W-infinity Symmetry in the Quantum Hall Effect

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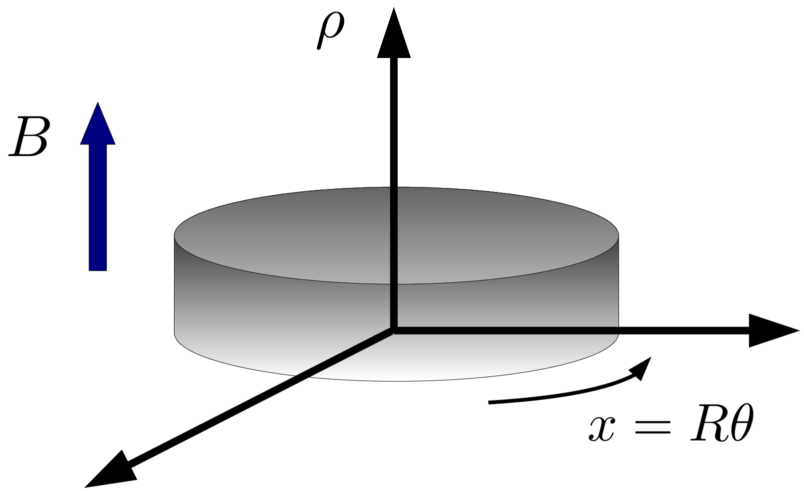
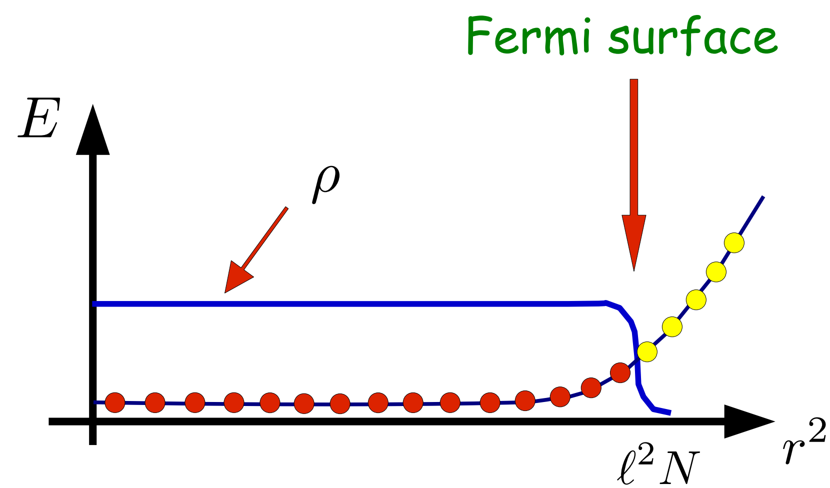
Outline

- QHE: bulk & edge
- W-infinity symmetry in bulk & edge
- W-infinity derivation of edge conformal theories
- Precise bulk – boundary map
- W-infinity description of bulk excitations (half-plane geometry)

Quantum Hall effect: edge excitations

Filled Landau level: bulk gap, incompressible fluid; massless edge fermion





edge ~ Fermi surface: linearize energy $\varepsilon(k) = \frac{v}{R}(k - k_F), \quad k \in \mathbb{Z}, \quad k_F = N$

→ set $r = R = \sqrt{\ell^2 N}$; massless chiral fermion in (1+1) dimensions $\psi(r, \theta, t)|_{r=R}$

→ fractional fillings $\nu = \frac{1}{3}, \frac{1}{5}, \ldots$ → interacting fermion → bosonization

→ c=1 conformal field theory (chiral Luttinger liquid, chiral boson)

Phenomenology of edge CFTs

- CFTs for other fractional plateaus? Endless possibilities in principle:

  $U(1)^n, \quad U(1) \times \frac{G}{H}, \cdots$ non-Abelian fusion rules & statistics, etc.

- however, observed plateaus are not many, mostly Jain states $\nu = \frac{n}{2pn \pm 1}, \quad n, p = 1, 2, \ldots$
- use the wavefunction - correlator correspondence + some physics inputs:

  e.g. Laughlin wf. $\Psi \sim \prod_{1=i<j}^{N} (z_i - z_j)^3 \quad \Longleftrightarrow \quad \langle \phi(z_1) \; \cdots \; \phi(z_N) \rangle$ CFT correlator (in-plane)

  → twenty years of extensive model building, experimental confirmations,...

Q: Is it possible to derive the relevant CFTs from symmetry principles only?

A: YES, by studying the W-infinity symmetry of quantum incompressible fluids

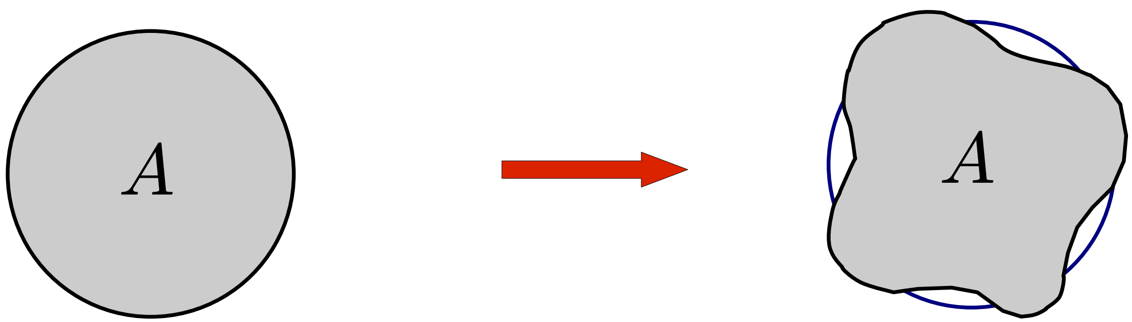
(AC, Trugenberger, Zemba '92;
Iso, Karabali, Sakita, '92)

→ See steps done so far...

