Problems 1 (Dimofte) SOLUTIONS

Moduli and 3d mirror symmetry

1. In the lecture, we saw that an abelian 3d gauge theory is quantum-equivalent to a free periodic scalar. A similar bit of logic can be used to show that

- A 2d periodic scalar is quantum-equivalent to a second 2d periodic scalar of inverse period. (This is known as T-duality.)
- A 4d abelian gauge theory is quantum-equivalent to a second 4d abelian gauge theory. (This electric-magnetic duality.)

The goal of this problem is to derive the 2d duality.

Let Σ be an arbitrary ccompact real 2d surface, and $\phi : \Sigma \to S^1$ a periodic scalar field, with the convention that $\phi \sim \phi + 1$, and with action

$$S = \int_\Sigma R^2\, d\phi \wedge *d\phi\,. \tag{1}$$

(Note that the effective period of ϕ is R. If we rescale $\tilde\phi = R\phi$ in order to make the kinetic term $\int_\Sigma d\tilde\varphi \wedge *d\tilde\phi$ canonical, then $\tilde\phi \sim \tilde\phi + R$.)

a) Argue that there is a 1-1 correspondence between smooth maps $\phi : \Sigma \to S^1$ and smooth one-forms $f \in \Omega^1(\Sigma)$ that satisfy $df = 0$ and $\int_\beta f \in \mathbb{Z}$ for all $\beta \in H_1(\Sigma, \mathbb{Z})$.

(This correspondence can and should be extended to larger classes of non-smooth maps, and you may think about how to do that, but we won't go into further details here.)

Given a smooth map $\phi : \Sigma \to S^1$ (which we think of as $\mathbb{R}/\mathbb{Z}$), define $f = d\phi$. Note that f is a real-valued 1-form, which is automatically closed. Moreover, if $\beta : S^1 \to \Sigma$ is a closed path, representing a first homology class, then the integral $\int_\beta f := \int_{\beta_*[S^1]} d\phi$ is the winding number (or degree) of the induced map $\phi\circ\beta : S^1 \to S^1$, and must therefore be an integer. One way to justify this is to let $dVol$ denote the volume form on the S^1 that is the target of the map ϕ, normalized so that $\int_{S^1} dVol = 1$. Then

$$\int_{\beta_*[S^1]} d\phi = \int_{(\phi\circ\beta)_*[S^1]} dVol = \deg(\phi\circ\beta) \tag{2}$$

Conversely, given a 1-form f satisfying $df = 0$, choose a basepoint $x_0 \in \Sigma$ and define

$$\phi(x) = \int_{\gamma(x_0,x)} f\,, \tag{3}$$

where γ is any path from x_0 to x. Given two different paths γ, γ', the difference

$$\phi(x) = \int_{\gamma'(x_0,x)} f - \int_{\gamma(x_0,x)} = \int_{\gamma'\cdot\gamma^{-1}} f \in \mathbb{Z} \tag{4}$$

is an integer (since $\gamma'\cdot\gamma^{-1}$ is a closed loop). Therefore, $\phi(x)$ is well-defined up to integers, and its image in $\mathbb{R}/\mathbb{Z} \simeq S^1$ is well defined.

The two operations (constructing ϕ from f, and f from ϕ) are clearly inverse to each other, by the fundamental theorem of calculus.

b) Perform a series of formal manipulations to relate the path integrals

$$\int_{\text{maps } \phi : \Sigma \to S^1} D\phi \; e^{\,i\int_\Sigma R^2\, d\phi \wedge * d\phi} \quad \leftrightarrow \quad \int_{\text{maps } \sigma : \Sigma \to S^1} D\sigma \; e^{\,i\int_\Sigma \frac{1}{4R^2} * d\sigma \wedge d\sigma}, \tag{5}$$

where $2R^2 d\phi = * d\sigma$, and $\sigma \sim \sigma + 2\pi$, whence σ has an effective period π/R.

