



# A SHORT INTRODUCTION TO SUPERSTRING THEORY

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## THE VENEZIANO AMPLITUDE

[Veneziano 1968]

$$A(s, t) = \frac{\Gamma(-\alpha(s)) \Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}$$

$$A(s, t) \simeq \frac{(-1)^n}{n! (n - 1 - \alpha' s)} \frac{\Gamma(-1 - \alpha' t)}{\Gamma(-1 - n - \alpha' t)}$$

polynomial of  
degree  $n$  in  $t$

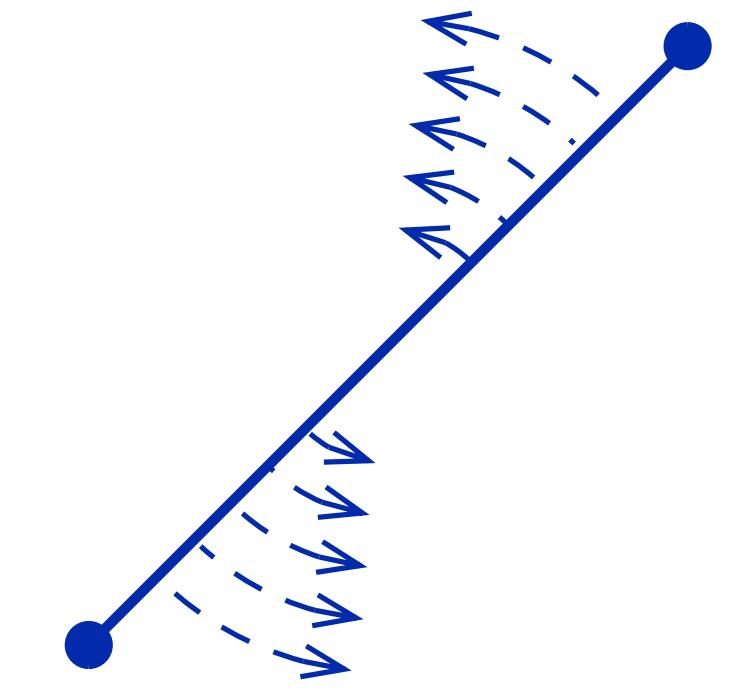
$$\alpha(y) = 1 + \alpha' y$$

$$M^2 = (n - 1)/\alpha'$$

$$A(s, t) \sim e^{-\alpha' s \log 2}$$

$$s, t \rightarrow \infty$$

Behind the Veneziano amplitude there is  
the mechanical model of a *vibrating string*



## RIGID RELATIVISTIC ROD

$$M = \int_{-\ell}^{\ell} \frac{\rho c^2 dr}{\sqrt{1 - v^2/c^2}} = \pi \rho c^2 \ell$$

$$J = \int_{-\ell}^{\ell} \frac{\rho c^2 r v dr}{\sqrt{1 - v^2/c^2}} = \frac{\pi}{2} \rho c^2 \ell^2$$