W-infinity symmetry

Area-preserving diffeomorphisms of classical incompressible fluids

$$\int d^2x \; \rho(x) = N = \rho_o A \quad \longrightarrow \quad \underline{A = \text{constant}}$$



$$\delta x^i(x), \quad i = 1, 2$$

$$\partial_i \delta x^i = 0$$

- fluctuations of the fluid described by generators of the symmetry
- recall canonical transformations of 2d phase space, using Poisson brackets

$$\delta z = \{z, \mathcal{L}(z,\bar{z})\} = i\frac{\partial \mathcal{L}}{\partial \bar{z}}, \qquad \delta \bar{z} = \{\bar{z}, \mathcal{L}\} = -i\frac{\partial \mathcal{L}}{\partial z}, \qquad z = x^1 + ix^2 \qquad \bar{z} = x^1 - ix^2$$

- generators in polynomial basis obey the classical w_∞ algebra

$$\mathcal{L}_{n,m} = z^n \bar{z}^m, \qquad \boxed{\{\mathcal{L}_{n,m}, \mathcal{L}_{k,l}\} = i(nl - mk)\mathcal{L}_{n+k-1,m+l-1}}$$

- droplet fluctuations at the edge

$$\rho_{g.s.}(z,\bar{z}) = \rho_0 \Theta(R^2 - |z|^2), \qquad \delta\rho_{g.s.} = \{\rho_{g.s.}, z^n \bar{z}^m\} \sim i(n-m)e^{i(n-m)\theta} \; \rho_0 \delta(r^2 - R^2)$$

W-infinity symmetry in Landau levels

- Non-cummuting coordinates $\{\bar{z}, z\} \Longrightarrow [\bar{z}, z] = \ell^2, \qquad \ell^2 = \frac{2\hbar c}{eB} \to 1$ (magnetic length)
- quantum generators $\mathcal{L}_{n,m} = \int d^2z \; \Psi^\dagger(z,\bar{z}) \; z^n \bar{z}^m \; \Psi(z,\bar{z}), \qquad n, m \geq 0,$

$\mathcal{W}_\infty$ algebra:

$$[\mathcal{L}_{n,m}, \mathcal{L}_{k,l}] = \hbar \, [mk - nl] \, \mathcal{L}_{n+k-1,m+l-1} + O\left(\hbar^2\right) + O\left(\hbar^3\right) + \cdots$$

- also known as GMP algebra in Fourier basis

$$\rho(k,\bar{k}) = \int d^2z \; \hat{\Psi}^\dagger(z,\bar{z}) \; e^{ik\bar{z}+\bar{k}z} \; \hat{\Psi}(z,\bar{z}), \qquad [\rho(k,\bar{k}), \rho(p,\bar{p})] = \left(e^{p\bar{k}/4} - e^{\bar{p}k/4}\right) \rho(k+p, \bar{k}+\bar{p}).$$

- generators create excitations

$$\mathcal{L}_{n,m}|\Omega\rangle = |\mathrm{excit}\rangle, \quad n > m, \quad \Delta J = n - m > 0, \qquad \mathcal{L}_{n,m}|\Omega\rangle = 0, \quad n < m$$

→ spectrum-generating algebra ≈ dynamical symmetry

- $\mathcal{L}_{n,n}$ mutually commuting: cons. charges ≈ density moments $\langle \mathcal{L}_{n,n} \rangle = \langle r^{2n} \rangle$
- but: large-N limit not well defined, moments explode $\langle r^{2n} \rangle = O(R^{2n}) = O(N^n)$

→ needs renormalization; a central extension is generated

Part 1: W-infinity symmetry on the edge

- Representation in 1+1d Weyl fermion theory: large-N limit well defined

bulk: $z = re^{i\theta}, \quad \bar{z} = \partial_z$ edge: $r = R,$ → $\hat{z} = Re^{i\theta}, \quad \hat{z}\partial_{\hat{z}} = i\partial_\theta$

$$\mathcal{L}_{i-k,i} = R^k \oint_{C_R} \frac{d\hat{z}}{i\hat{z}} \; \psi^\dagger(\hat{z}) \; \hat{z}^{-k} \left(\hat{z}\partial_{\hat{z}}\right)^i \psi(\hat{z}), \qquad \mathcal{L}_{i-k,i}|\Omega\rangle = 0, \qquad 0 < k < N \to \infty$$

edge momentum

- algebra regularized by normal ordering, acquires a central extension
- includes current algebra and Virasoro for $i = 0, 1$ → CFT data
- extends to fractional filling by bosonization
- higher-spin currents: $i = 0, 1, 2, \ldots \quad J = \partial\phi, \quad T = (\partial\phi)^2, \quad W^{(3)} = (\partial\phi)^3, \ldots$
- all representations are known; charges are finite, $c = n = 1, 2, \ldots$ (Kac, Radul '93-95)

- special degenerate representations match excitations of Jain states

→ W-infinity "minimal models" (AC, Trugenberger, Zemba '95-'99)

W-infinity minimal models

- Generic W-infinity representations lead to known $\widehat{U(1)}^n$ CFTs with $c = n$; conformal dimensions span n-dimensional lattices (K Gram matrices)
- degenerate reps. occur for $U(n)$ extended symmetry; irreps amount to the projection

$$\widehat{U(1)} \times \widehat{SU(n)}_1 \implies \widehat{U(1)} \times \frac{\widehat{SU(n)}_1}{SU(n)}$$

corresponding CFTs are in one-to-one correspondence with Jain fillings $\nu = \frac{n}{2pn \pm 1}$ and their spectra of edge excitations

- projection partially suppresses excitations within the n layers:

