Exercises

Exercise 1 Example of generating series.

By finding all the genus 0 maps with one boundary of arbitrary length composed only of triangles, check that

$$W_{0,1}(x) = \frac{t}{x} + t^2\left(\frac{1}{x^3} + \frac{t_3}{x^2}\right) + t^3\left(\frac{2}{x^5} + \frac{4t_3}{x^4} + \frac{t_3^2}{x^3}\right) + O(t^4).$$

Exercise 2 Brown's lemma.

1. Show that $N_{0,1}(k) \in t^2\mathbb{Q}[t_3,\dots,t_d][\![t]\!]$ for $k \geq 1$.
2. Recalling that $N_{0,1}(0) := t$, show that

$$P_{0,1}(x) = t\frac{V'(x)}{x} + O(t^2)$$

   where $P_{0,1}(x)$ is the polynomial part of $V'(x)W_{0,1}(x)$ and $V'(x) = x - \sum_{j=3}^{d} t_j x^{j-1}$.
3. Show that the zeros of $V'(x)^2 - 4P_{0,1}(x)$ take the form

$$\forall i = 1,\dots,d-1\,,\ X_i \pm 2\frac{\sqrt{P_{0,1}(X_i)}}{V''(X_i)} + O(\sqrt{t})$$

   where $V'(X_i) = 0$ for all $i \in [\![1, d-1]\!]$.
4. Prove that the zeros of $V'(x)^2 - 4P_{0,1}(x)$ are further constrained by

$$a_1 \sim 2\sqrt{t} + o(\sqrt{t}) \qquad , \qquad b_1 \sim -2\sqrt{t} + o(\sqrt{t})$$

   and

$$\forall i \in [\![2, d-1]\!]\,,\ a_i \sim X_i + O(t) \qquad , \qquad b_i \sim X_i + O(t).$$

5. Assume that, for $i \neq 1$, $a_i - b_i \neq 0$ to some order in t. Show that this implies that $W_{0,1}(x)$has a pole at $x = X_i$ to some order in t [1] .
6. Conclude that

$$V'(x)^2 - 4P_{0,1}(x) = M(x)^2(x-a_1)(x-b_1).$$

Exercise 3 Rational parameterisation.

Let us consider the function

$$x(z) = \frac{a+b}{2} + \frac{a-b}{4}\left(z + \frac{1}{z}\right)$$

where $a = \alpha + 2\gamma$ and $b = \alpha - 2\gamma$ as well as the polynomial

$$V'(x) = x - \sum_{j=3}^{d} t_j x^{j-1}.$$

1. Look at the behavior of $\sqrt{(x-a_i)(x-b_i)}$ as $a_i, b_i \to X_i$.

1. Show that one can write

$$V'(x(z)) = \sum_{k=0}^{d-1} u_k \left(z^k + \frac{1}{z^k} \right)$$

where

$$u_k = \alpha\delta_{k,0} + \gamma\delta_{k,1} - \sum_{l=2}^{d-1} \sum_{j=k}^{\left[\frac{l+k}{2}\right]} t_{l+1} \frac{l!}{j!\,(j-k)!\,(l+k-2j)!} \gamma^{2j-k} \alpha^{l+k-2j}.$$

2. Prove that $W_{0,1}(x) = \frac{t}{x} + O(x^{-2})$ together with $W_{0,1}(x(z)) = \sum_{k=1}^{d-1} u_k z^{-k}$ imply that

$$\begin{cases} u_0 = 0 \\ u_1 = \frac{t}{\gamma} \end{cases}.$$

Exercise 4 Triangulations.

Let us study triangulations by setting $d = 3$ and considering $V'(x) = x - t_3 x^2$.

1. Using the expression of u_k of exercise 3, show that the constraints $u_0 = 0$ and $u_1 = \frac{t}{\gamma}$ imply that

$$\begin{cases} \alpha = \frac{1}{2t_3}\left(1 - \frac{t}{\gamma^2}\right) \\ \frac{t^2}{\gamma^6} - \frac{1}{\gamma^2} + 8t_3^2 = 0 \end{cases}.$$

2. Defining $r := \frac{\gamma^2}{t}$, show that

$$\begin{cases} \alpha = \frac{1-r}{2t_3} \\ r - r^3 = 8tt_3^2 \end{cases}$$

and $r = 1 + O(t)$.