$$v/c = r/\ell$$

Regge behaviour

$$\frac{J}{M^2} = \frac{1}{\pi \rho c^2} \equiv \frac{1}{\alpha'}$$

## OUTLINE

(CLOSED) SUPERSTRINGS IN D=10

1

OPEN STRINGS AND D-BRANES

2

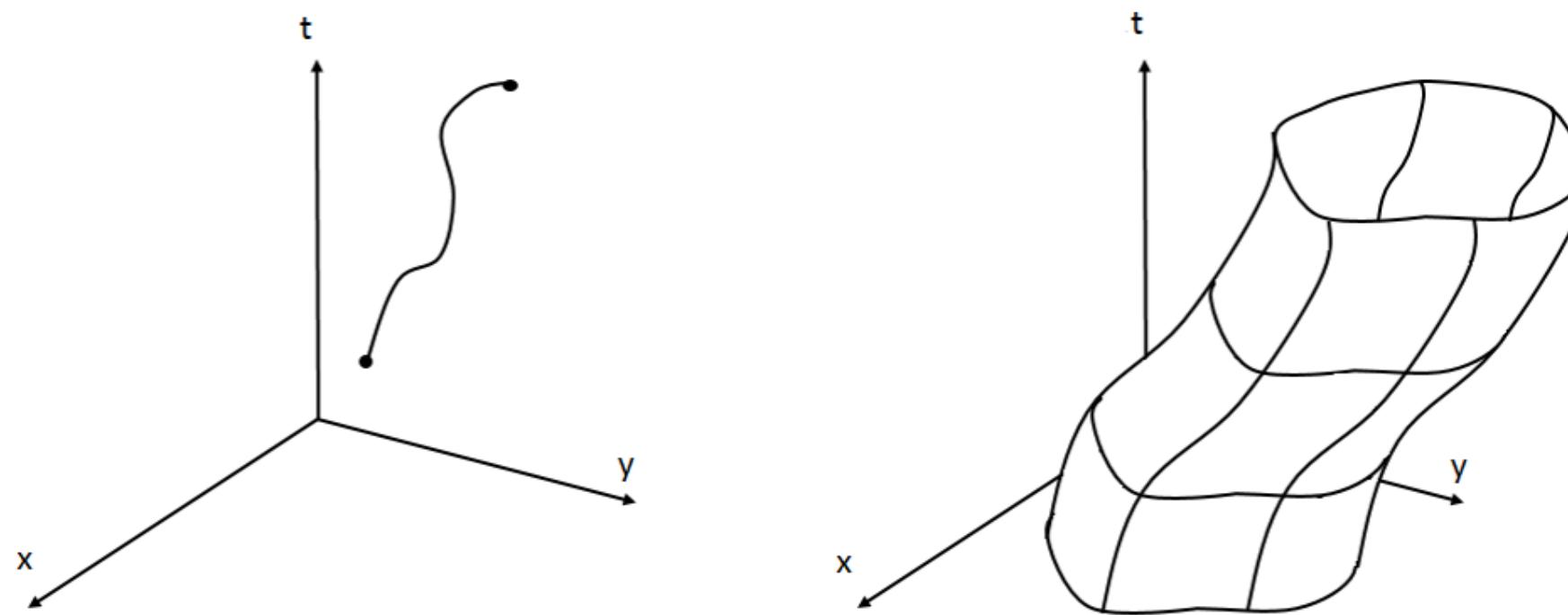
COMPACTIFICATION

3

SUPERSYMMETRY BREAKING

4

# THE MECHANICAL MODEL OF A VIBRATING STRING



$$X^\mu = X^\mu(\sigma, \tau)$$

$$\begin{aligned}\sigma &\in [0, \pi] \\ \tau &\in (-\infty, +\infty)\end{aligned}$$

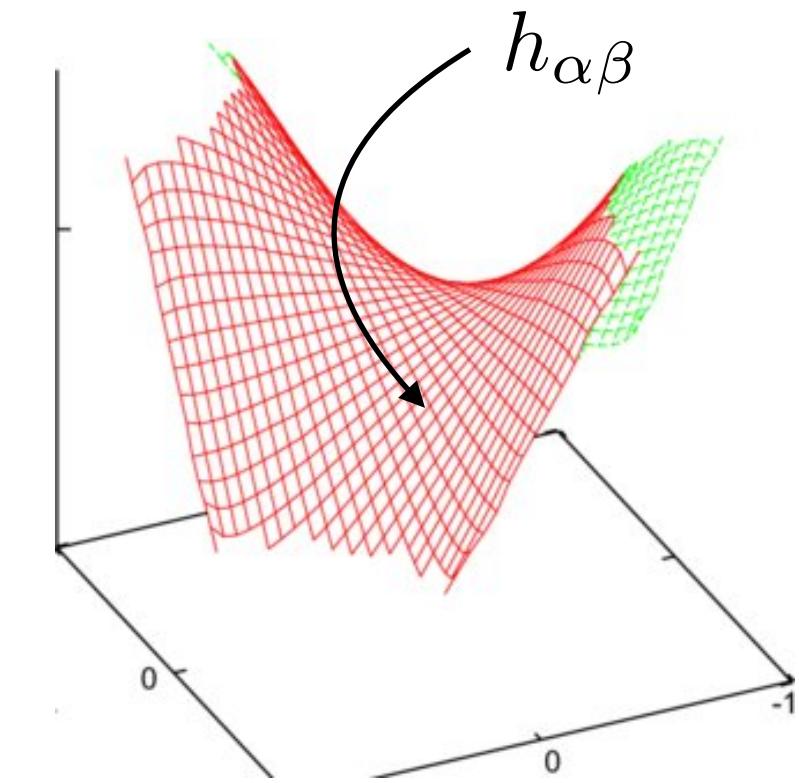
$$S = \text{Area} = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{\dot{X}^2 X'^2 - (\dot{X} \cdot X')^2}$$

Nambu-Goto Action

# THE MECHANICAL MODEL OF A VIBRATING STRING

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

(Brink-Di Vecchia-Howe) Polyakov Action



## SYMMETRIES

D-dimensional Poincaré

2d diffeomorphism

2d Weyl transformation

$$\delta X^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu$$

$$\delta\sigma^\alpha = \xi^\alpha(\sigma)$$

$$\delta h_{\alpha\beta} = \omega(\sigma) h_{\alpha,\beta}$$

## THE MECHANICAL MODEL OF A VIBRATING STRING

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

Equations of motion

$$\square X^\mu = 0 \qquad T_{\alpha\beta} = 0$$

(dynamics)

(constraint)

Boundary conditions

$$X^\mu(\sigma + \pi, \tau) = X^\mu(\sigma, \tau) \qquad \partial_\sigma X^\mu = 0 \quad \text{or} \quad \partial_\tau X^\mu = 0$$

(closed)

(open)

## CLOSED STRINGS

left and right waves decouple in 2 dimensions

$$X^\mu(\sigma, \tau) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma)$$

this simple observation is essential in the  
construction of the *heterotic string* ...

hold on!