→ reduced multiplicities of edge states, derivation of Jain wfs

- drawback: W-infinity minimal models are not Rational CFTs

Ex: c=2 minimal model

$$\widehat{U(1)} \times \widehat{SU(2)}_1 \longrightarrow \widehat{U(1)} \times \frac{\widehat{SU(2)}_1}{SU(2)} = \widehat{U(1)} \times \mathrm{Vir}$$

Vir = SU(2) Casimir subalgebra

$$c = 2, \quad \frac{1}{\nu} = 2p + \frac{1}{2}, \quad \nu = \frac{2}{4p+1} = \frac{2}{5}, \ldots$$

- keep excitations symmetric w.r.t. two layers only
- neutral part is described by the degenerate Virasoro reps. at $c = 1$
- fields characterized by dimension $h = \frac{k^2}{4}$ and spin $s = \frac{k}{2}$; NO s_z
- electron has $s = \frac{1}{2}$
- identify electrons of two layers using Dotsenko-Fateev screening operators

$$V_\pm = e^{\pm \frac{i}{\sqrt{2}}\phi} \Longrightarrow V_- \sim V_+, \quad (s_z \sim -s_z) \qquad V_+ = J_0^+ V_-, \quad J_n^+ = \oint du J^+(u) u^{-n-1}$$

- projection by adding a non-local term to the CFT Hamiltonian (AC, Zemba '97))
- matching Jain wavefunctions: $V_+ \rightarrow J_{-1}^+ V_- = \partial_z V_-$ (+ antisymmetrization)

(A.C. '13, H. Hansson et al, '07-'11)

- some open questions: e.g. statistics of excitations ??

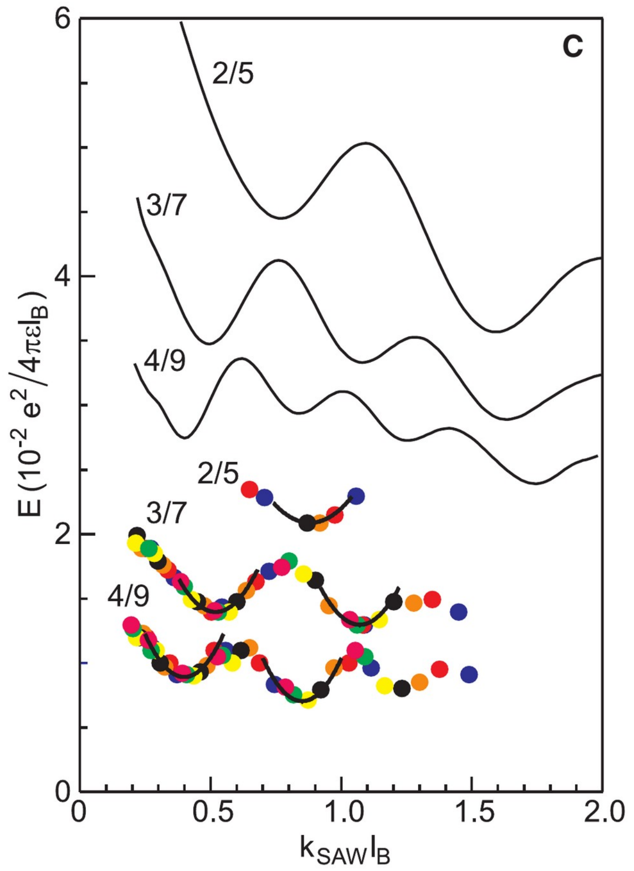
Part 2: W-infinity symmetry in the bulk

- Edge physics is universal and exact in the low-energy limit (CFT);

  matches topological bulk data: charges, statistics, top. ord. (Chern-Simons theory)
- what about bulk dynamics ? ("simple" low-energy d.o.f., universality,..?)

  ➡ composite fermion, density waves, magneto-roton minimum, higher-spin excit....
- approaches:

  - superfluid ansatz (Girvin, MacDonald, Platzman)
  - composite fermion numerics (Jain et al.)
  - two-dimensional metric (Haldane et al.)
  - hydrodynamics (Wiegmann et al.)
  - higher-spin d.o.f. (D.T. Son et al., AC et al.)



➡ exploit W-infinity symmetry in the bulk

Precise bulk-boundary map

(AC, Maffi, '18, '21)

- Laplace transform w.r.t. r^2

$$\rho_k(\lambda) = \int_0^\infty dr\, r e^{-\lambda r^2} \int_0^{2\pi} d\theta\, \rho(r,\theta) e^{-ik\theta}$$

- precise map to the edge by simultaneous limit of coordinate r and momentum k

$$r = R + x, \quad |x| < 1, \qquad k = R^2 + k', \quad |k'| < R, \qquad R \propto \sqrt{N} \to \infty$$

