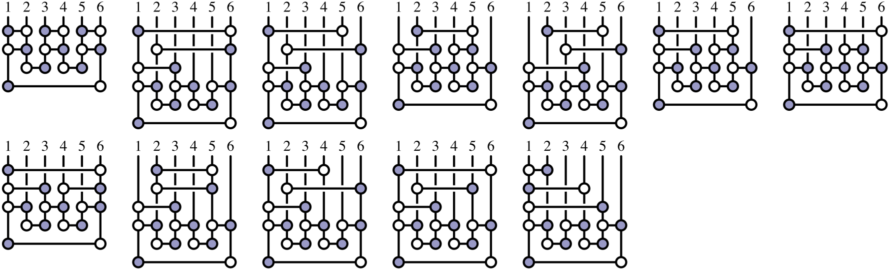

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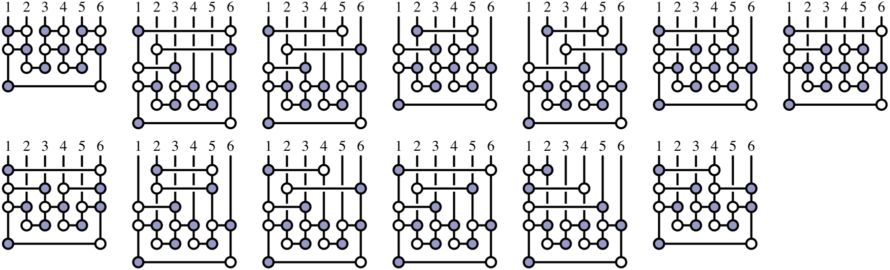

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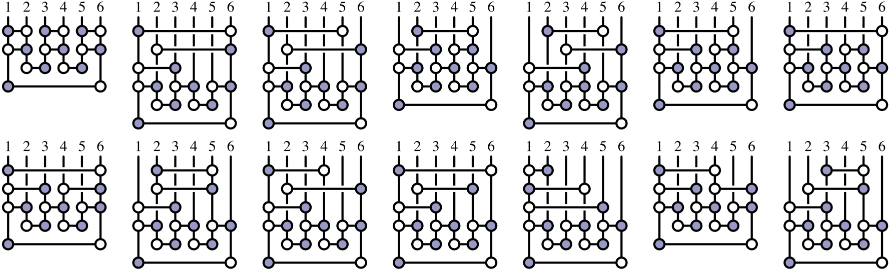

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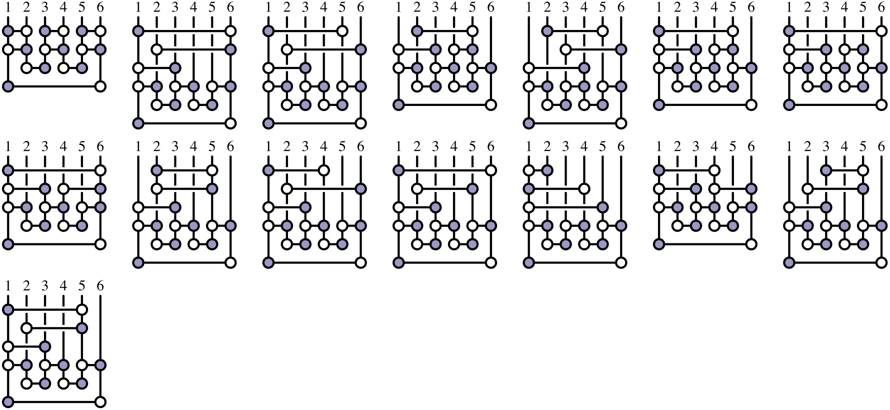

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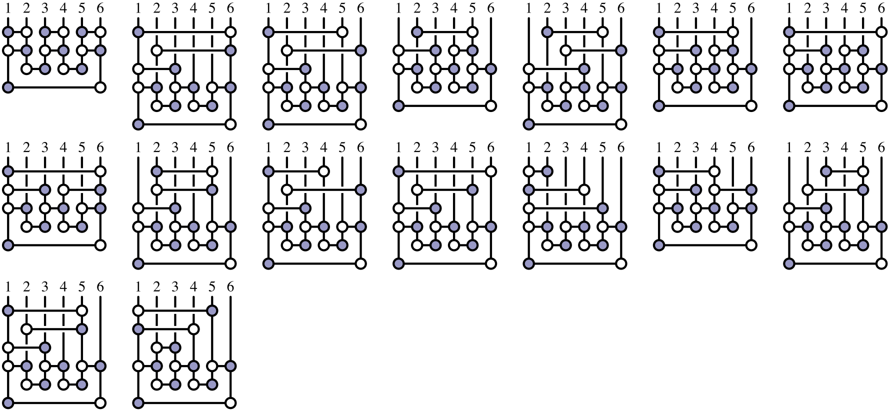

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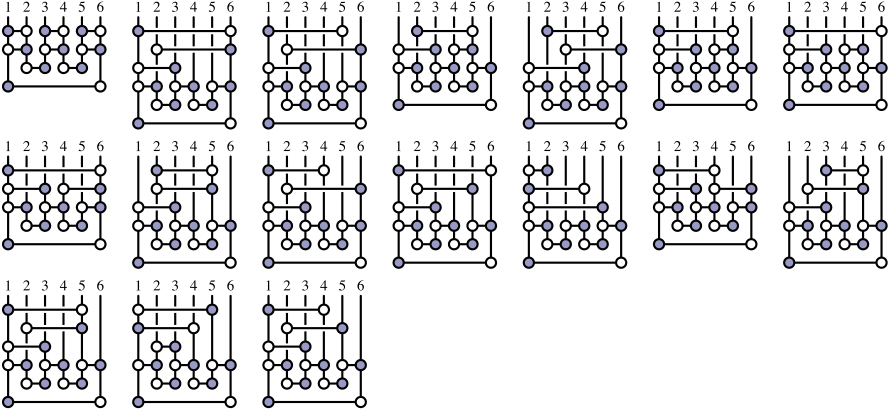

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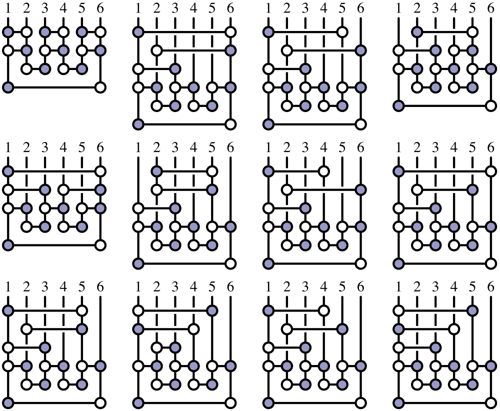

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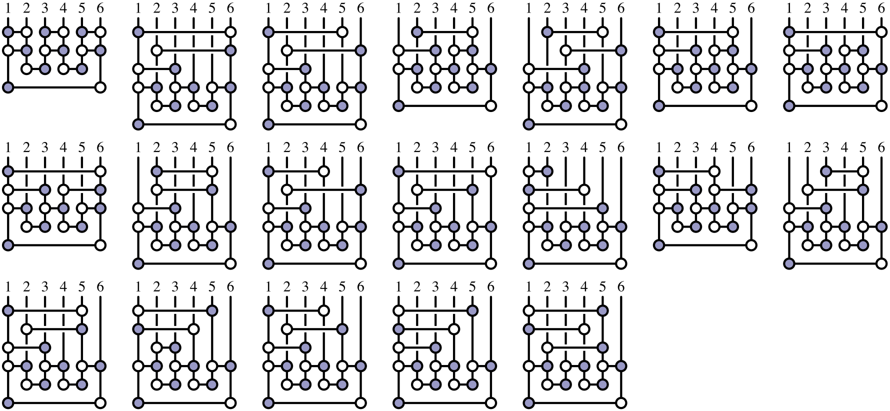