## THE FERMIONIC STRING (SUPERSTRING)

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau [ \partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu} - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu ]$$

$$\delta X^\mu = \bar{\epsilon} \psi^\mu, \quad \delta \psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon$$

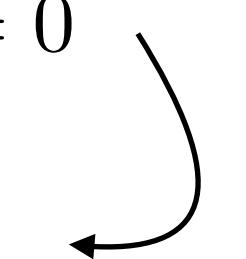
$$\rho_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

SUSY

Dirac equation decouples the two components

$$\psi^\mu(\sigma, \tau) = \begin{pmatrix} \psi_-(\tau - \sigma) \\ \psi_+(\tau + \sigma) \end{pmatrix}$$

$$\begin{aligned} \rho^\alpha \partial_\alpha \psi^\mu &= 0 \\ \partial_\pm \psi_\mp &= 0 \end{aligned}$$


Independent supersymmetries for left and right moving modes

$$\mathcal{N} = (\mathcal{N}_L, \mathcal{N}_R)$$

# QUANTISATION

$$X^\mu(\sigma, \tau) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-2in(\tau+\sigma)} + \frac{\tilde{\alpha}_n^\mu}{n} e^{-2in(\tau-\sigma)}$$

$$\psi_\pm^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^\mu(\tilde{b}_r^\mu) e^{-2ir\sigma^\pm}$$

anti-periodic (NS)

$$\psi_\pm^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} d_n^\mu(\tilde{d}_n^\mu) e^{-2in\sigma^\pm}$$

periodic (R)

# QUANTISATION

$$[\alpha_n^\mu, \alpha_m^\nu] = n \delta_{n+m} \eta^{\mu\nu}$$

$$\{b_r^\mu, b_s^\nu\} = \delta_{r+s} \eta^{\mu\nu}$$

anti-periodic (NS)

$$\{d_n^\mu, d_m^\nu\} = \delta_{n+m} \eta^{\mu\nu}$$

periodic (R)

$$(\alpha_{-n}^\mu)^k (b_{-r}^\nu)^\ell |0\rangle$$

states

$$(\alpha_{-n}^\mu)^k (d_{-m}^\nu)^\ell |0\rangle$$

How to impose the energy momentum constraint?

“Gupta-Bleuler” quantisation

$$T_{\alpha\beta} |\phi\rangle = 0$$

## LIGHT-CONE QUANTISATION

The gauge choice

$$h_{\alpha\beta} = \eta_{\alpha\beta}$$

does not fixes the gauge symmetry

$$(\delta_{\text{diff}} + \delta_{\text{Weyl}})\eta_{\alpha\beta} = 0 \quad \Rightarrow \quad \text{conformal symmetry}$$

This residual symmetry allows to solve the constraint  
only the transverse oscillators are physical

$$\alpha_n^\mu \rightarrow \alpha_n^i, \quad b_r^\mu \rightarrow b_r^i, \quad d_n^\mu \rightarrow d_n^i \quad i = 1, \dots, D-2$$

## THE SPECTRUM: THE NAÏVE WAY

$$M^2 = \frac{4}{\alpha'} \left[ N_X + \tilde{N}_X + N_\psi + \tilde{N}_\psi + \Delta_L + \Delta_R \right] \quad \textcolor{red}{mass-shell condition}$$

$$N_X + N_\psi + \Delta_L = \tilde{N}_X + \tilde{N}_\psi + \Delta_R \quad \textcolor{green}{level matching}$$

**R-R**

$|0, \tilde{0}\rangle$

$T$

$M^2 < 0$

$b_{-1/2}^i \tilde{b}_{-1/2}^j |0, \tilde{0}\rangle$

$g_{\mu\nu}, B_{\mu\nu}, \phi$

$M^2 = 0 \quad \text{IF} \quad D = 10$

Ramond vacuum:  $|\Omega\rangle = |8_s\rangle + |8_c\rangle$

**NS-NS**

$|\Omega, \tilde{\Omega}\rangle = (|8_s\rangle + |8_c\rangle) \otimes (|\tilde{8}_s\rangle + |\tilde{8}_c\rangle)$