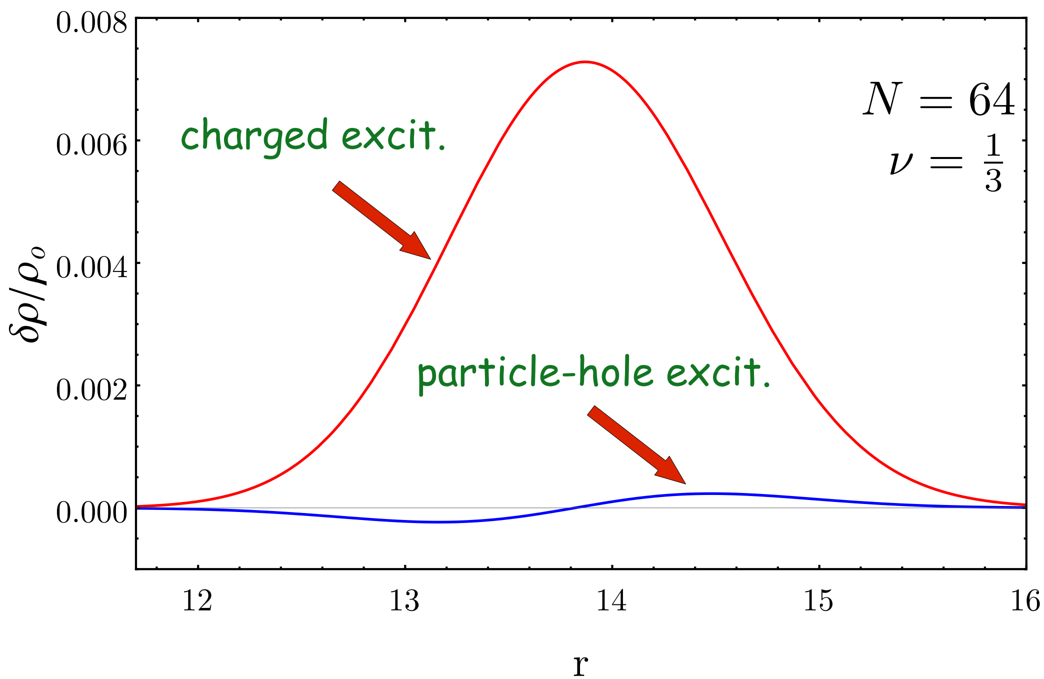
- $\rho_k(\lambda)$ once renormalized, matches earlier CFT: generating function of $\mathcal{L}_{i-k,i}$

➡ Compute analytic density profiles of excitations using W-infinity algebra

- ex: charged excitation $\nu = \frac{1}{m}$

$$\langle \delta\rho(r) \rangle \propto \frac{Q}{R} e^{-\frac{2x^2}{\sqrt{m}}}, \qquad Q = \frac{n}{m}$$

- Gaussian localized at edge
- $\ell^2 \to \sqrt{m}\, \ell^2$ scaling
- universal (built from CFT data)



➡ $O(1/R)$ "small"

"Large" edge excitations → "half" plane

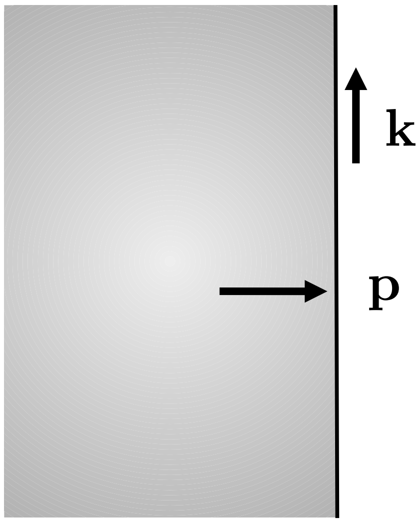
- Range of earlier limit can be extended beyond CFT
- large excitations stay finite for $R \to \infty$ in terms of sizes, momenta, energies

$$r = R + x, \quad x = O(1), \qquad k \to R^2 + k, \quad k = O(R), \qquad \mathbf{k} = \frac{k}{R} = O(1)$$

- from Laplace to Fourier: $\lambda = -\frac{\partial}{\partial r^2} \sim -\frac{1}{2R}\frac{\partial}{\partial x} = -\frac{i\mathbf{p}}{2R}$ $\qquad \mathbf{p} = O(1)$
- W-infinity algebra:

$$\rho_k(\mathbf{p}) \to \rho(\vec{k}), \quad \vec{k} = (\mathbf{k}, \mathbf{p}), \qquad \vec{a} = (\mathbf{1}, \mathbf{0})$$

$$\left[\rho(\vec{k}), \rho(\vec{k}')\right] = 2i \sin(\vec{k} \times \vec{k}'/4)\rho(\vec{k} + \vec{k}') + c\, \delta[(\vec{k} + \vec{k}') \cdot \vec{a}]\, \frac{4\sin(\vec{k} \times \vec{k}'/4)}{(\vec{k} + \vec{k}') \times \vec{a}}$$



- like GMP algebra, but renormalized & central extended
- Use Haldane short-range bulk potential

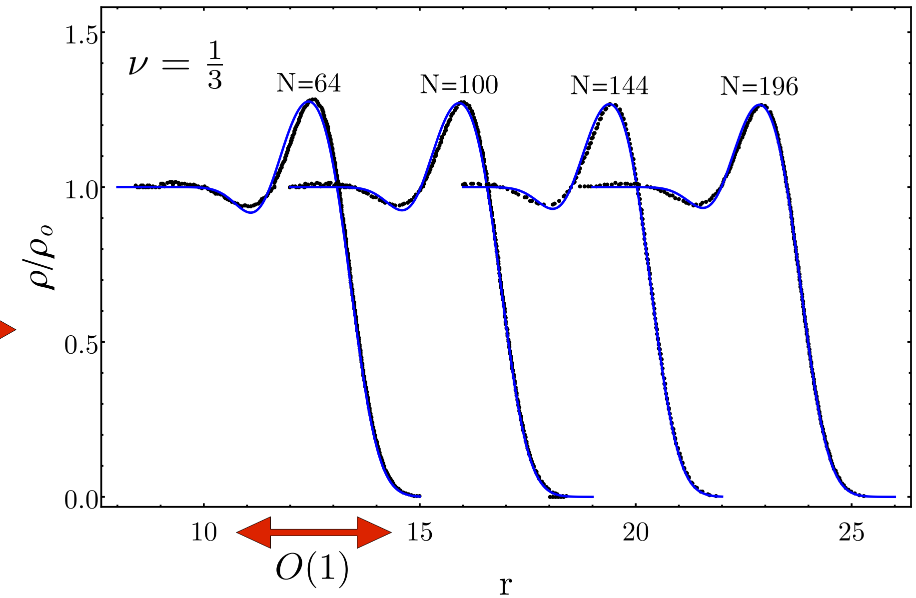
$$H = \int d^2k\, e^{-\vec{k}^2/4}\, (\vec{k}^2 - 2)\, \rho(-\vec{k})\rho(\vec{k})$$