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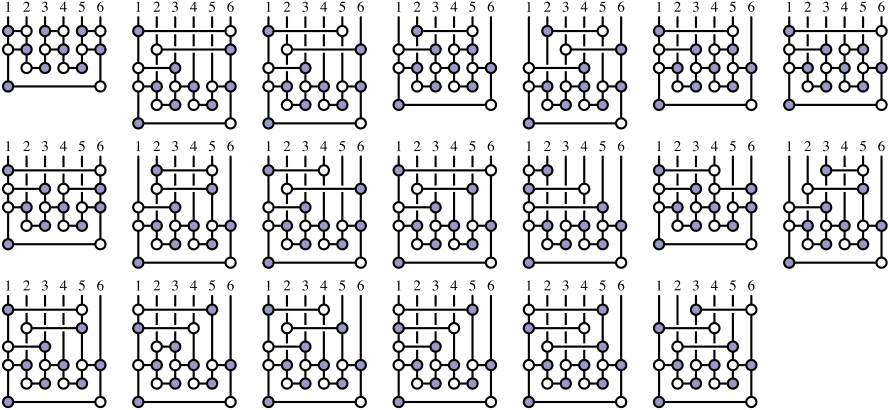

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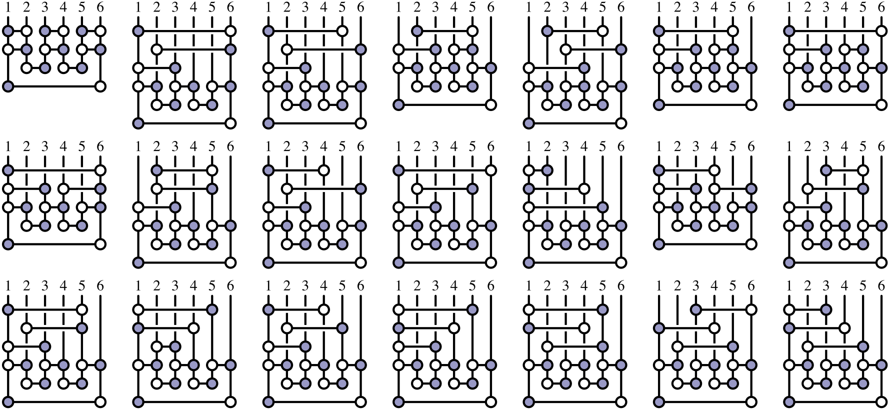

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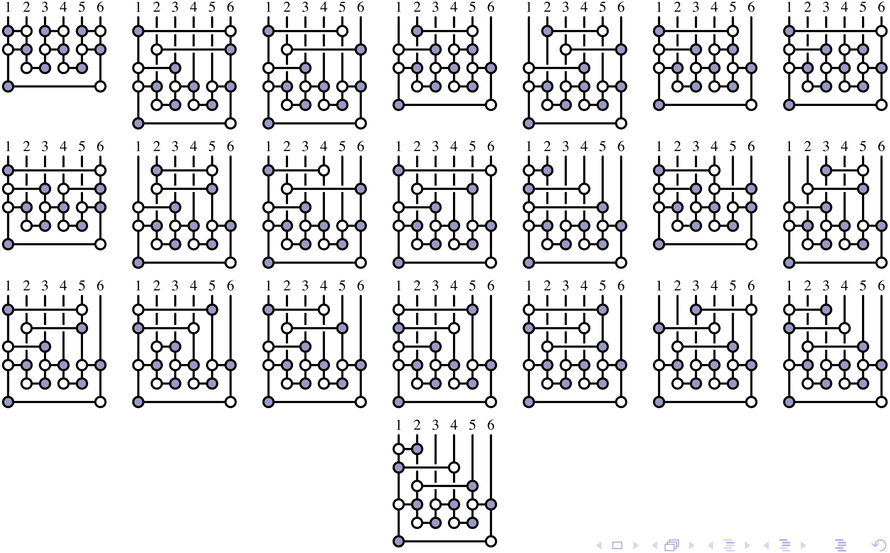

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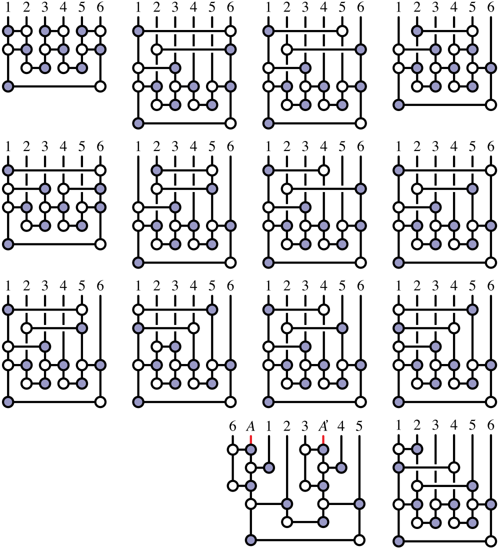

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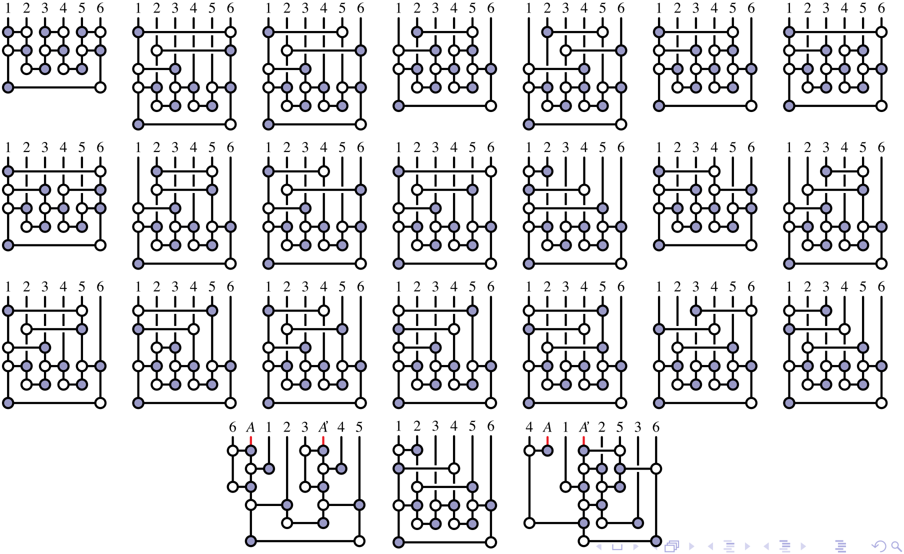