Hint: as in the lectures, let $f = d\phi$, and first rewrite the initial path integral as $\int Df \cdots$, with some additional fields that force $df = 0$ and $\int_\beta f \in \mathbb{Z}$ for all $\beta \in H_1(\Sigma, \mathbb{Z})$.

Caveat: both here and in the lectures, we are working in Euclidean signature, and the actions in the path integrals shouldn't have an "i" in front. However, these manipulations in Euclidean signature require a somewhat tricky analytic continuation, as they lead to relations like $2R^2 d\phi = i * d\sigma$. I am avoiding that in the lectures and exercise by being sloppy with some signs and factors of i.

We have:

$$\int_{\phi \in \Omega^0(\Sigma, \mathbb{R}/\mathbb{Z})} D\phi \; e^{\,i\int_\Sigma R^2\, d\phi \wedge * d\phi} = \int_{f \in \Omega^1(\Sigma) \text{ s.t. } df = 0, \int_\beta f \in \mathbb{Z}} Df \; e^{\,i\int_\Sigma R^2 f \wedge * f} \tag{6}$$

$$= \int_{f \in \Omega^1(\Sigma), \sigma \in \Omega^0(\Sigma)} Df\, D\sigma \sum_{\alpha \in H^1(\Sigma, \mathbb{Z})} e^{\,i\int_\Sigma \left[R^2 f \wedge * f - \sigma df + 2\pi\alpha \wedge f\right]} \tag{7}$$

$$= \int_{f \in \Omega^1(\Sigma), \sigma \in \Omega^0(\Sigma)} Df\, D\sigma \sum_{\alpha \in H^1(\Sigma, \mathbb{Z})} e^{\,i\int_\Sigma \left[R^2 f \wedge * f - f \wedge (d\sigma + 2\pi\alpha)\right]} \tag{8}$$

$$= \int_{f \in \Omega^1(\Sigma), \sigma \in \Omega^0(\Sigma, \mathbb{R}/2\pi\mathbb{Z})} Df\, D\sigma \; e^{\,i\int_\Sigma \left[R^2 f \wedge * f - f \wedge d\sigma\right]} \tag{9}$$

(using the EOM for the quadratic functional, $2R^2 * f = d\sigma$)

$$= \int_{\sigma \in \Omega^0(\Sigma, \mathbb{R}/2\pi\mathbb{Z})} D\sigma \; e^{\,i\int \frac{1}{4R^2} * d\sigma \wedge d\sigma}. \tag{10}$$

2. Consider the abelian 3d gauge theory from lectures, and its dual form

$$\int_{U(1)\text{ gauge fields on } M/\text{gauge equiv.}} DA\, e^{-i\int_M \frac{1}{g^2} dA\wedge *dA} \quad\leftrightarrow\quad \int_{\gamma:M\to S^1} D\gamma\, e^{i\int_M \frac{g^2}{4} d\gamma\wedge *d\gamma} \tag{11}$$

where $*dA = \frac{g^2}{2}d\gamma$, and $\gamma\sim\gamma+1$, whence γ has effective period $g/2$.

Let 3d spacetime be $M=\mathbb{R}^3$. Consider the local operator $e^{2\pi i n\gamma(0)}$, where $n\in\mathbb{Z}$ and $\gamma(0)$ denotes the value of γ at the origin $0\in\mathbb{R}^3$. Show that an insertion of this operator in the path integral is equivalent to replacing spacetime with a punctured 3-space $\mathbb{R}^3\backslash 0$, and performing the path integral over gauge fields A on $\mathbb{R}^3\backslash 0$ whose field strength $F=dA$ satisfies

$$\oint_{S^2} F = 2\pi n \tag{12}$$

for any 2-sphere S^2 surrounding the origin. In other words,

$$\int_{U(1)\text{ gauge fields on } \mathbb{R}^3\backslash 0 \text{ s.t. } \oint_{S^2}F=2\pi n} DA\, e^{-i\int_{\mathbb{R}^3\backslash 0} \frac{1}{g^2} dA\wedge *dA} \quad\leftrightarrow\quad \int_{\gamma:R^3\to S^1} D\gamma\, e^{i\int_{\mathbb{R}^3} \frac{g^2}{4} d\gamma\wedge *d\gamma}\, e^{2\pi i n\gamma(0)}\,. \tag{13}$$