Density profile of large excitations

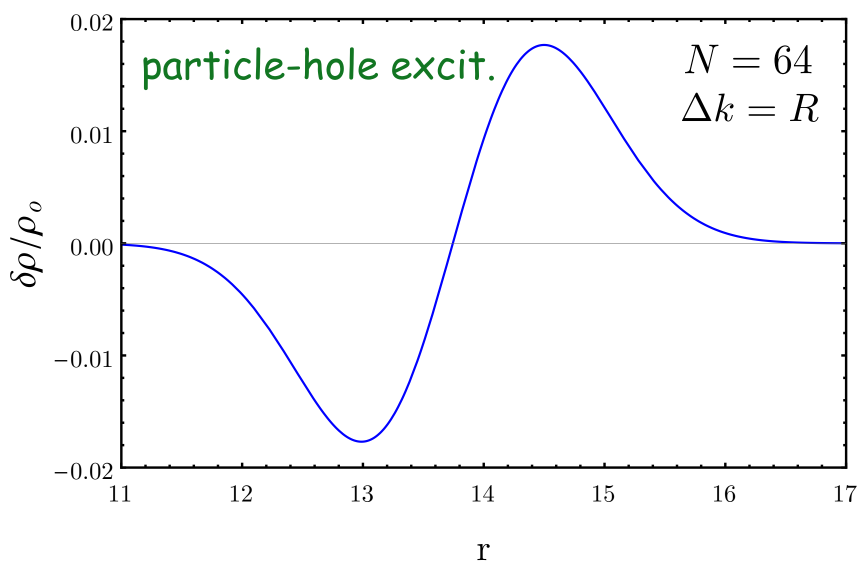
- Laughlin "overshoot", boundary effect that stay constant as $R^2 \propto N \to \infty$

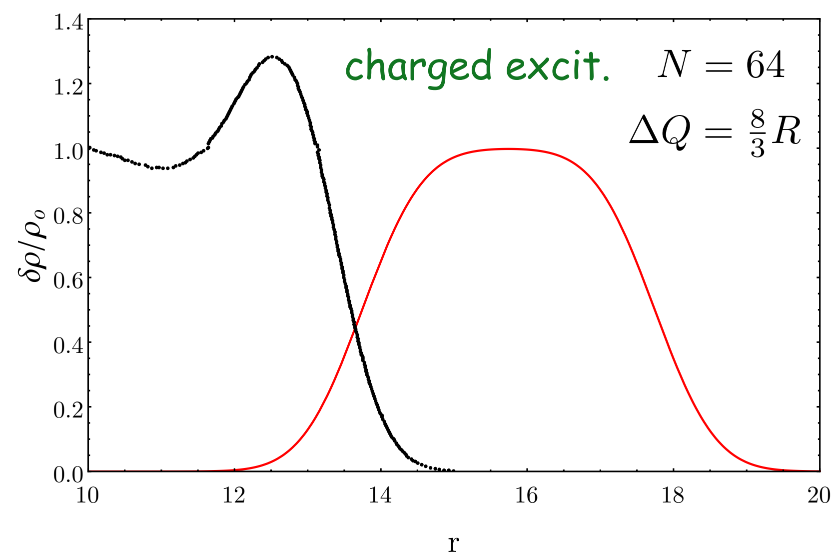
  $$\Delta r = O(1) \;\leftrightarrow\; \Delta k = O(R) = O(\sqrt{N})$$

  one-bump phenom. fit $\Delta k = O(\sqrt{N})$

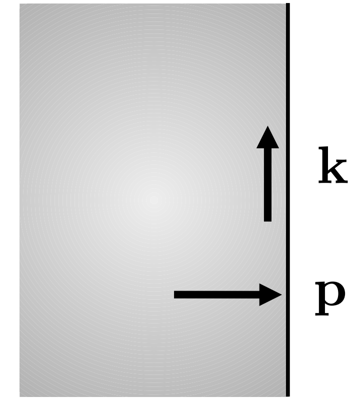


- W-infinity algebra: analytic results for large excitations (ansatz states)





"Edge reconstruction"



- Two-body repulsive potential has a small attractive exchange term
  - ➡ thin shell expelled from the droplet at distance $\Delta r = O(1)$
  - ➡ "edge reconstruction" (Chamon, Wen '94) ; "edge roton" (Jolad, Sen, Jain '10)
- analytic spectrum of large particle-hole excitations $|\mathrm{ex}\rangle = \rho(\mathbf{k},\mathbf{p})|\Omega\rangle$