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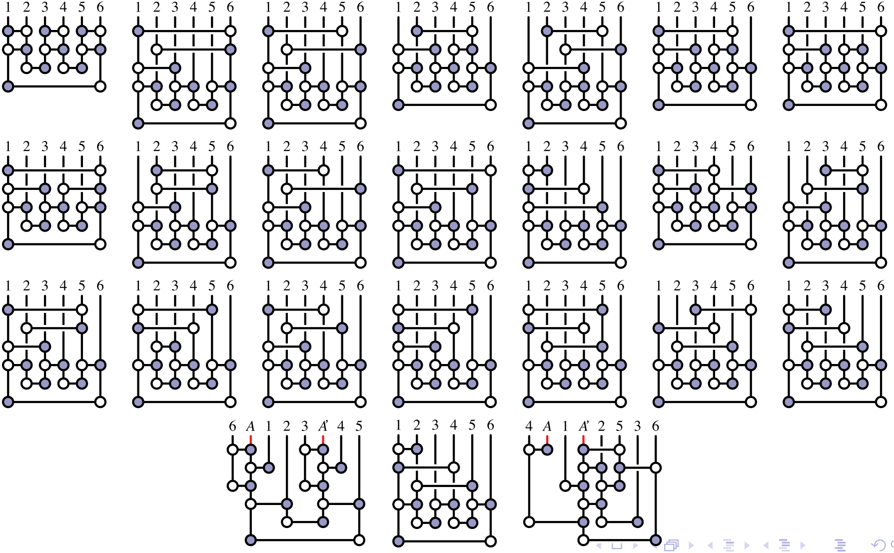

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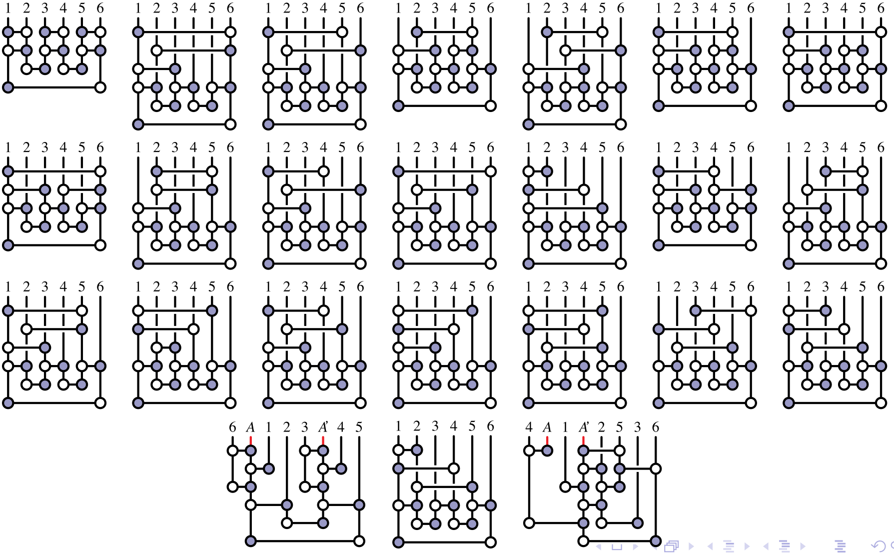

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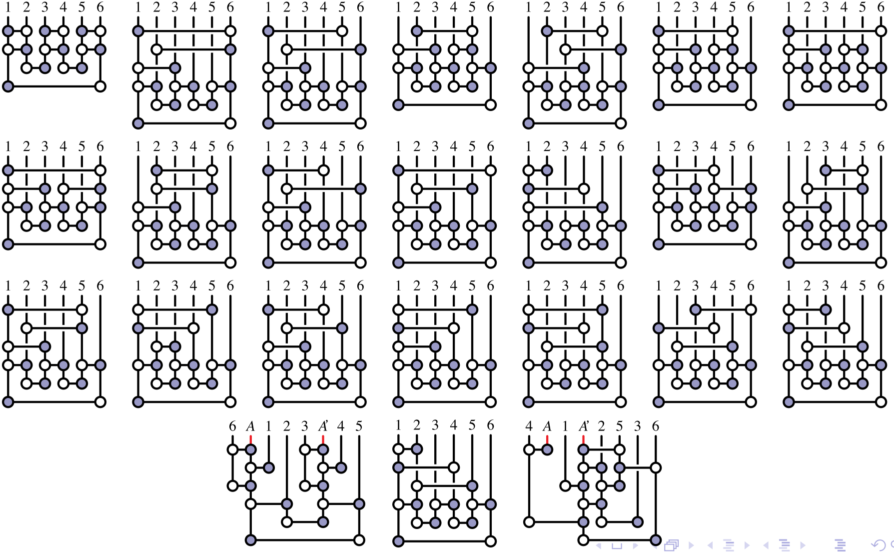

Enumeration of All (ten) 'Leading Singularities' of $G(3,6)$

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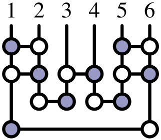
$$f_1 \equiv \oint\limits_{(123)=0} \Omega_1 = \left.\frac{\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(234)(345)(456)(561)(612)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$= \frac{\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 23\rangle\,[56]\,\langle 3|4{+}5|6]\,s_{456}\,\langle 1|5{+}6|4]\,\langle 12\rangle\,[45]}$$



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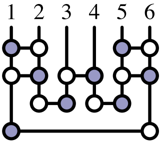
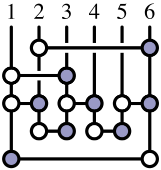


$$f_2 \equiv \oint_{(123)=0} \Omega_2 = \left.\frac{(235)\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(136)(156)(234)(245)(256)(345)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

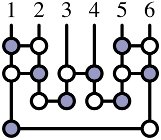
$$= \frac{\langle 23\rangle\,[64]\;\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 13\rangle\,[45]\,\langle 1|5{+}6|4]\langle 23\rangle\,[56]\,\langle 2|4{+}5|6]\langle 2|5{+}6|4]\langle 3|4{+}5|6]}$$



Enumeration of All (ten) 'Leading Singularities' of $G(3,6)$

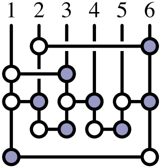
$$f_1 \equiv \oint_{(123)=0} \Omega_1 = \left.\frac{\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\!\cdot\!\widetilde{\lambda})}{(234)(345)(456)(561)(612)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$= \frac{\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\!\cdot\!\widetilde{\lambda})}{\langle 23\rangle\,[56]\,\langle 3|4{+}5|6]s_{456}\langle 1|5{+}6|4]\langle 12\rangle\,[45]}$$



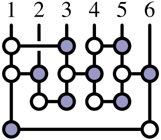
$$f_2 \equiv \oint_{(123)=0} \Omega_2 = \left.\frac{(235)\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\!\cdot\!\widetilde{\lambda})}{(136)(156)(234)(245)(256)(345)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$= \frac{\langle 23\rangle\,[64]\;\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\!\cdot\!\widetilde{\lambda})}{\langle 13\rangle\,[45]\,\langle 1|5{+}6|4]\langle 23\rangle\,[56]\,\langle 2|4{+}5|6]\langle 2|5{+}6|4]\langle 3|4{+}5|6]}$$



$$f_3 \equiv \oint_{(123)=0} \Omega_4 = \left.\frac{(145)\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\!\cdot\!\widetilde{\lambda})}{(124)(136)(156)(245)(345)(456)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

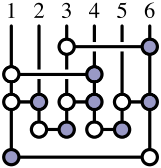
$$= \frac{\langle 1|4{+}5|6]\;\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\!\cdot\!\widetilde{\lambda})}{\langle 12\rangle\,[56]\,\langle 13\rangle\,[45]\,\langle 1|5{+}6|4]\langle 2|4{+}5|6]\langle 3|4{+}5|6]s_{456}}$$



Enumeration of All (ten) 'Leading Singularities' of $G(3,6)$

$$f_4 \equiv \oint_{(123)=0} \Omega_5 = \left.\frac{(135)\,\delta^{3\times4}\big(C^{*}\!\cdot\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{(124)(145)(156)(236)(345)(356)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

