SYMPLECTIC DUALITY AND THE AFFINE GRASSMANNIAN: PROBLEM SET 1

All algebraic varieties will be over $\mathbb{C}$.

1. Recall that for a conical symplectic resolution $\pi : \widetilde{X} \to X$ we denote by $\mathfrak{s}_X$ the Cartain algebra of the group of symplectic automorphisms of X (commuting with the $\mathbb{C}^*$-action) and by $\mathfrak{t}_X$ the vector space $H^2(\widetilde{X}, \mathbb{C})$ (which is supposed to depend only on X). In the following examples of supposedly dual pairs verify that $\mathfrak{s}_X, \mathfrak{t}_X$ are swapped (on the level of dimensions). Verify also that $\dim H^*(\widetilde{X}, \mathbb{C}) = \dim H^*(\widetilde{X}^*, \mathbb{C})$ in all of these examples, but they are not necessarily isomorphic as algebras. In all of these examples describe the deformation of X (resp. of X^*) parametrized by $\mathfrak{t}_X$ (resp. by $\mathfrak{t}_{X^*}$).

a) $\widetilde{X} = T^*\mathbb{P}^{n-1}$ (what is X in this case?), $X^* = \mathbb{C}^2/\mathbb{Z}_n$, $\widetilde{X}^*$ - its "minimal resolution" (here $\mathbb{Z}_n$ acts on $\mathbb{C}^2$ "hyperbolically").

b) $X = X^* = Sym^(\mathbb{C}^2), \widetilde{X} = \widetilde{X}^* = Hilb^n(\mathbb{C}^2)$ (the last thing is the Hilbert scheme of points on $\mathbb{C}^2$, which classifies ideals of codimension n in $\mathbb{C}[x, y]$).

2. Let $\widetilde{X} = T^* \operatorname{Gr}(k, n)$ (the Grassmannian of k-planes in $\mathbb{C}^n$). Describe X, the spaces $\mathfrak{t}_X, \mathfrak{s}_X$ and the corresponding deformation.

3. Let $G = SL(2)$. Consider the affine Grassmannian $\operatorname{Gr}_G = G((t))/G[[t]]$. For a number $n \geq 0$ let Gr_G^n be the $G[[t]]$ orbit of the element g_n

$$\begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}$$

Prove by an explicit calculation that $\dim \operatorname{Gr}_G^n = 2n$ and the the closure of Gr_G^n contains Gr_G^m if and only if $n \geq m$.

4. Show that the cohomology of $\overline{\operatorname{Gr}}_G^n$ has dimension $2n + 1$. Try to prove that in this case cohomology coincides with intersection cohomology. Thus the intersections cohomology of $\overline{\operatorname{Gr}}_G^n$ looks like the representation of $PGL(2)$ with highest weight $2n$.

5. Let us still be in the setting of problem 4. Let $n \geq m$. Let W_m be the orbit of g_m under the group $G[t^{-1}]_1$ - the subgroup of $G[t^{-1}]$ consisting of elements whose value at ∞ is 1. Let W_m^n be the intersection of W_m with Gr_G^n, $\overline{W}_m^n$ – the intersection of W_m with the closure Gr_G^n. Show that this is a transversal slice to Gr_G^m inside $\overline{\operatorname{Gr}}_G^n$. Compute explicitly the surface $\overline{W}_{n-2}^n$ (and show that it is a Kleinian singularity).

6. Let $G = GL(n)$. Describe explicitly the orbits $\operatorname{Gr}_G^\lambda$ when λ is the i-th fundamental weights of $GL(n)$. Show that it is closed and that it is isomorphic to the Grassmannian $\operatorname{Gr}(i, n)$. What does it tell you about the relation between cohomology of this orbit and the corresponding fundamental representation of $GL(n)$?

7*. Consider the equivaiant derived category $D_{\mathbb{C}^*}(\mathbb{C})$. Show (at least by an a not completely formal argument) that it is equivalent to the derived category of the abelian category of $\mathbb{C}^*$-equivariant perverse sheaves on $\mathbb{C}$.