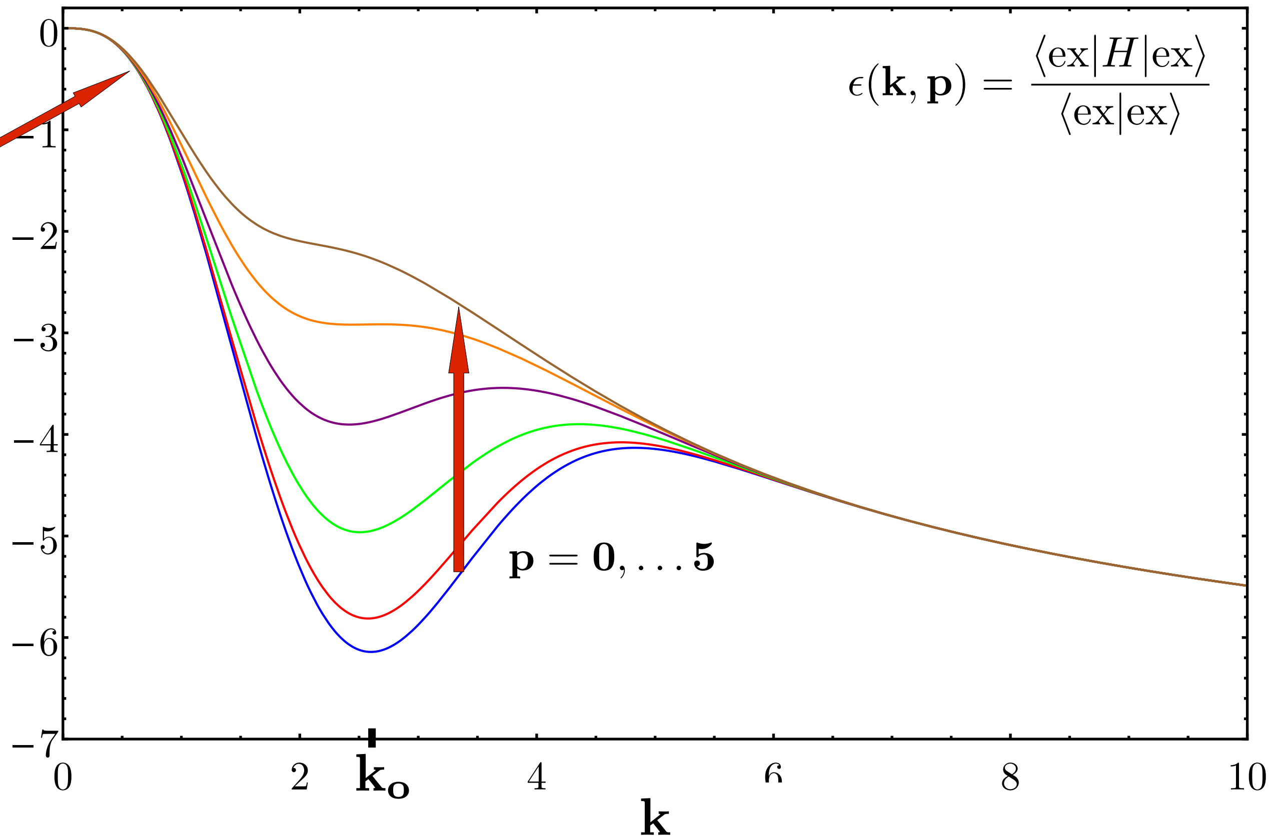
capillary waves

$\epsilon(\mathbf{k}) \sim -\mathbf{k}^3$

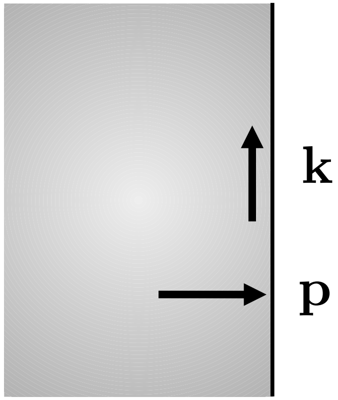
$\epsilon(\mathbf{k},\mathbf{p})$

local minimum at

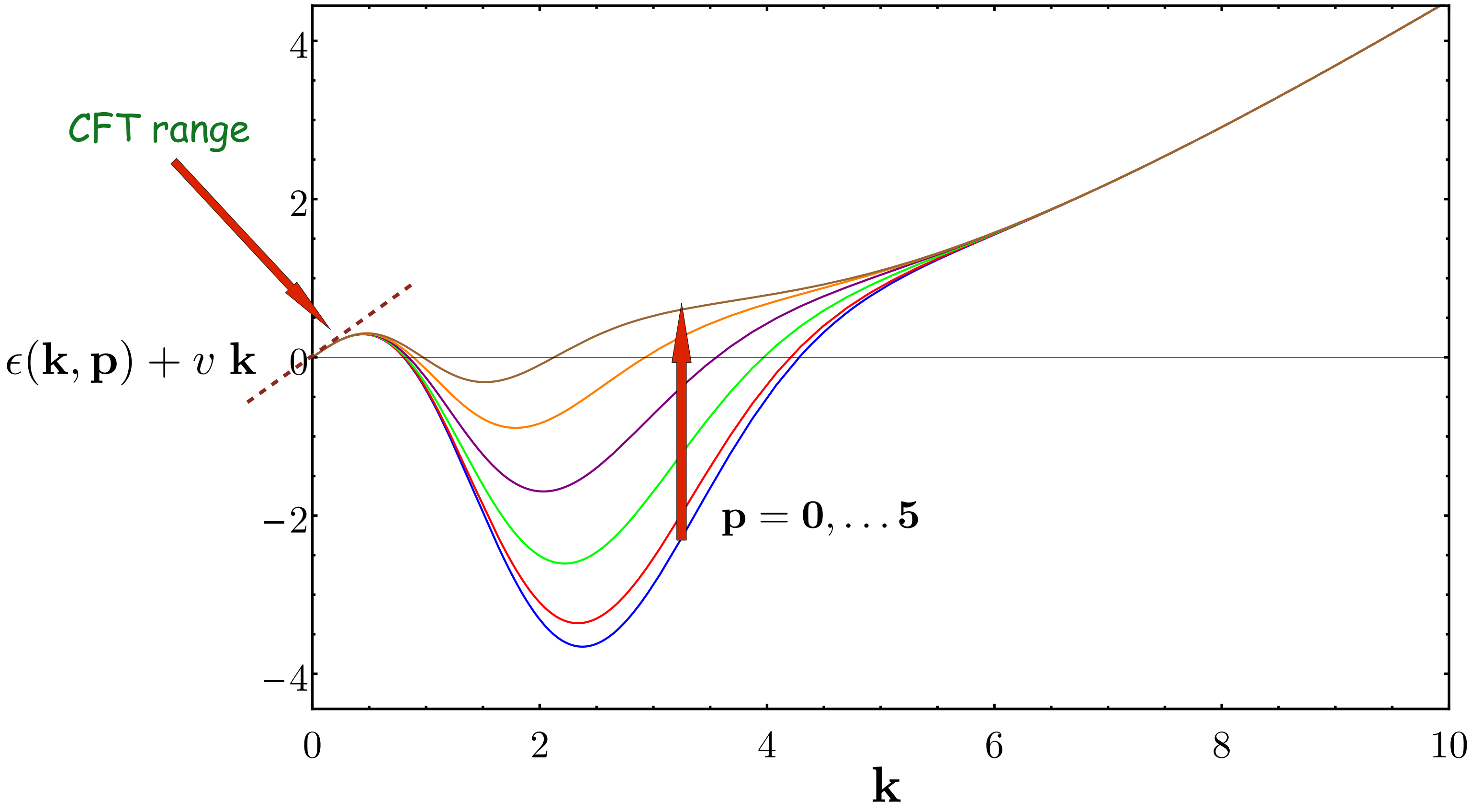
finite distance $\sim \mathbf{k_o}$



"Edge roton"