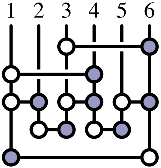
$$= \frac{\langle 13\rangle\,[64]\;\delta^{3\times4}\big(C^{*}\!\cdot\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{\langle 12\rangle\,[56]\,\langle 1|4{+}5|6]\langle 1|5{+}6|4]\langle 23\rangle\,[45]\,\langle 3|4{+}5|6]\langle 3|5{+}6|4]}$$



Enumeration of All (ten) 'Leading Singularities' of $G(3,6)$

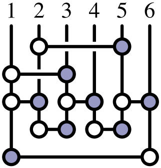
$$f_4 \equiv \oint_{(123)=0} \Omega_5 = \frac{(135)\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(124)(145)(156)(236)(345)(356)}\Bigg|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$= \frac{\langle 13\rangle\,[64]\;\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 12\rangle\,[56]\,\langle 1|4{+}5|6]\langle 1|5{+}6|4]\langle 23\rangle\,[45]\,\langle 3|4{+}5|6]\langle 3|5{+}6|4]}$$



$$f_5 \equiv \oint_{(123)=0} \Omega_9 = \frac{(125)\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(134)(156)(245)(256)(16(25)\bigcap(34))}\Bigg|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

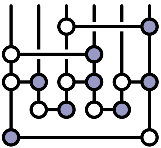
$$= \frac{\langle 12\rangle\,[64]\;\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 13\rangle\,[56]\,\langle 1|5{+}6|4]\langle 2|4{+}5|6]\langle 2|5{+}6|4]\big(\langle 23\rangle\,[56]\,\langle 1|5{+}6|4]-\langle 12\rangle\,[45]\,\langle 3|4{+}5|6]\big)}$$



Enumeration of All (ten) 'Leading Singularities' of $G(3,6)$

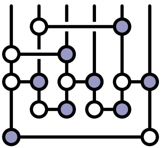
$$f_4 \equiv \oint_{(123)=0} \Omega_5 = \left.\frac{(135)\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(124)(145)(156)(236)(345)(356)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$= \frac{\langle 13\rangle\,[64]\;\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 12\rangle\,[56]\,\langle 1|4{+}5|6]\langle 1|5{+}6|4]\langle 23\rangle\,[45]\,\langle 3|4{+}5|6]\langle 3|5{+}6|4]}$$



$$f_5 \equiv \oint_{(123)=0} \Omega_9 = \left.\frac{(125)\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(134)(156)(245)(256)(16(25)\bigcap(34))}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

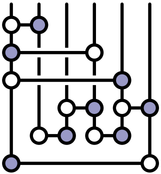
$$= \frac{\langle 12\rangle\,[64]\;\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 13\rangle\,[56]\,\langle 1|5{+}6|4]\langle 2|4{+}5|6]\langle 2|5{+}6|4]\big(\langle 23\rangle\,[56]\,\langle 1|5{+}6|4]-\langle 12\rangle\,[45]\,\langle 3|4{+}5|6]\big)}$$



$$C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$f_6 \equiv \oint_{(123)=0} \Omega_{12} = \left.\frac{(134)^2(456)\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(124)(145)(146)(156)(234)(345)(346)(356)}\right|_{C^*}$$

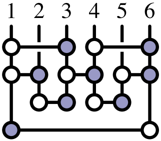
$$= \frac{\langle 13\rangle^2 s_{456}\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 12\rangle\langle 1|4{+}5|6]\langle 1|4{+}6|5]\langle 1|5{+}6|4]\langle 23\rangle\langle 3|4{+}5|6]\langle 3|4{+}6|5]\langle 3|5{+}6|4]}$$



Enumeration of All (ten) 'Leading Singularities' of $G(3,6)$

$$f_7 \equiv \oint\limits_{(123)=0} \Omega_{13} = \left.\frac{(145)^2\,\delta^{3\times4}\big(C^{*}\!\cdot\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{(125)(134)(146)(156)(245)(345)(456)}\right|_{C^{*}} \qquad C^{*}\equiv\begin{pmatrix}\lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1\\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2\\ 0 & 0 & 0 & [56] & [64] & [45]\end{pmatrix}$$

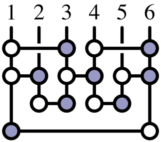
$$= \frac{\langle 1|4{+}5|6]^2\,\delta^{3\times4}\big(C^{*}\!\cdot\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{\langle 12\rangle\,[64]\,\langle 13\rangle\,[56]\,\langle 1|4{+}6|5]\langle 1|5{+}6|4]\langle 2|4{+}5|6]\langle 3|4{+}5|6]s_{456}}$$



Enumeration of All (ten) 'Leading Singularities' of $G(3,6)$

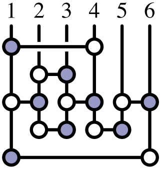
$$f_7 \equiv \oint_{(123)=0} \Omega_{13} = \left.\frac{(145)^2\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(125)(134)(146)(156)(245)(345)(456)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$= \frac{\langle 1|4{+}5|6]^2\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 12\rangle\,[64]\,\langle 13\rangle\,[56]\,\langle 1|4{+}6|5]\langle 1|5{+}6|4]\langle 2|4{+}5|6]\langle 3|4{+}5|6]\,s_{456}}$$



$$f_8 \equiv \oint_{(14(23)\cap(56))=0} \Omega_{16} = \int \frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_8}{\alpha_8}\;\delta^{3\times4}(C(\alpha)\cdot\widetilde{\eta})\,\delta^{3\times2}(C(\alpha)\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp}(\alpha))$$

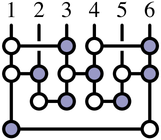
$$C(\alpha) \equiv \begin{pmatrix} 1 & \alpha_6 & \alpha_6\,\alpha_7 & 0 & 0 & \alpha_1 \\ 0 & 1 & \alpha_5{+}\alpha_7 & 0 & \alpha_2 & \alpha_2\,\alpha_4 \\ \alpha_8 & 0 & 0 & 1 & \alpha_3 & \alpha_3\,\alpha_4 \end{pmatrix}$$



Enumeration of All (ten) 'Leading Singularities' of $G(3,6)$

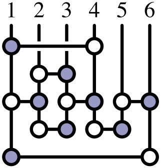
$$f_7\equiv\oint_{(123)=0}\Omega_{13}=\left.\frac{(145)^2\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(125)(134)(146)(156)(245)(345)(456)}\right|_{C^*}\qquad C^*\equiv\begin{pmatrix}\lambda_1^1&\lambda_2^1&\lambda_3^1&\lambda_4^1&\lambda_5^1&\lambda_6^1\\\lambda_1^2&\lambda_2^2&\lambda_3^2&\lambda_4^2&\lambda_5^2&\lambda_6^2\\0&0&0&[56]&[64]&[45]\end{pmatrix}$$

$$=\frac{\langle1|4{+}5|6]^2\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle12\rangle\,[64]\,\langle13\rangle\,[56]\,\langle1|4{+}6|5]\langle1|5{+}6|4]\langle2|4{+}5|6]\langle3|4{+}5|6]s_{456}}$$



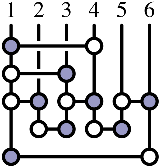
$$f_8\equiv\oint_{(14(23)\cap(56))=0}\Omega_{16}=\int\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_8}{\alpha_8}\;\delta^{3\times4}(C(\alpha)\cdot\widetilde{\eta})\,\delta^{3\times2}(C(\alpha)\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp}(\alpha))$$

$$C(\alpha)\equiv\begin{pmatrix}1&\alpha_6&\alpha_6\,\alpha_7&0&0&\alpha_1\\0&1&\alpha_5{+}\alpha_7&0&\alpha_2&\alpha_2\,\alpha_4\\\alpha_8&0&0&1&\alpha_3&\alpha_3\,\alpha_4\end{pmatrix}$$



$$f_9\equiv\oint_{(14(23)\cap(56))=0}\Omega_{18}=\int\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_8}{\alpha_8}\;\delta^{3\times4}(C(\alpha)\cdot\widetilde{\eta})\,\delta^{3\times2}(C(\alpha)\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp}(\alpha))$$

