Exercises 2

Exercise 1 Airy curve.

Let us consider the curve $y^2 = x$, i.e.

$$\begin{cases} x(z) = z^2 \\ y(z) = z \\ \omega_{0,2}(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2} \end{cases} .$$

This very simple curve is often refered to as the Airy curve since the generating function

$$\Psi(x(z), \hbar) := \exp \left[\sum_{g,n \geq 0} \frac{\hbar^{2g-2+n}}{n!} \int_0^z \cdots \int_0^z \omega_g, n \right]$$

is a WKB solution to Airy equation

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - x \right] \Psi(x, \hbar) = 0.$$

1. Compute $\omega_{0,3}$, $\omega_{0,4}$ and $\omega_{1,1}$.
2. In the examples above, check that the result takes the form

$$\omega_{g,n}(z_1, \ldots, z_n) = (-2)^{2g-2+n} \sum_{\mathbf{d} \in \mathbb{N}^n} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^n \frac{(2d_i + 1)!!\, dz_i}{z_i^{2d_i+2}}$$

where the coefficients $\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{g,n}$ are non-vanishing scalars only if $\sum_{i=1}^n d_i = 3g - 3 + n$.

This property actually follows from the fact that these coefficients are intersection of cohomology classes on the Deligne-Mumford compactification of the moduli space of curves :

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i}$$

where ψ_i are degree 2 cohomology classes and $\overline{\mathcal{M}}_{g,n}$ has dimension $3g - 3 + n$.

Use question 1 to compute $< \tau_0\tau_0\tau_0 >_{0,3}$, $< \tau_1\tau_0\tau_0 >_{0,4}$ and $< \tau_1 >_{1,1}$.

3. In order to prove that the coefficients $\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{g,n}$ are indeed intersection numbers, check that the topological recursion is equivalent to

$$\forall 2g - 2 + n > 0\,,\ \omega_{g,n}(z_1, \ldots, z_n) = (-2)^{2g-2+n} \sum_{\mathbf{d} \in \mathbb{N}^n} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^n \frac{(2d_i + 1)!!\, dz_i}{z_i^{2d_i+2}}$$

where the coefficients $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{g,n}$ satisfy

$$
\begin{aligned}
(2k_1+1)!!\, \langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n} \quad = \quad & \sum_{j=2}^{n} \tfrac{(2k_1+2k_j-1)!!}{(2k_j-1)!!} \left\langle \tau_{k_2} \dots \tau_{k_j+k_1-1} \dots \tau_{k_n} \right\rangle_{g,n-1} \\
& + \tfrac{1}{2} \sum_{i+j=k_1-2} (2i+1)!!\,(2j+1)!! \Big[\left\langle \tau_i \tau_j \tau_{k_2} \dots \tau_{k_n} \right\rangle_{g-1,n+1} + \\
& \sum_{\substack{A \sqcup B = \{k_2,\dots,k_n\} \\ g_1+g_2=g}}^{\text{stable}} \langle \tau_i \tau_A \rangle_{g_1,|A|+1} \langle \tau_j \tau_B \rangle_{g_2,|B|+1} \Big]
\end{aligned}
$$

where the term stable in the sum means that one sets $\langle \tau_k \rangle_{0,1} = \langle \tau_k \tau_l \rangle_{0,2} = 0$ in the sum.

4. Show that the formula above is equivalent to Witten-Kontsevich conjecture (or Virasoro constraints for the Gromov-Witten invariants of the point). Namely, check that it is equivalent to

$$\forall n \geq -1\,,\ L_n(e^F) = 0$$

order by order in $\hbar$ where

$$F := \sum_{g=0}^{\infty} \hbar^{2g-2} \sum_{\{k_j\}} \left\langle \prod_{j=0}^{\infty} \tau_j^{k_j} \right\rangle_{g,\cdot} \prod_{j=1}^{\infty} \frac{t_j^{k_j}}{k_j!}$$

where the sum is over all sequence of non-negative integer with only finitely non-zero terms and

$$L_n := -\frac{(2n+3)!!}{2^{n+1}} \frac{\partial}{\partial t_{n+1}} + \sum_{j=0}^{\infty} \frac{(2j+2n+1)!!}{(2j-1)!!\, 2^{n+1}} t_j \frac{\partial}{\partial t_{j+n}} + \frac{\hbar^2}{2} \sum_{j=0}^{n-1} \frac{(2j+1)!!\,(2n-2j-1)!!}{2^{n+1}} \frac{\partial^2}{\partial t_j\, \partial t_{n-j-1}}$$

satisfy $[L_n, L_m] = (n-m) L_{n+m}$.

Exercise 2 Computation of the first correlation functions.

Let us consider the curve

$$\begin{cases} x(z) = \alpha + \gamma \left(z + \frac{1}{z}\right) \\ y(z) \\ \omega_{0,2}(z_1,z_2) = \frac{dz_1\, dz_2}{(z_1-z_2)^2} \end{cases}$$

where $y(z)$ is an arbitrary function of z holomorphic at $z = \pm 1$.

Show that the topological recursion builds the correlation functions

$$\omega_{0,3}(z_1,z_2,z_3) = -\frac{1}{2\gamma y'(1)} \frac{1}{(z_1-1)^2(z_2-1)^2(z_3-1)^2} + \frac{1}{2\gamma y'(-1)} \frac{1}{(z_1+1)^2(z_2+1)^2(z_3+1)^2}$$

and

$$
\begin{aligned}
\omega_{1,1}(z) \quad = \quad & -\tfrac{1}{16\gamma y'(1)} \left(\tfrac{1}{(z-1)^4} + \tfrac{1}{(z-1)^3} - \tfrac{1+\frac{y''(1)}{y'(1)}+\frac{y'''(1)}{3y'(1)}}{2(z-1)^2} \right) \\
& + \tfrac{1}{16\gamma y'(-1)} \left(\tfrac{1}{(z+1)^4} - \tfrac{1}{(z+1)^3} + \tfrac{1-\frac{y''(-1)}{y'(-1)}+\frac{y'''(-1)}{3y'(-1)}}{2(z+1)^2} \right).
\end{aligned}
$$