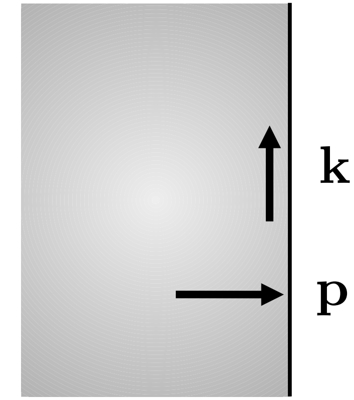


Add a (shallow) boundary potential to the spectrum

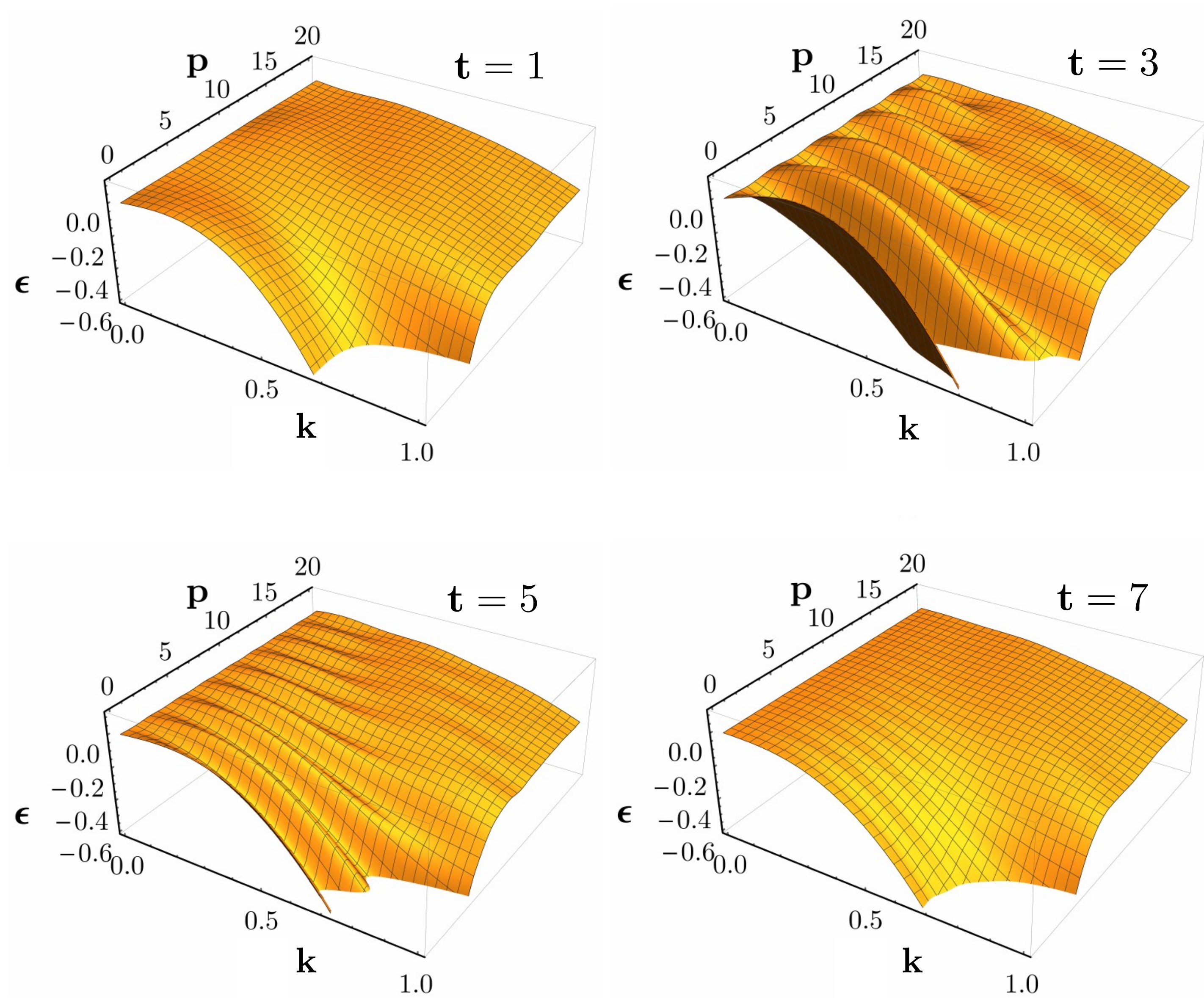


(cf. "freezing at the edge", Cardoso, Stephan, Abanov '20)