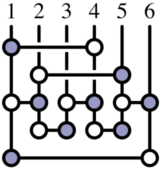
$$C(\alpha)\equiv\begin{pmatrix}1&\alpha_5&\alpha_7&0&0&\alpha_1\\0&1&\alpha_4&0&\alpha_2&\alpha_2\,\alpha_6\\\alpha_8&0&0&1&\alpha_3&\alpha_3\,\alpha_6\end{pmatrix}$$



Enumeration of All (ten) 'Leading Singularities' of $G(3,6)$

$$f_{10} \equiv \oint_{z=0} \Omega_{20} = \int \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_8}{\alpha_8} \; \delta^{3\times4}\big(C(\alpha)\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C(\alpha)\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}(\alpha)\big)$$

$$C(\alpha) \equiv \begin{pmatrix} \alpha_6\,\alpha_8 & \alpha_1 & 1 & \alpha_6 & \alpha_1\,\alpha_7 & 0 \\ \alpha_8 & 0 & 0 & 1 & \alpha_5 & \alpha_4 \\ \alpha_3 & \alpha_2 & 0 & 0 & \alpha_2\,\alpha_7 & 1 \end{pmatrix}$$



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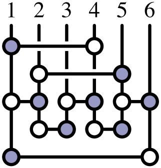
$$C(\alpha) \equiv \begin{pmatrix} \alpha_6\,\alpha_8 & \alpha_1 & 1 & \alpha_6 & \alpha_1\,\alpha_7 & 0 \\ \alpha_8 & 0 & 0 & 1 & \alpha_5 & \alpha_4 \\ \alpha_3 & \alpha_2 & 0 & 0 & \alpha_2\,\alpha_7 & 1 \end{pmatrix}$$



On-Shell Physics/Grassmannian Geometry Correspondence

$$f_\Gamma \equiv \prod_i \Big(\sum_{h_i, q_i} \int d^3\mathrm{LIPS}_i \Big) \prod_v \mathcal{A}_v \equiv \int \Omega_C \; \delta(C, p, h)$$

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On-Shell Physics

- on-shell diagrams
- physical symmetries
  - trivial symmetries (identities)

$\Leftrightarrow$

Grassmannian Geometry

- $\{$strata $C \in G(k,n)$, volume-form $\Omega_C\}$
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

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Important Open Questions (for math *and* physics)

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Grassmannian Geometry

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  - cluster coordinate mutations

Important Open Questions (for math *and* physics)

- how many functions exist?

On-Shell Physics/Grassmannian Geometry Correspondence

$$f_\Gamma \equiv \prod_i \Big(\sum_{h_i, q_i} \int d^3\mathrm{LIPS}_i \Big) \prod_v \mathcal{A}_v \equiv \int \Omega_C \; \delta(C, p, h)$$

On-Shell Physics

- on-shell diagrams
- physical symmetries
  - trivial symmetries (identities)

$\Leftrightarrow$

Grassmannian Geometry

- $\{$strata $C \in G(k,n)$, volume-form $\Omega_C\}$
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

Important Open Questions (for math *and* physics)

- how many functions exist? (how to name them?)

On-Shell Physics/Grassmannian Geometry Correspondence

$$f_\Gamma \equiv \prod_i \Big(\sum_{h_i, q_i} \int d^3\mathrm{LIPS}_i \Big) \prod_v \mathcal{A}_v \equiv \int \Omega_C \; \delta(C, p, h)$$

On-Shell Physics

- on-shell diagrams
- physical symmetries
  - trivial symmetries (identities)

$\Leftrightarrow$

Grassmannian Geometry

- $\{$strata $C \in G(k,n)$, volume-form $\Omega_C\}$
- volume-preserving diffeomorphisms
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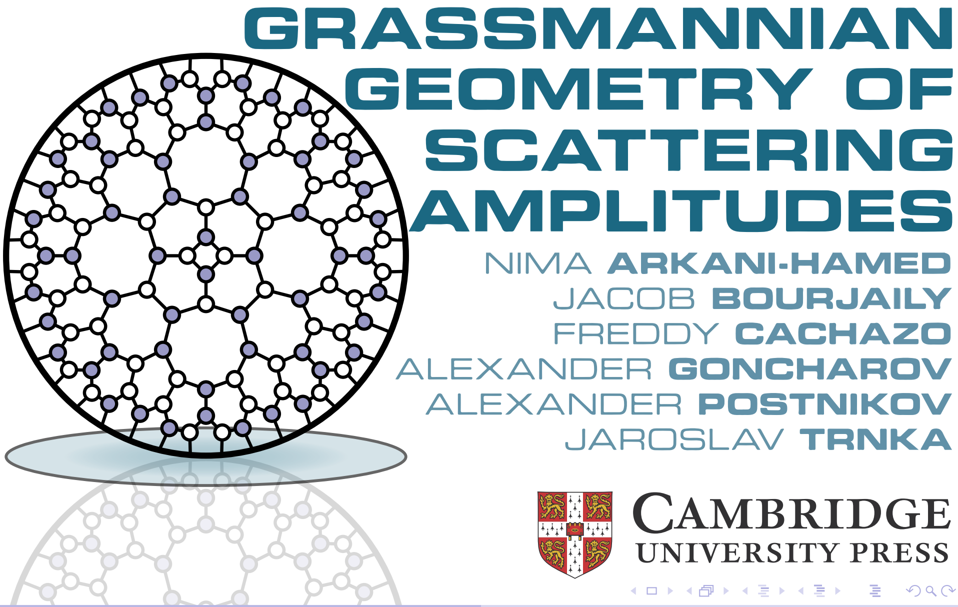
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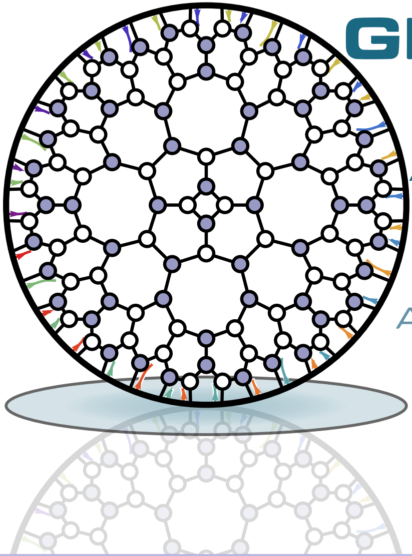

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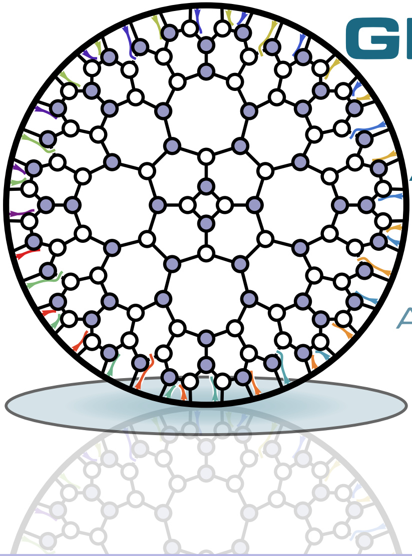

