Problems 2 (Dimofte)
Branches and twists

Review

When G is abelian, the algebra of functions on the Coulomb branch $\mathbb{C}[\mathcal{M}_C]$ is simply given by

$$\mathbb{C}\big[\varphi \in \mathfrak{t}_{\mathbb{C}}, \{v_a\}_{a\in \mathrm{cochar}(G)}\big]\Big/\Big(v_a v_b = v_{a+b} \prod_{\substack{w\,\in\,\mathrm{weights}(V)\\ \langle w,a\rangle\langle w,b\rangle<0}} \langle w,\varphi\rangle^{\min(|\langle w,a\rangle|,|\langle w,b\rangle|)}\Big) \tag{1}$$

(Recall that "functions of $\varphi \in \mathfrak{t}_{\mathbb{C}}$" should be interpreted as the polynomial algebra generated by weights $\langle w,\varphi\rangle$, where w runs over a basis of the weight lattice of G.)

(Exercise for the intrepid: prove that the same description arises from the BFN construction!)

Example

Consider $G = U(1)$ and $V = \mathbb{C}_{(1)} \oplus \mathbb{C}_{(1)} \oplus \mathbb{C}_{(1)}$ (three copies of $\mathbb{C}$ on which $U(1)$ acts with weight 1). Then there is a single variable φ (the coordinate on $\mathfrak{t}_{\mathbb{C}} = \mathbb{C}$) and the above description of the Coulomb branch simplifies to

$$\mathbb{C}[\mathcal{M}_C] = \mathbb{C}[\varphi, v_1, v_{-1}]/(v_1 v_{-1} = \varphi^3) \tag{2}$$

This is the ring of functions on the symplectic singularity $\mathbb{C}^2/\mathbb{Z}_3$. Functions on the Higgs branch are given by the $GL(1)$ invariants of functions of $\vec{X} = (X_1, X_2, X_3) \in \mathbb{C}^3$ and $\vec{Y} = (Y_1, Y_2, Y_3) \in \mathbb{C}^3$ (where $GL(1)$ acts with weight 1 on $\vec{X}$ and weight -1 on $\vec{Y}$), modulo the complex moment map constraint $\mu_{\mathbb{C}} = \vec{X}\cdot\vec{Y} = 0$,

$$\mathbb{C}[\mathcal{M}_H] = \big(\mathbb{C}[\vec{X},\vec{Y}]/(\vec{X}\cdot\vec{Y})\big)^{GL(1)}. \tag{3}$$

The $GL(1)$ invariant functions are generated by bilinears $X_i Y_j$. It is clarifying to assemble them in a matrix

$$m = \begin{pmatrix} X_1Y_1 & X_1Y_2 & X_1Y_3 \\ X_2Y_1 & X_2Y_2 & X_2Y_3 \\ X_3Y_1 & X_3Y_2 & X_3Y_3 \end{pmatrix}. \tag{4}$$

The matrix obviously has rank 1; and $\mu_{\mathbb{C}} = 0$ is equivalent to $\mathrm{Tr}(m) = 0$. One can then argue that

$$\mathcal{M}_H = \{m \in \mathfrak{sl}_3 \,|\, \mathrm{rank}(m) = 1\}, \tag{5}$$

otherwise known as the minimal nilpotent orbit in $\mathfrak{sl}_3$. Its resolution is $T^*\mathbb{P}^2$.

1. Consider $G = U(1)^2$ and $V = \mathbb{C}_{(1,0)} \oplus \mathbb{C}_{(-1,1)} \oplus \mathbb{C}_{(0,-1)}$. The subscripts denote the weights of $U(1)\times U(1)$ acting on each factor. Check that the Higgs branch of this theory is $\mathbb{C}^2/\mathbb{Z}_3$ and that the Coulomb branch is the minimal nilpotent orbit of $\mathfrak{sl}_3$, consistent with the fact that this theory is 3d-mirror to the one on the previous page.