3. In order to enumerate maps composed of triangles, one is interested in the expansion of these quantities as series in t and t_3. Let us thus write $r = \sum_{n=0}^{\infty} c_n (8tt_3^2)^n$ for some coefficients c_n to be determined. Remark that one can write

$$c_n = \operatorname*{Res}_{t\to 0} r \cdot \frac{d(8tt_3^2)}{(8tt_3^2)^{n+1}}.$$

Using a simple change of variable for the computation of the residue, show that

$$c_n = -\frac{1}{2} \frac{\Gamma\left(\frac{3n-1}{2}\right)}{n!\,\Gamma\left(\frac{n+1}{2}\right)}$$

where the Γ function satisfies

$$\forall n \in \mathbb{N}\,,\ \Gamma(n + \frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \qquad \text{and} \qquad \Gamma(n) = (n-1)!.$$

4. Let us remind that the number of triangulations of a disc with boundary of length l reads
$$N_{0,1}(l) = -\operatorname*{Res}_{x\to\infty} W_{0,1}(x) x^l dx.$$
Show that it implies that
$$N_{0,1}(l) = \sum_{\substack{j+k<l \\ j<k<j+3}} \frac{(k-j)l!}{(j+1)!\,(k+1)!\,(l-1-j-k)!} \alpha^{l-1-k-j} \gamma^{k+j+2} u_{k-j}.$$

5. Combining the computations above check that
$$N_{0,1}(l) = \frac{t}{8t_3^3}\left(-\frac{1}{r} + 4 - 4r + r^3\right) = \frac{2t^2}{t_3} \sum_{n\geq 1} (8tt_3^2)^n \frac{\Gamma\left(\frac{3n}{2}+1\right)}{(n+2)!\,\Gamma\left(\frac{n}{2}+1\right)}.$$

6. What is the number of triangulations with one marked face among a total of $2n$ faces ?

Exercise 5 Computation of the first correlation functions.

Let us consider the curve
$$\begin{cases} x(z) = \alpha + \gamma\left(z + \frac{1}{z}\right) \\ y(z) \end{cases}$$
where $y(z)$ is an arbitrary function of z holomorphic at $z = \pm 1$.

Show that the topological recursion builds the correlation functions
$$\omega_{0,3}(z_1,z_2,z_3) = -\frac{1}{2\gamma y'(1)} \frac{1}{(z_1-1)^2(z_2-1)^2(z_3-1)^2} + \frac{1}{2\gamma y'(-1)} \frac{1}{(z_1+1)^2(z_2+1)^2(z_3+1)^2}$$
and
$$\begin{aligned} \omega_{1,1}(z) &= -\frac{1}{16\gamma y'(1)}\left(\frac{1}{(z-1)^4} + \frac{1}{(z-1)^3} - \frac{1+\frac{y''(1)}{y'(1)}+\frac{y'''(1)}{3y'(1)}}{2(z-1)^2}\right) \\ &\quad + \frac{1}{16\gamma y'(-1)}\left(\frac{1}{(z+1)^4} - \frac{1}{(z+1)^3} + \frac{1-\frac{y''(-1)}{y'(-1)}+\frac{y'''(-1)}{3y'(-1)}}{2(z+1)^2}\right). \end{aligned}$$

Exercise 6 Airy curve.

Let us consider the curve $y^2 = x$, i.e.
$$\begin{cases} x(z) = z^2 \\ y(z) = z \\ \omega_{0,2}(z_1,z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2} \end{cases}.$$

1. Compute $\omega_{0,3}$, $\omega_{0,4}$ and $\omega_{1,1}$.
2. Check that the topological recursion is equivalent to
$$\forall 2g-2+n>0\,,\ \omega_{g,n}(z_1,\dots,z_n) = (-2)^{2g-2+n} \sum_{\mathbf{d}\in\mathbb{N}^n} \langle \tau_{d_1}\dots\tau_{d_n}\rangle_{g,n} \prod_{i=1}^n \frac{(2d_i+1)!!\,dz_i}{z_i^{2d_i+2}}$$

where the coefficients $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{g,n}$ satisfy

$$\begin{aligned}(2k_1+1)!!\, \langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n} &= \sum_{j=2}^{n} \frac{(2k_1+2k_j-1)!!}{(2k_j-1)!!} \left\langle \tau_{k_2} \dots \tau_{k_j+k_1-1} \dots \tau_{k_n} \right\rangle_{g,n-1} \\ &+ \frac{1}{2} \sum_{i+j=k_1-2} (2i+1)!!\,(2j+1)!! \Big[\left\langle \tau_i \tau_j \tau_{k_2} \dots \tau_{k_n} \right\rangle_{g-1,n+1} + \\ &\sum^{\text{stable}}_{\substack{A \sqcup B = \{k_2,\dots,k_n\} \\ g_1+g_2=g}} \left\langle \tau_i \tau_A \right\rangle_{g_1,|A|+1} \left\langle \tau_j \tau_B \right\rangle_{g_2,|B|+1} \Big]\end{aligned}$$

where the term stable in the sum means that one sets $\langle \tau_k \rangle_{0,1} = \langle \tau_k \tau_l \rangle_{0,2} = 0$ in the sum.