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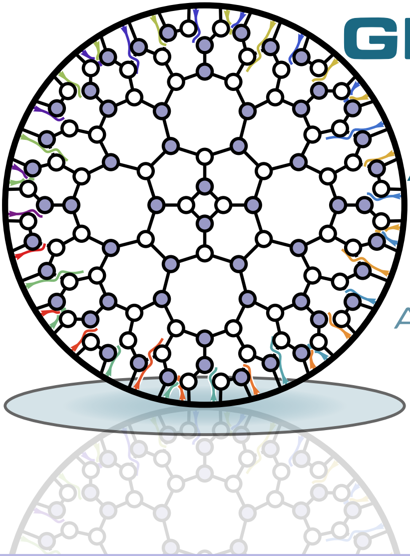

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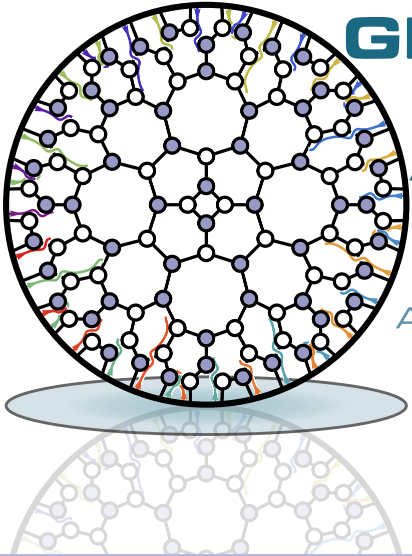

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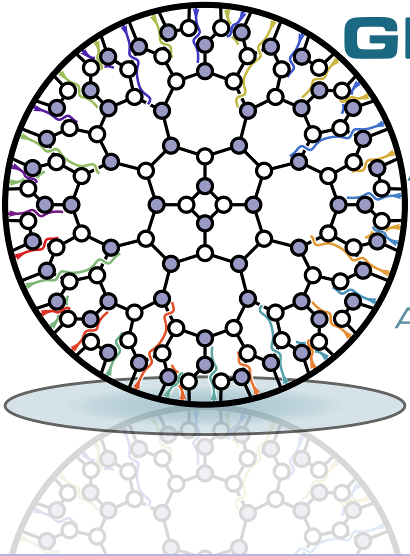

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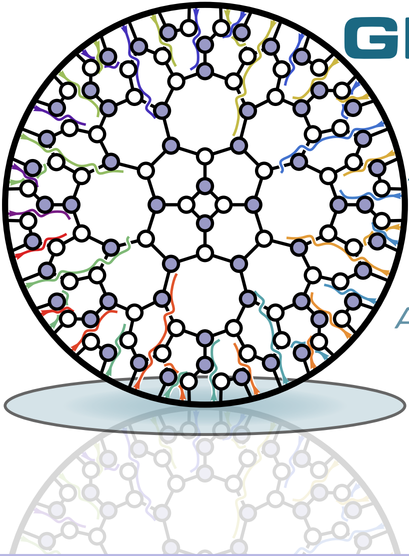

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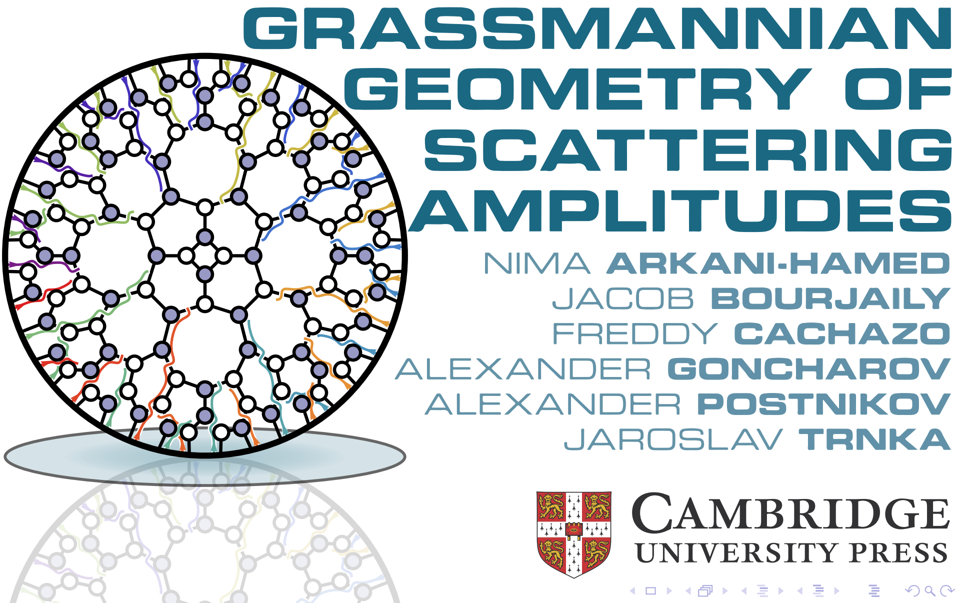
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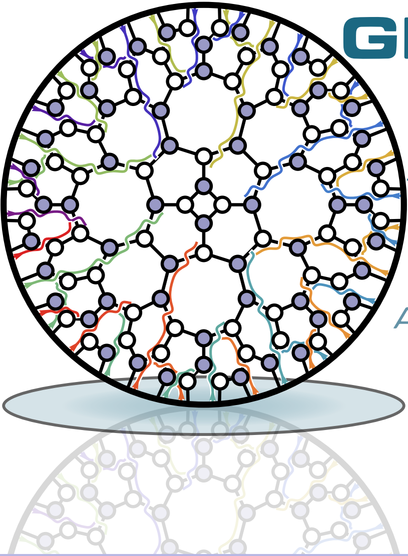

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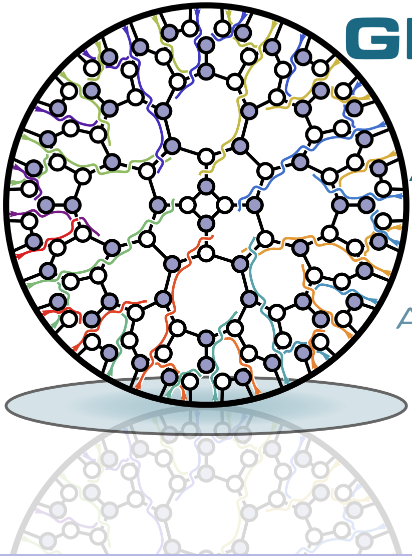

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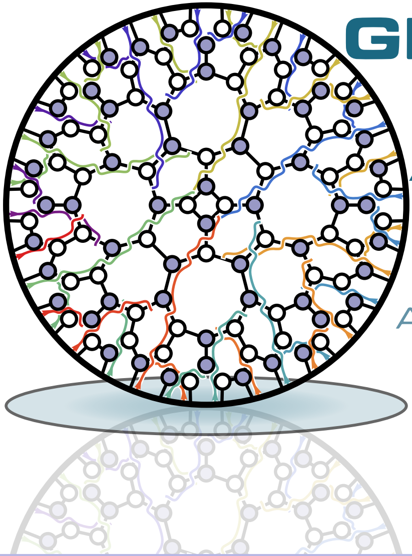