(Problem 1, continued)
Hint H: For the Higgs branch, let $(X_1, X_2, X_3) \in V$ and $(Y_1, Y_2, Y_3) \in V^*$. The complex moment map for $U(1) \times U(1)$ now has two components

$$\mu_{\mathbb{C}} = \big(X_1Y_1 - X_2Y_2\,,\ X_2Y_2 - X_3Y_3\big)\,. \tag{6}$$

Find three $U(1) \times U(1)$-invariant monomials U, V, W of the X's and Y's that, upon setting $\mu_{\mathbb{C}} = 0$, satisfy

$$UV = W^3\,. \tag{7}$$

Hint C: Functions on the Coulomb branch turn out to be generated by weights $\langle(1,0),\varphi\rangle =: \varphi_1$ and $\langle(0,1),\varphi\rangle =: \varphi_2$ together with three pairs of monopole operators $v_{\pm(1,0)}, v_{\pm(1,1)}, v_{\pm(0,1)}$. How do these monopole operators multiply? Show that the following matrix has rank one:

$$\begin{pmatrix} \varphi_1 & v_{(1,0)} & v_{(1,1)} \\ v_{(-1,0)} & -\varphi_1 + \varphi_2 & v_{(0,1)} \\ v_{(-1,-1)} & v_{(0,-1)} & -\varphi_2 \end{pmatrix} \tag{8}$$

2. Recall that the global symmetry groups of (G, V) gauge theory acting as tri-Hamiltonian isometries of the Higgs and Coulomb branch, called "flavor symmetries," are

$$F_H = N_{USp(T^*V)}(G)/G\,, \qquad F_C = \mathrm{Hom}(\pi_1(G), U(1)) \tag{9}$$

For this exercise, it suffices to replace $USp(T^*V)$ with the smaller group $U(V)$ of all unitary isometries of V; *i.e.* $F_H = N_{U(V)}(G)/G$.

Find F_H and F_C for the theories with $(G, V) = (U(1), \mathbb{C}^3_{(1)})$ and $(G, V) = (U(1)^2, \mathbb{C}_{(1,0)} \oplus \mathbb{C}_{(-1,1)} \oplus \mathbb{C}_{(0,-1)})$. Assuming that the answer is compatible with 3d mirror symmetry, when and how does F_C get enhanced in the IR to a nonabelian group?

3. The $\mathcal{N} = 4$ SUSY algebra on $\mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$, with coordinates (z, t), is the super-Lie algebra generated as a vector space by three even elements $\partial_z, \partial_{\bar z}, \partial_t$ and eight odd elements $Q_\alpha^{a\dot a}$, for $a, \dot a, \alpha \in \{+, -\}$, with super Lie brackets given by

$$[Q_+^{a\dot a}, Q_+^{b\dot b}] = -2\epsilon^{ab}\epsilon^{\dot a\dot b}\partial_{\bar z}\,, \quad [Q_-^{a\dot a}, Q_-^{b\dot b}] = 2\epsilon^{ab}\epsilon^{\dot a\dot b}\partial_z\,, \tag{10}$$

$$[Q_+^{a\dot a}, Q_-^{b\dot b}] = [Q_-^{a\dot a}, Q_+^{b\dot b}] = 2\epsilon^{ab}\epsilon^{\dot a\dot b}\partial_t \tag{11}$$

(and all other Lie brackets vanishing), where $\epsilon^{+-} = -\epsilon^{-+} = 1$ and $\epsilon^{++} = \epsilon^{--} = 0$. (Not important for this problem: the Q's are elements of the tensor product of the 2d rep of $SU(2)_C$, the 2d rep of $SU(2)_H$, and the 2d rep of the 3d spin group $\mathrm{Spin}(3) = SU(2)$, corresponding to the respective indices $a, \dot a, \alpha$.

Check that $Q_{HT} := Q_+^{+\dot +}$ is nilpotent, satisfying $[Q_+^{+\dot +}, Q_+^{+\dot +}] = 0$; and find the image $\mathrm{im}[Q_{HT}, -]$ in the SUSY algebra.
Check that $Q_A := Q_+^{+\dot +} + Q_-^{+\dot -}$ remains nilpotent and find its image.
Check that $Q_B := Q_+^{+\dot +} + Q_-^{-\dot +}$ also remains nilpotent and find its image.

This problem (hopefully) justifies why Q_{HT} is a holomorphic twist and Q_A, Q_B are topological twists.