Hint: work with the γ path integral. Absorb the operator insertion into the action by writing

$$\gamma(0) = \int_{\mathbb{R}^3} \delta^{(3)}(x)\gamma(x)\,, \tag{14}$$

where $\delta^{(3)}$ is a Dirac delta-function three-form supported at the origin. It suffices to show that the equations of motion for the γ action modified by the $e^{2\pi i n\gamma(0)}$ operator insertion end up setting $dF = 2\pi n\delta^{(3)}(0)$.

Following the hint, we'll write the integrand on the RHS as

$$\exp\left[i\int_{\mathbb{R}^3}\left[\frac{g^2}{4}d\gamma\wedge *d\gamma + 2\pi n\delta^{(3)}\gamma\right]\right] \tag{15}$$

The action has effectively been modified by the second term. The equations of motion now give

$$\frac{\delta}{\delta\gamma}\left[\frac{g^2}{4}d\gamma\wedge *d\gamma + 2\pi n\delta^{(3)}\gamma\right] = 0 \quad\Rightarrow\quad \frac{g^2}{2}d*d\gamma = 2\pi n\delta^{(3)} \tag{16}$$

Using the relation between γ and F, this implies

$$dF = 2\pi n\delta^{(3)}\,, \tag{17}$$

whence $\oint_{S^2}F = \int_{B^3}dF = 2\pi n$ as desired. Note that the solution to $dF = 2\pi n\delta^{(3)}$ is

$$F = *d\frac{n}{|x|}\,. \tag{18}$$

3. Recall from the lecture that the Coulomb branch of a 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with n hypermultiplets of charge 1 (*i.e.* $G = GL(1)$ and $V = \mathbb{C}^n$) has a ring of functions generated by monopole operators $v_\pm$ and a complex scalar φ, subject to the relation

$$v_+ v_- = \varphi^n \,. \tag{19}$$

Similarly, a 3d $\mathcal{N} = 4$ "pure" $PSU(2)$ gauge theory has an adjoint ($\mathfrak{psl}_2$-valued) scalar field Φ that can generically be diagonalized

$$\Phi = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi \end{pmatrix} . \tag{20}$$

At such generic points in the moduli space, where $\varphi \neq 0$, the gauge group is broken to the stabilizer of Φ, namely a maximal torus $U(1) \subset PSU(2)$. Then there are two abelian monopole operators v_+, v_- that now satisfy

$$v_+ v_- = -\frac{1}{\varphi^2} \,. \tag{21}$$

(This is the analogue of (19), when instead of hypermultiplet matter there are vectormultiplet fields charged under the maximal torus $U(1)$. In this case, these fields are the two off-diagonal components of Φ, with charges +1 and -1 under the maximal torus.)

The global functions on the Coulomb branch of the $SU(2)$ gauge theory are generated by the Weyl-invariant combinations

$$V := v_+ + v_- \,, \qquad U := \varphi(v_+ - v_-) \,, \qquad \Psi := \varphi^2 \,. \tag{22}$$

Show that (21) implies

$$U^2 - \Psi V^2 = 4 \,. \tag{23}$$

Note: this is the defining equation of the so-called Atiyah-Hitchin manifold, the smooth affine variety

$$\mathcal{M}_{\text{Atiyah-Hitchin}} = \text{Spec}\, \mathbb{C}[U, V, \Psi]/(U^2 - \Psi V^2 - 4) \,. \tag{24}$$

The Atiyah-Hitchin describes the global Coulomb branch of $PSU(2)$ gauge theory.

This is fairly straighforward. Just plug (22) into the expression $U^2 - \Psi V^2$ and then use (21) to simplify.