$M^2 = 0$

**R-NS and NS-R**

$b_{-1/2}^i |0\rangle \otimes (|\tilde{8}_s\rangle + |\tilde{8}_c\rangle)$

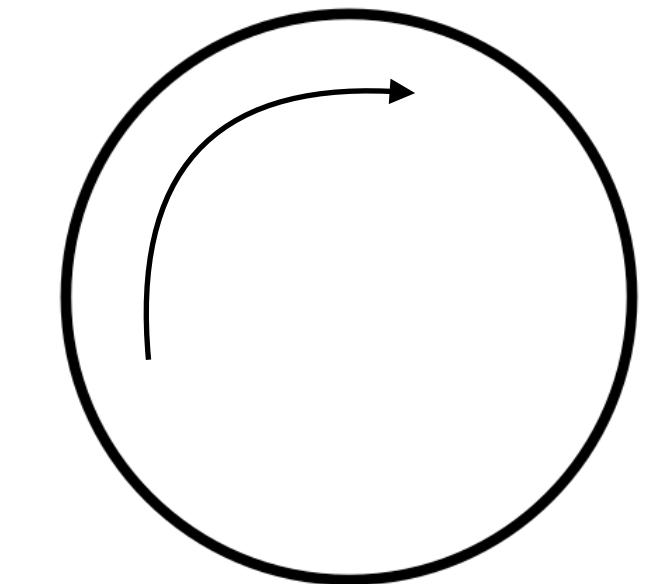
$M^2 = 0$

spacetime fermions

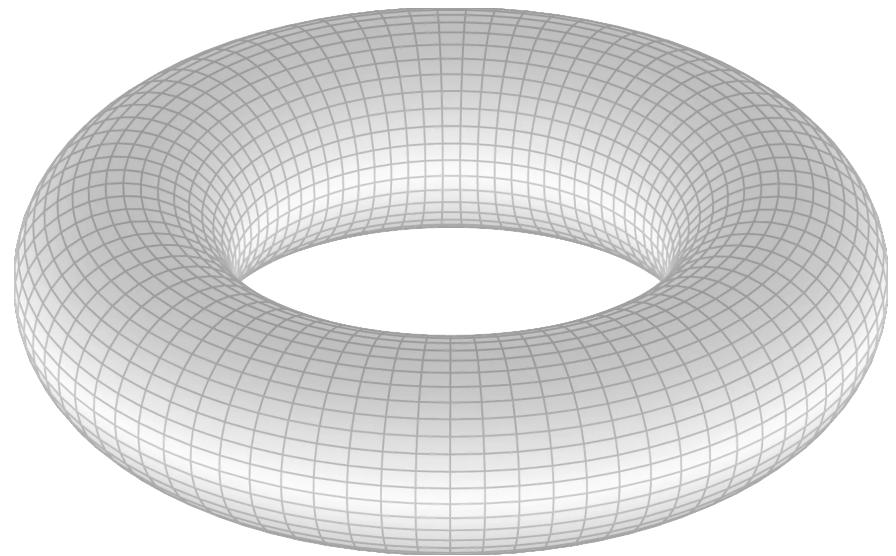
# TOWARDS THE CORRECT SPECTRUM

Vacuum to vacuum amplitude in Quantum Field Theory

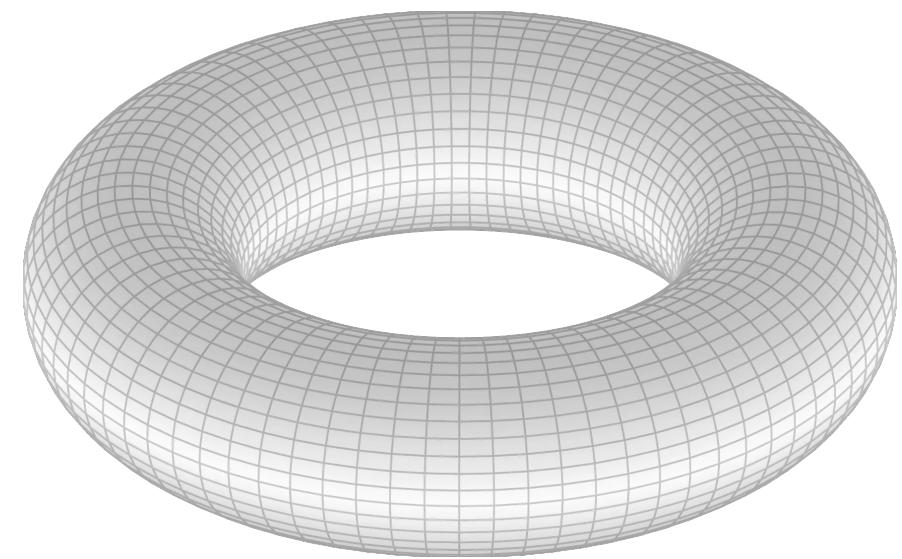
$$Z = \int_0^\infty \frac{dt}{t} \text{tr} \left[ e^{-\pi t(\square + M^2)} \right]$$



In string theory ...