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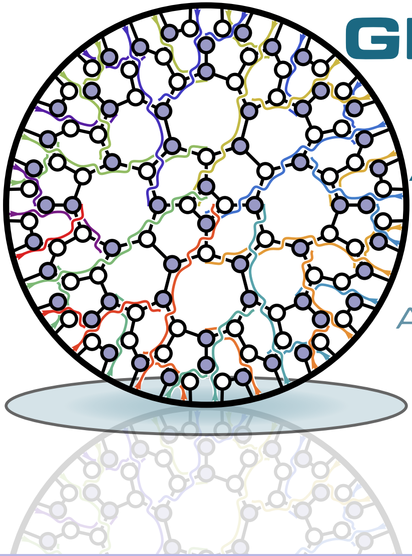

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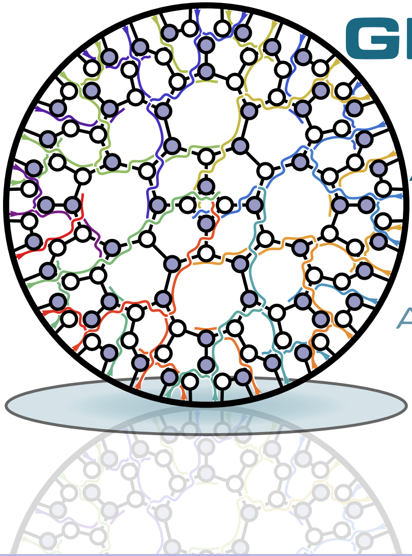

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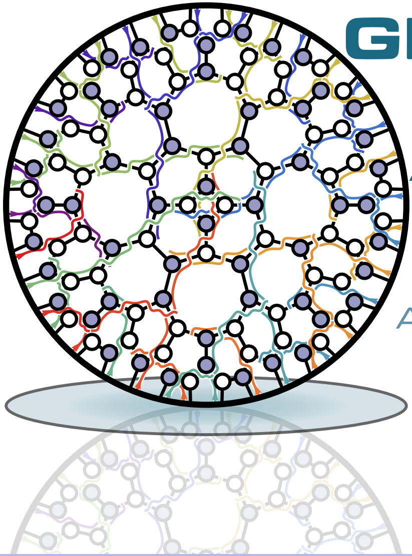

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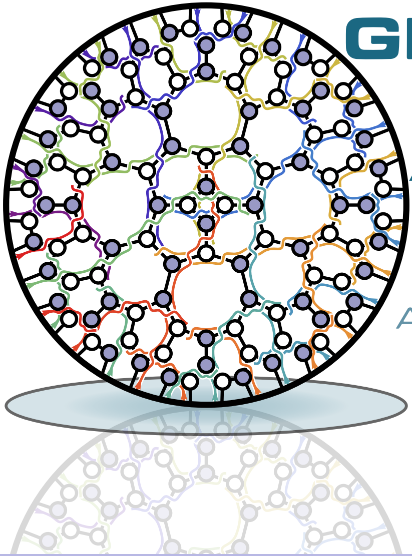

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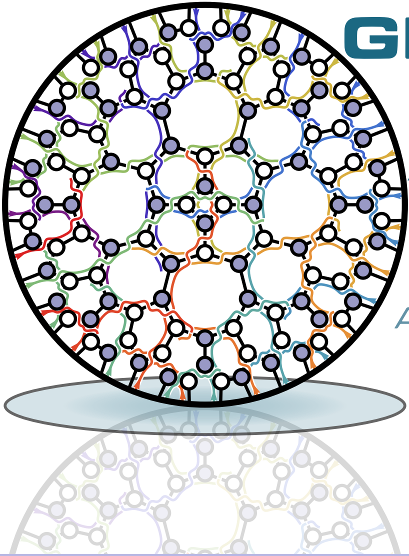

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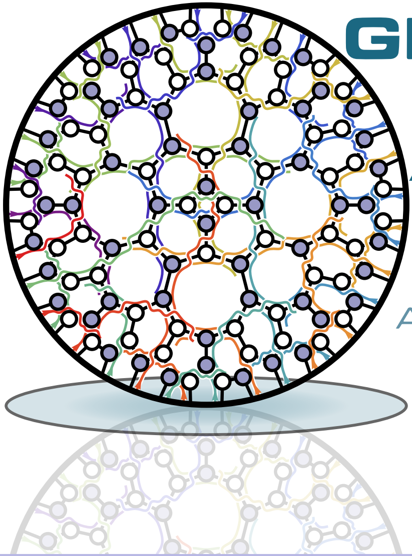

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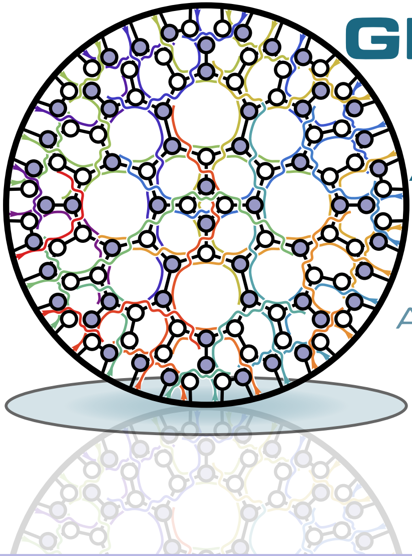

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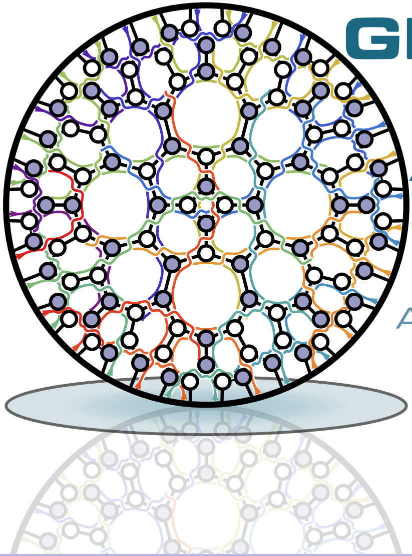

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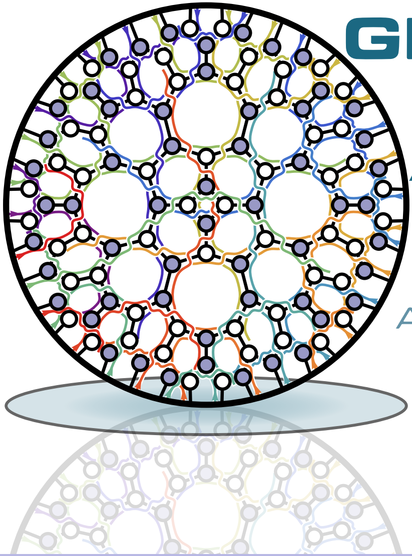

