Geometric and analytic aspects of the quantum Hall effect
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A bulk gap in the presence of edge states for a truncated Haldane pseudopotential

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Base on joint work with

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   - Methods for lower bounding spectral gap and ground state energy

Background: Haldane Pseudopotentials

Haldane pseudopotentials were originally introduced as Hamiltonian models for the fractional quantum Hall effect for $\nu = 1/(p+2)$ with $p \geq 0$ odd.

- Laughlin '83: Ansatz for many-body ground state wave function Ψ_p.
- Haldane '83: Pseudopotential $W^p \geq 0$ obtained via projection onto lowest Landau level of repulsive, short-range, radially symmetric pair potential:

$$W^p = \sum_{i<j} P_{LLL} v_p(z_i - z_j) P_{LLL}, \quad v_p \propto \Delta^p \delta, \quad z = x + iy$$

  Tailored so $\Psi_p \in \ker W^p$.
- Haldane-Rezayi '85, Trugman-Kivelson '85, Lee-Papic-Thomale '17, ...: More generalized study of various pseudopotentials on different 2D geometries.
- Regnault-Jolicoeur '04, Cooper '08,...: Also model rapidly rotating Bose gases (p even).
- Lewin-Seiringer '09, Seiringer-Yngvason '20: Haldane pseudopotentials realized as scaling limit.
- Johri-Papic-Schmitteckert-Bhatt-Haldane '12: Properties of pseudopotentials robust under change of geometry.

Properties and Conjectures:

$$\Lambda = \text{ physical space}, \quad N = \text{ number of particles}, \quad \nu \approx \frac{N}{|\Lambda|} \text{ filling factor}$$

1. Ground States: Zero energy states $\psi \in \mathcal{G}_\Lambda = \ker W_\Lambda^p \subset \oplus_N \mathcal{H}_\Lambda^N$ satisfy $\nu \leq \nu(p) := \dfrac{1}{p+2}$.
   - E.g. Ψ_p has maximal filling $\nu(p)$.
2. Spectral Rigidity: For states with higher fillings $\nu > \nu(p)$:

$$E_0(\mathcal{H}_\Lambda^N) = \inf_{0 \neq \psi \in \mathcal{H}_\Lambda^N} \frac{\langle \psi | W_\Lambda^p \psi \rangle}{\|\psi\|^2} \propto |\Lambda| \times \text{ increasing function of } \nu$$

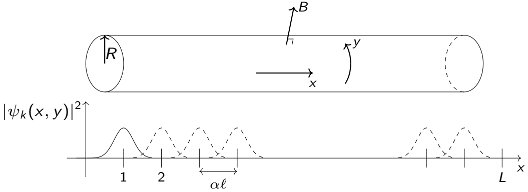
   - Determines Yrast line for Bose gases: Viefers-Hansson-Reimann '00, Regnault-Jolicoeur '04, Lewin-Seiringer '09,...
3. Spectral Gap Conjecture: Haldane '83, Haldane-Rezayi '85, Rougerie '19,...

$$\gamma := \inf_\Lambda \mathrm{gap}(W_\Lambda^p) > 0 \quad \text{where} \quad \mathrm{gap}(W_\Lambda^p) = \inf_{\substack{0 \neq \psi \in \oplus_N \mathcal{H}_\Lambda^N \\ \psi \perp \mathcal{G}_\Lambda}} \frac{\langle \psi | W_\Lambda^p \psi \rangle}{\|\psi\|^2}$$

   - The gap also implies the incompressibility of the FQH fluid as $E_0(\mathcal{H}_\Lambda^N) = 0$ for $\nu \leq \nu(p)$ and $E_0(\mathcal{H}_\Lambda^N) > \gamma$ for $\nu > \nu(p)$ (see, e.g. Rougerie '19).

4 Anyonic Excitations with Fractional Charge and their topological stability: Hastings-Michalakis '15, Haah '16, Cha-Naaijkens-Nachtergaele '20,...

The Haldane Pseudopotential



magnetic length: ℓ

ratio: $\alpha = \ell/R$

Figure: The Landau orbitals. A magnetic flux $2\pi\beta$ along the cylinder axis shifts the orbitals by $\beta\alpha\ell$ (not shown).