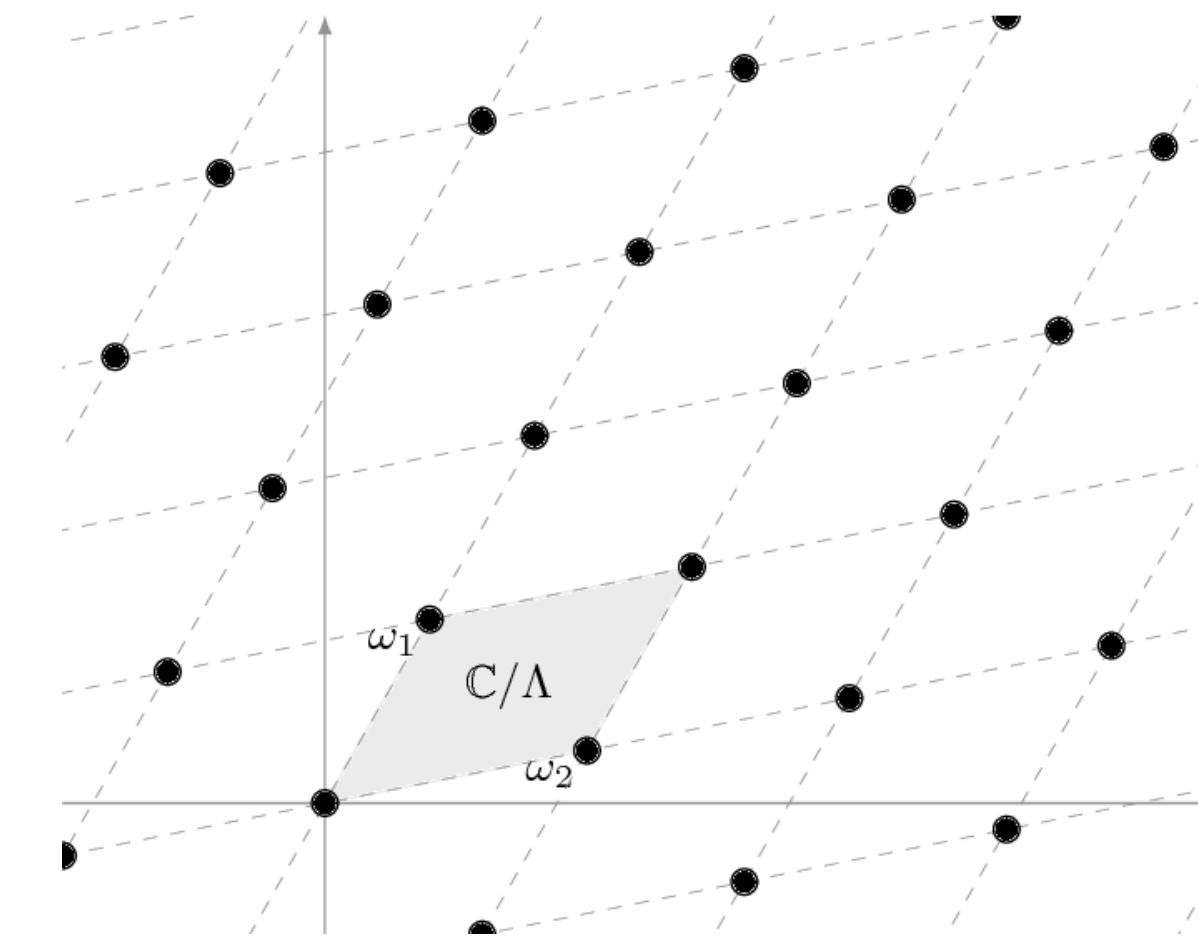
# THE TORUS



homeomorphism of the torus:

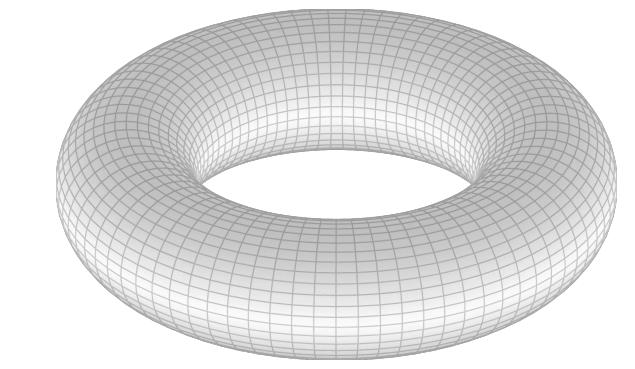
$$\text{SL}(2; \mathbb{Z}) : \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

$$\left[ \tau \equiv \frac{\omega_1}{\omega_2} \rightarrow \frac{a\tau + b}{c\tau + d} \right]$$