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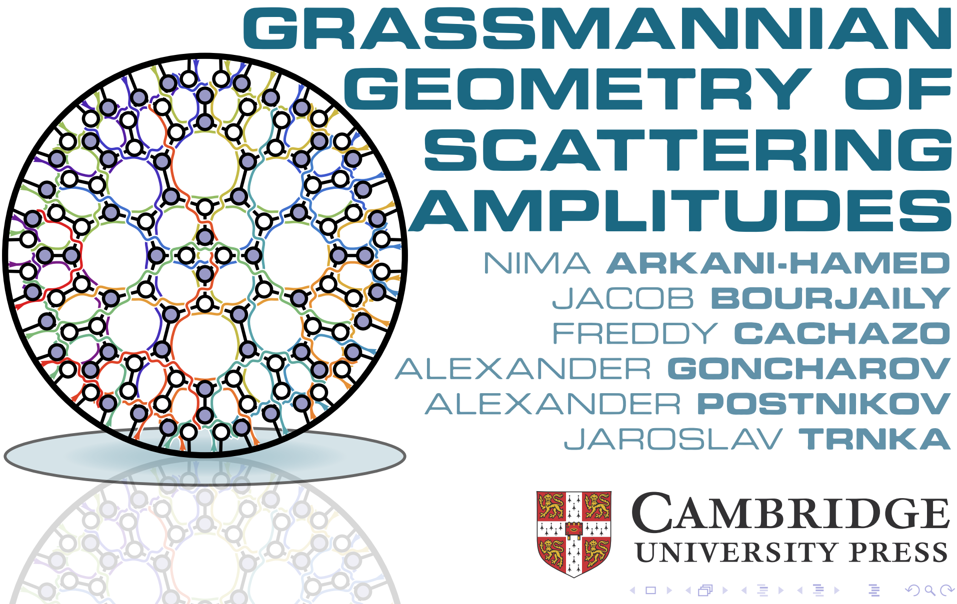
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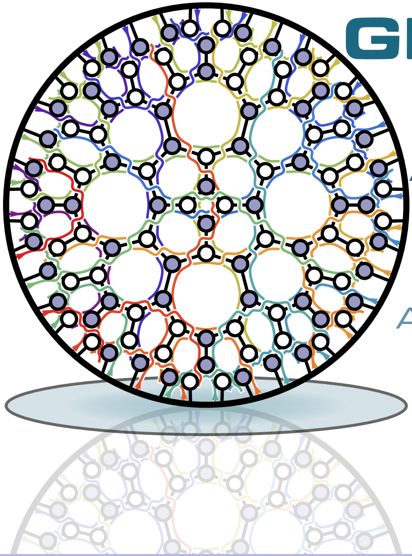

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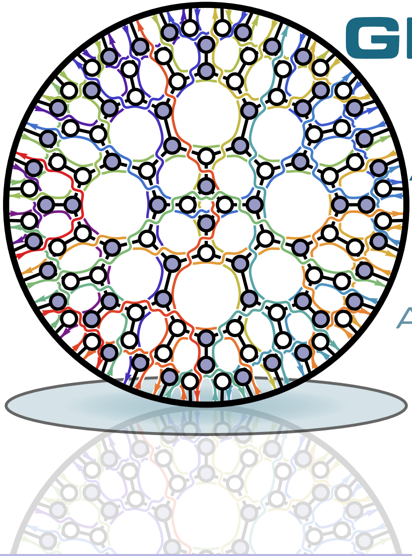

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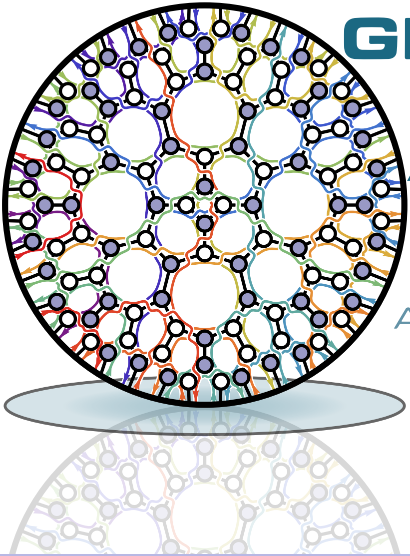

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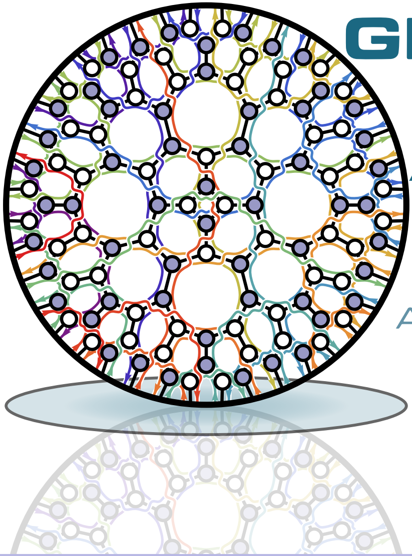

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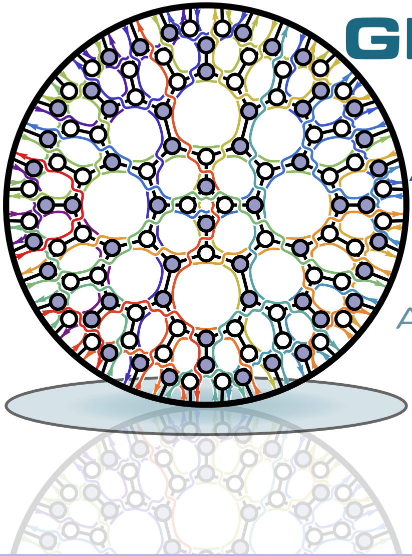

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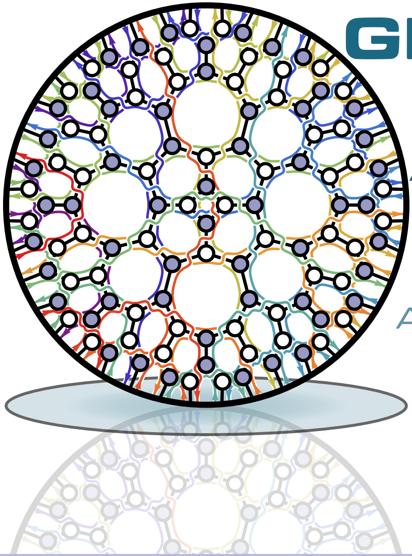

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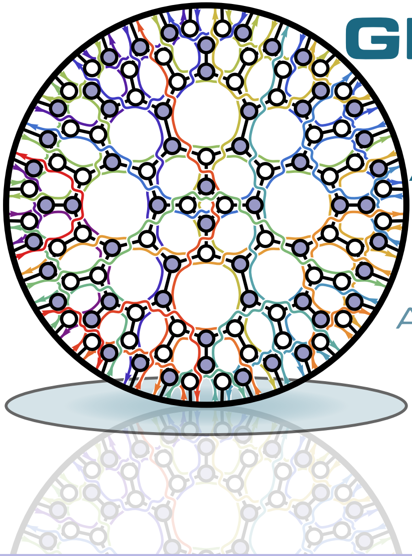

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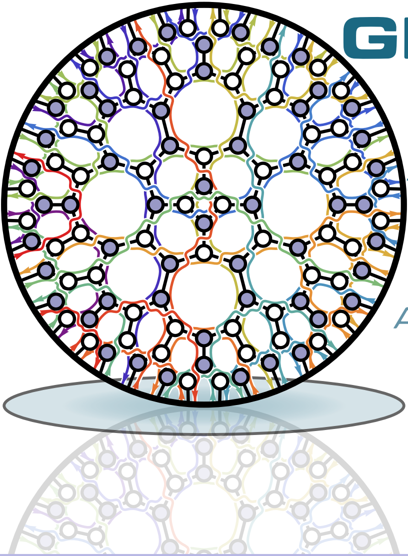

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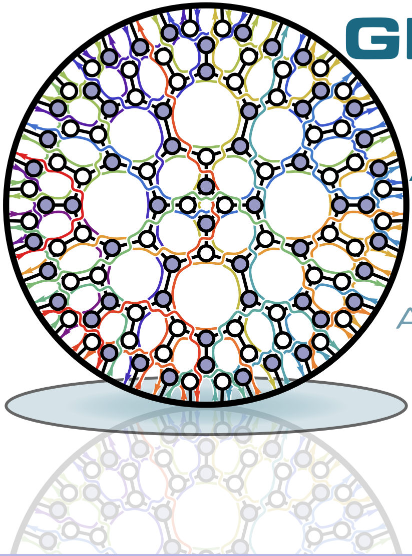

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