3. Show that the formula above is equivalent to Witten-Kontsevich conjecture (or Virasoro constraints for the Gromov-Witten invariants of the point). Namely, check that it is equivalent to

$$\forall n \geq -1\,,\; L_n(e^F) = 0$$

order by order in $\hbar$ where

$$F := \sum_{g=0}^{\infty} \hbar^{2g-2} \sum_{\{k_j\}} \left\langle \prod_{j=0}^{\infty} \tau_j^{k_j} \right\rangle_{g,\cdot} \prod_{j=1}^{\infty} \frac{t_j^{k_j}}{k_j!}$$

where the sum is over all sequence of non-negative integer with only finitely non-zero terms and

$$L_n := -\frac{(2n+3)!!}{2^{n+1}} \frac{\partial}{\partial t_{n+1}} + \sum_{j=0}^{\infty} \frac{(2j+2n+1)!!}{(2j-1)!!\, 2^{n+1}} t_j \frac{\partial}{\partial t_{j+n}} + \frac{\hbar^2}{2} \sum_{j=0}^{n-1} \frac{(2j+1)!!\,(2n-2j-1)!!}{2^{n+1}} \frac{\partial^2}{\partial t_j\, \partial t_{n-j-1}}$$

satisfy $[L_n, L_m] = (n-m) L_{n+m}$.

Exercise 7 Properties.

Show the following properties of the correlation functions $\omega_{g,n}$ built by topological recursion.

1. For $2g-2+n>0$, $\omega_{g,n}$ has poles only at the branch points and vanishing residue there.
2. Linear loop equations :

$$\forall 2g-2+n > 0\,,\; \forall \lambda \in \mathbb{C}\,,\quad \sum_{z \in x^{-1}(\lambda)} \omega_{g,n}(z, z_2, \dots, z_n) = 0$$

and

$$\forall 2g-2+n > 0\,,\; \forall \lambda \in \mathbb{C}\,,\quad \sum_{z \in x^{-1}(\lambda)} \omega_{g,n}(z_1, z, z_3, \dots, z_n) = 0$$

3. Periods :

$$\forall i = 1, \dots g\,,\; \forall j = 1, \dots n\,,\; \oint_{z_j \in \mathcal{A}_i} \omega_{g,n}(z_1, \dots, z_n) = 0.$$

This will be proved by reminding that it is true by definition for $\omega_{0,2}$.

4. Quadratic loop equations : for $(k,g) \neq (0,1)$

$$P_{g,k}(\lambda,\mathbf{z}) := \frac{1}{dx^2} \sum_{z\in x^{-1}(\lambda)} \left[\omega_{g-1,k+2}(z,z,\mathbf{z}) + \sum_{h=0}^{g} \sum_{A\sqcup B=\mathbf{z}} \omega_{h,|A|+1}(z,A)\omega_{g-h,|B|+1}(z,B)\right]$$

where $\mathbf{z} = \{z_1,\dots,z_k\}$, is a rational function of λ without any pole at the image of the branch points.

In order to prove it, using the linear loop equations, first show that

$$\begin{aligned}\omega_{g,k+1}(z,\mathbf{z}) &= -\frac{1}{2}\sum_{a\in\mathcal{R}} \operatorname*{Res}_{Z\to a} \frac{\int_{\sigma_a(Z)}^{Z}\omega_{0,2}(z,\cdot)}{2(\omega_{0,1}(Z)-\omega_{0,1}(\sigma_a(Z)))}\Big[P_{g,k}(x(Z),\mathbf{z})dx(Z)^2 \\ &\quad -2\sum_{p\in x^{-1}(x(Z))}\omega_{0,1}(p)\omega_{g,k+1}(p,\mathbf{z})\Big].\end{aligned}$$

Then prove that the last term in the RHS is actually itself equal to $\omega_{g,k+1}(z,\mathbf{z})$ implying that

$$0 = \sum_{a\in\mathcal{R}} \operatorname*{Res}_{Z\to a} \frac{\int_{\sigma_a(Z)}^{Z}\omega_{0,2}(z,\cdot)}{2(\omega_{0,1}(Z)-\omega_{0,1}(\sigma_a(Z)))} P_{g,k}(x(Z),\mathbf{z})dx(Z)^2.$$

Finally, this implies for any $m\in\mathbb{N}$ and any $b\in\mathcal{R}$ that

$$0 = \operatorname*{Res}_{z\to b}(y(z)-y(z))(x(z)-x(b))^m \sum_{a\in\mathcal{R}} \operatorname*{Res}_{Z\to a} \frac{\int_{\sigma_a(Z)}^{Z}\omega_{0,2}(z,\cdot)}{2(\omega_{0,1}(Z)-\omega_{0,1}(\sigma_a(Z)))} P_{g,k}(x(Z),\mathbf{z})dx(Z)^2.$$

To conclude prove that it is equivalent to

$$0 = \operatorname*{Res}_{z\to b}(x(z)-x(b))^m P_{g,k}(x(z),\mathbf{z})dx(z)$$

leading to the result.

5. Symmetry : $\omega_{g,n}$ is invariant under permutation of its n variables. One just has to prove the symmetry under the permutation of the two first variables. For doing so, run two steps of the recursion in two different ways. First consider the recursion starting with the first variable and then run the recursion with the second variable for the correlation functions obtained after the first step. Then do the same thing starting with the second variable and then running the recursion for the first variable in the second step. Then prove that both procedure give the same result using the properties above.