## GEOMETRICAL CONSTRAINT

The spectrum of closed-(super)string theory  
*must not depend*  
on the choice of elementary cell for the one-loop torus amplitude

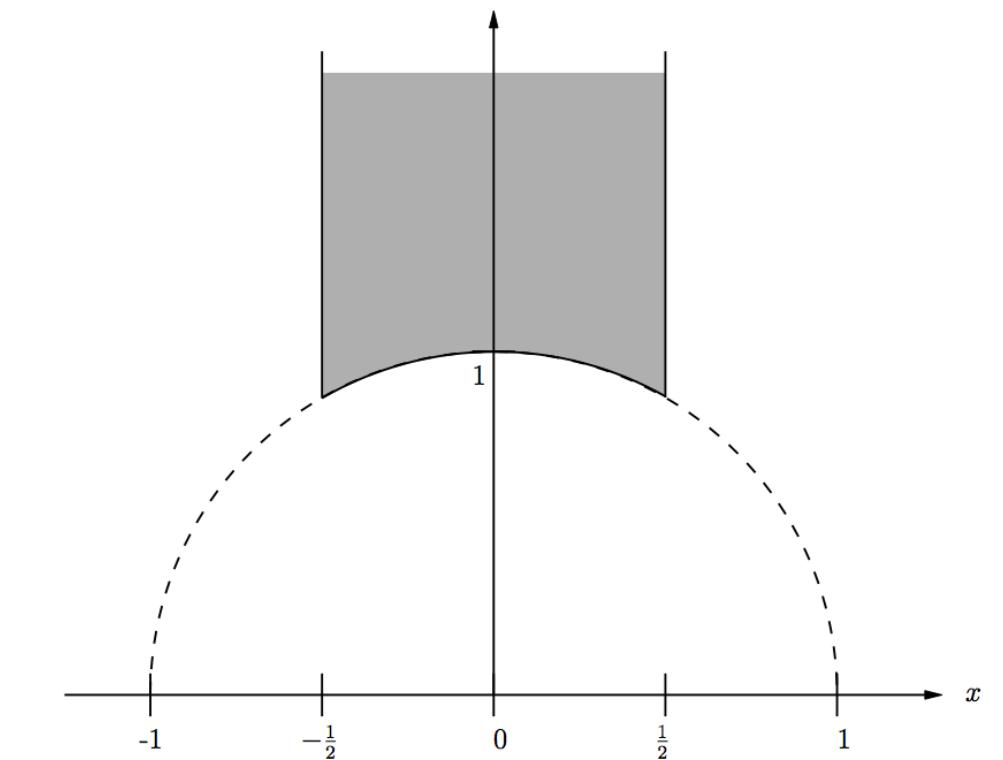


$$\text{tr} \left[ e^{-\pi t(\square + M^2)} \right]$$

must be invariant under the action of the **modular group**  $\text{SL}(2; \mathbb{Z})$

## THE SPECTRUM: THE CORRECT WAY — MODULAR INVARIANCE

$$Z = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left[ NS \otimes \tilde{NS} + R \otimes \tilde{R} - NS \otimes \tilde{R} - R \otimes \tilde{NS} \right]$$



Modular invariance highly constrains the way  
left and right moving oscillators can be combined

In D=10 only two (supersymmetric solutions): **IIA** and **IIB** superstrings

No SUSY: **0A** and **0B**

## **TYPE IIA SUPERSTRING**

$$Z = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^6} \frac{1}{(\eta\bar{\eta})^8} (V_8 - S_8)(\bar{V}_8 - \bar{C}_8)$$

At low energy: **N=(1,1) (type IIA) SUPERGRAVITY** in D=10

SUGRA :  $\{g_{\mu\nu}, \phi, B_{\mu\nu}, C_\mu, C_{\mu\nu\rho}; \psi_L^\mu, \psi_R^\mu, \lambda_L, \lambda_R\}$

## **TYPE IIB SUPERSTRING**

$$Z = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^6} \frac{1}{(\eta\bar{\eta})^8} (V_8 - S_8)(\bar{V}_8 - \bar{S}_8)$$

At low energy:  **$N=(2,0)$  (type IIB) SUPERGRAVITY in D=10**

SUGRA :  $\{g_{\mu\nu}, \phi, B_{\mu\nu}, C, C_{\mu\nu}, C_{\mu\nu\rho\sigma}^{(+)}; 2\psi_L^\mu, 2\lambda_R\}$

## BRANES IN TYPE II SUPERSTRINGS

Notice that type II superstrings include  $(p+1)$ -forms  
among the massless excitations

$(p+1)$ -form potentials are probed by  
 $p$ -dimensional extended objects: ***p-branes***

**String theory is not a theory of just strings!**

## THE HETEROtic STRINGS

Recall the decoupling of left and right moving modes?

$$X^\mu(\sigma, \tau) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma)$$

This allows for generalisation of the previous construction

## CLOSED STRINGS

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$$X^\mu(\sigma, \tau) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma)$$

this simple observation is essential in the  
construction of the *heterotic string* ...

hold on!