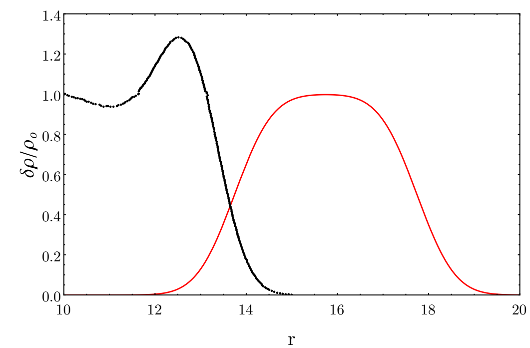
Large charged excitations



Adding charge leads to oscillating spectrum w.r.t. bulk momentum $\mathbf{p}$



$$Q = \frac{\mathbf{t}R}{3}, \quad \nu = \frac{1}{3}$$



Conclusions

- OLD: W-infinity symmetry characterizes the Jain states
- NEW: W-infinity symmetry determines the density profile of edge excitations

→ analytic, universal shapes from CFT; $\ell^2 \to \sqrt{m}\,\ell^2$ scaling for $\nu = 1/m$

- approach extends to "large" excitations having finite limit for $R \to \infty$

→ analytic "half-bulk" physics:

- ansatz excitations showing edge reconstruction
- charged excitations showing oscillating spectrum w.r.t. bulk momentum

BOLD STATEMENT:

W-infinity approach ≈ NR bosonization ≈ composite fermion

Perspectives

- Need better bulk ansatzes:

  ➡ bulk density wave for magneto-roton minimum: (quadrupole deformation)

  (cf. Liu, Gromov, Papic '18,'21; Gromov, Son 18)

- need numerical checks (cf. Cardoso, Stephan, Abanov '20)

- extend approach to Jain states

- W-infinity algebra for torus geometry

Formulas

- Large charged excitation: $|\{\mathbf{n},\mathbf{p};\mathbf{t}\}\rangle = \rho(-\mathbf{n},\mathbf{p})|Q\rangle, \qquad Q=\frac{\mathbf{t}R}{m}, \qquad \nu=\frac{1}{m}.$
- Energy spectrum:

$$\varepsilon(\mathbf{n},\mathbf{p};\mathbf{t}) = \frac{\langle\{\mathbf{n},\mathbf{p};\mathbf{t}\}|H|\{\mathbf{n},\mathbf{p};\mathbf{t}\}\rangle}{\langle\{\mathbf{n},\mathbf{p};\mathbf{t}\}|\{\mathbf{n},\mathbf{p};\mathbf{t}\}\rangle}$$

$$\begin{aligned}\varepsilon(\mathbf{n},\mathbf{p};\mathbf{t}) = \frac{\sqrt{m}\,e^{-\frac{\mathbf{n}^2\sqrt{m}}{4}}}{\mathbf{n}}\Bigg\{ &-2\int_0^{\mathbf{n}} d\mathbf{k}\, e^{-\frac{\mathbf{k}^2\sqrt{m}}{4}}(\mathbf{n}-\mathbf{k})\left[\mathbf{k}^2 e^{\frac{\mathbf{n}^2\sqrt{m}}{4}} + (\mathbf{n}^2-\mathbf{k}^2)\cos\left(\frac{\mathbf{p}\mathbf{k}\sqrt{m}}{2}\right)\right] \\ &-\int_0^{\mathbf{n}} d\mathbf{k}\int_0^{\mathbf{n}} d\mathbf{k}' e^{-\frac{(\mathbf{k}-\mathbf{k}')^2\sqrt{m}}{4}}\left(\mathbf{n}^2-(\mathbf{k}-\mathbf{k}')^2\right)\cos\left(\frac{\mathbf{p}(\mathbf{k}-\mathbf{k}')\sqrt{m}}{2}\right) \\ &+2\frac{\sqrt{m}-1}{m}\int_0^{\mathbf{n}} d\mathbf{k}\int_0^{\frac{\mathbf{t}}{\sqrt{m}}} d\mathbf{k}'\left[e^{-\frac{(\mathbf{k}-\mathbf{k}')^2\sqrt{m}}{4}}\left(\mathbf{n}^2-(\mathbf{k}-\mathbf{k}')^2\right)\cos\left(\frac{\mathbf{p}(\mathbf{k}-\mathbf{k}')\sqrt{m}}{2}\right) - (\mathbf{k}'\to-\mathbf{k}')\right]\Bigg\}\end{aligned}$$

- Density profile:

$$\langle\delta\rho(x)\rangle = \frac{1}{\pi}\frac{\langle\{\mathbf{n},\mathbf{p};\mathbf{t}\}|\rho_0(\mathbf{s})|\{\mathbf{n},\mathbf{p};\mathbf{t}\}\rangle}{\langle\{\mathbf{n},\mathbf{p};\mathbf{t}\}|\{\mathbf{n},\mathbf{p};\mathbf{t}\}\rangle}$$

$$\begin{aligned}\langle\delta\rho(x)\rangle = \frac{1}{m\pi}\Bigg\{ &\frac{1}{2}\left(\mathrm{erf}\left(\frac{\mathbf{t}-2x}{\sqrt{2}m^{\frac{1}{4}}}\right) + \mathrm{erf}\left(\frac{\sqrt{2}x}{m^{\frac{1}{4}}}\right)\right) \\ &+\frac{1}{\sqrt{2\pi}\mathbf{n}R}\int_0^{\mathbf{n}} d\mathbf{k}\left[\left(e^{-\frac{2}{\sqrt{m}}\left(x-\frac{t-\mathbf{k}\sqrt{m}}{2}\right)^2} - e^{-\frac{2}{\sqrt{m}}\left(x-\frac{t+\mathbf{k}\sqrt{m}}{2}\right)^2}\right) - (\mathbf{t}=0)\right] \\ &+\frac{1}{\sqrt{2\pi}\mathbf{n}R}\int_0^{\mathbf{n}} d\mathbf{k}\left(e^{-\frac{2}{\sqrt{m}}\left(x-\frac{\mathbf{k}\sqrt{m}}{2}\right)^2} - e^{-\frac{2}{\sqrt{m}}\left(x-\frac{(\mathbf{k}-\mathbf{n})\sqrt{m}}{2}\right)^2}\right)\Bigg\}\end{aligned}$$