## THE FERMIONIC STRING (SUPERSTRING)

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau [ \partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu} - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu ]$$

$$\delta X^\mu = \bar{\epsilon} \psi^\mu, \quad \delta \psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon$$

SUSY

Dirac equation decouples the two components

$$\psi^\mu(\sigma, \tau) = \begin{pmatrix} \psi_-(\tau - \sigma) \\ \psi_+(\tau + \sigma) \end{pmatrix}$$

$$\partial_- \psi_+ = 0$$

$$\partial_+ \psi_- = 0$$

Independent supersymmetries for left and right moving modes

$$\mathcal{N} = (1, 1)$$

$$\begin{aligned} \delta X_{R,L} &= i\epsilon_\pm \psi_\pm \\ \delta \psi_\pm &= \epsilon_\pm \partial_\pm X \end{aligned}$$

## THE HETEROtic STRINGS

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left[ \partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu} - 2i\psi_-^\mu \partial_+ \psi_-{}_\mu - 2i\lambda_+^A \partial_- \lambda_+^A \right] \quad A = 1, \dots, 32$$

This two-dimensional action has only a right-moving supersymmetry,  
while the left-moving waves provide extra degrees of freedom

$$\mathcal{N} = (0, 1)$$

The light spectrum (an example)

$$\alpha_{-1}^i \tilde{b}_{-1/2}^j |0, \tilde{0}\rangle \rightarrow g_{\mu\nu}, B_{\mu\nu}, \phi$$

$$\lambda_{-1/2}^A \lambda_{-1/2}^B \tilde{b}_{-1/2}^j |0, \tilde{0}\rangle \rightarrow A_\mu^{[AB]}$$

## THE HETEROtic STRINGS

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left[ \partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu} - 2i\psi_-^\mu \partial_+ \psi_-{}_\mu - 2i\lambda_+^A \partial_- \lambda_+^A \right] \quad A = 1, \dots, 32$$

This two-dimensional action has only a right-moving supersymmetry,  
while the left-moving waves provide extra degrees of freedom

$$\mathcal{N} = (0, 1)$$

The light and heavy spectrum:

Again, modular invariance of the one-loop vacuum amplitude  
highly constrains the spectrum

# THE HETEROtic STRINGS

SO(32) string

$$Z = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^6} \frac{1}{(\eta\bar{\eta})^8} (V_8 - S_8)(\bar{O}_{32} + \bar{S}_{32})$$

$E_8 \times E_8$  string

$$Z = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^6} \frac{1}{(\eta\bar{\eta})^8} (V_8 - S_8)(\bar{O}_{16} + \bar{S}_{16})(\bar{O}_{16} + \bar{S}_{16})$$

At low energy: **N=(1,0) SUPERGRAVITY** in D=10 with gauge group

$$G = SO(32) \quad \text{or} \quad E_8 \times E_8$$

SUGRA :  $\{g_{\mu\nu}, \phi, B_{\mu\nu}; \psi_L^\mu, \lambda_R\}$

VECTOR :  $\{A_\mu; \lambda_L\}$

## THE STRING COUPLING CONSTANT

String perturbation theory is described by the path integral

$$\begin{aligned}\mathcal{A} &= \sum_{g=0}^{\infty} \int_{\Sigma_g} \mathcal{D}X \mathcal{D}h e^{-\int (\partial X)^2 - \int \phi R^{(2)}} V_1 \dots V_n \\ &= \sum_{g=0}^{\infty} g_s^{2g-2} \int_{\Sigma_g} \mathcal{D}X \mathcal{D}h e^{-\int (\partial X)^2} V_1 \dots V_n\end{aligned}$$

The string coupling constant is  $g_s = e^{\langle \phi \rangle}$

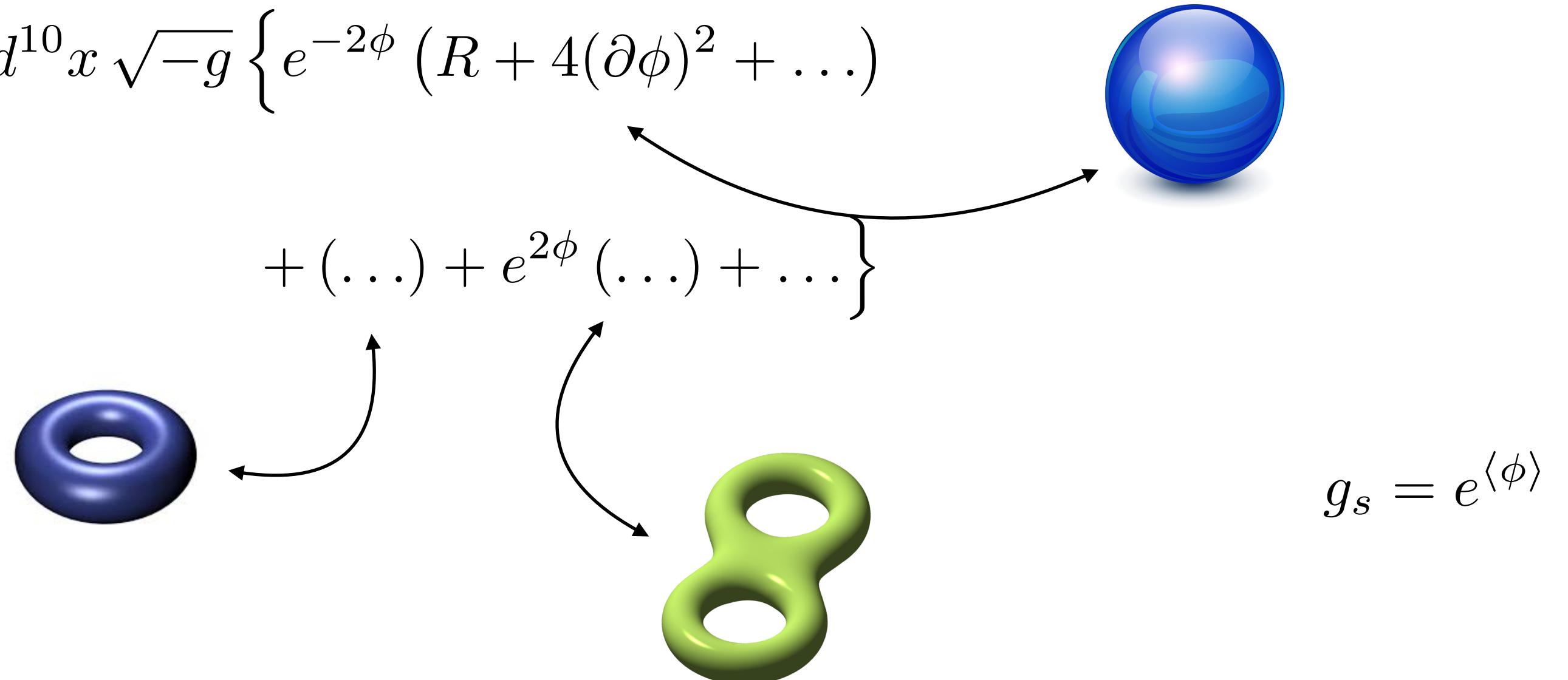
# THE LOW-ENERGY EFFECTIVE ACTION

The *Einstein frame*

$$S = \int d^{10}x \sqrt{-g} (R + (\partial\phi)^2 + \dots)$$

The *string frame*

$$S = \int d^{10}x \sqrt{-g} \left\{ e^{-2\phi} (R + 4(\partial\phi)^2 + \dots) + (\dots) + e^{2\phi} (\dots) + \dots \right\}$$



# THE LEEA FOR THE HETEROtic STRINGS

$$S = \int d^{10}x \sqrt{-g_H} e^{-2\phi_H} \left( R(g_H) + 4(\partial\phi_H)^2 - \tfrac{1}{12}H_H^2 - \tfrac{1}{4}\text{Tr}F_H^2 + \text{fermions} \right)$$



in the *Einstein frame*

$$g_{H\mu\nu} \rightarrow e^{\phi_H/2} g_{H\mu\nu}$$

$$S = \int d^{10}x \sqrt{-g_H} \left( R(g_H) + (\partial\phi_H)^2 - \tfrac{1}{12}e^{-\phi_H} H_H^2 - \tfrac{1}{4}e^{-\phi_H/2}\text{Tr}F_H^2 + \text{fermions} \right)$$

## **SUMMARY PART ONE**

**CONSISTENT SUPERSTRING VACUA RESPECT MODULAR INVARIANCE**

**DECOUPLING OF WAVES ALLOWS EXOTIC CONSTRUCTIONS**

**STRING THEORY IS NOT ONLY A THEORY OF STRINGS**

**THE DILATON PLAYS THE ROLE OF STRING COUPLING CONSTANT**