In second quantization the $\nu = 1/3$ Haldane pseudopotential acts on the fermionic Fock space $\mathcal{F} = \bigoplus_{N\geq 0} S_-(\mathcal{H}_{LLL}^{\otimes N})$ generated by the lowest Landau level. In the cylinder geometry:

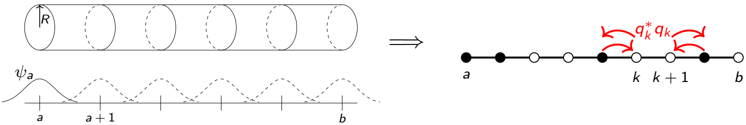
$$\mathcal{H}_{LLL} = \mathrm{span}\left\{\psi_k \propto \exp\left(ik\frac{\alpha y}{\ell}\right)\exp\left(-\frac{1}{2}\left[\frac{x}{\ell} - k\alpha\right]^2\right) : k \in \mathbb{Z}\right\}.$$

For $\nu = 1/3$ (i.e. $p = 1$): $\quad W^{(1)} = \sum_{s\in\mathbb{Z}/2} B_s^* B_s, \quad B_s = \sum_{k\in s+\mathbb{Z}} k e^{-(\alpha k)^2} c_{s-k} c_{s+k}.$

We consider the model given from truncating B_s to $|k| \leq 3/2$ Jansen '11, Nakamura-Wang-Bergholtz '12. Depending on s, this produces electrostatic and dipole hopping terms:

$$s \in \mathbb{Z}: \ \tilde{B}_s = 2e^{-\alpha^2} c_{s-1} c_{s+1}, \qquad s \in \mathbb{Z} + \frac{1}{2}: \ \tilde{B}_s = e^{-\frac{\alpha^2}{4}}\left(c_{s-\frac{1}{2}} c_{s+\frac{1}{2}} + 3e^{-2\alpha^2} c_{s-\frac{3}{2}} c_{s+\frac{3}{2}}\right)$$

The Finite-Volume Truncated 1/3-Model



Truncated Hamiltonian (with OBC): Fix $\lambda \in \mathbb{C}$ and $\kappa > 0$. For any $\Lambda = [a, b] \subseteq \mathbb{Z}$

$$H_\Lambda = \sum_{k=a}^{b-2} n_k n_{k+2} + \kappa \sum_{k=a+1}^{b-2} q_k^* q_k$$

$$n_k = c_k^* c_k, \quad q_k = c_k c_{k+1} - \lambda c_{k-1} c_{k+2}$$

$$\mathcal{H}_\Lambda = \mathrm{span}\left\{|\mu_a, \ldots, \mu_b\rangle : \mu_k \in \{0,1\}\right\}, \qquad \mu_k = \text{ occupation of } \psi_k$$

Symmetries:

Particle number: $N_\Lambda = \sum_{k=a}^{b} n_k$, Center of mass: $M_\Lambda = \sum_{k=a}^{b} k n_k$

Physical regime: $\kappa = e^{3\alpha^2/2}/4$ and $\lambda = -3e^{-2\alpha^2}$ where $\alpha = \frac{\ell}{R}$.

Tao-Thouless limit: $\lambda \to 0$ and $\kappa = \mathcal{O}(1)$ as $R \to 0$.

Uniform ground state gap: The model is uniformly gapped if there is $L > 0$ so that

$$\gamma := \inf_{\substack{\Lambda = [a,b], \\ |\Lambda| \geq L}} \mathrm{gap}(H_\Lambda) > 0, \quad \mathrm{gap}(H_\Lambda) = E_1(\mathcal{H}_\Lambda) - E_0(\mathcal{H}_\Lambda) = E_1(\mathcal{H}_\Lambda) > 0$$

Uniform Spectral Gap

$$\gamma(\kappa, \lambda) := \inf_{|\Lambda| \geq L} \operatorname{gap}(H_\Lambda).$$

Nachtergaele-Warzel-Y. '21 (OBC Gap) There is a constant $C(\lambda, \kappa) = \mathcal{O}(1)$ so that for all $0 < |\lambda| < 5.3$ and $L \geq 11$

$$\operatorname{gap}(H_{[1,L]}) \geq C(\lambda, \kappa) \min_{k=7,8,9} \operatorname{gap}(H_{[1,k]}) \propto |\lambda|^2$$

For $\lambda = 0$: $\operatorname{gap}(H_\Lambda) = \min\{1, \kappa\}$.

Example Edge Mode for $0 < |\lambda| << 1$: Lowest eigenvalue of H_Λ in invariant subspace $\operatorname{span}\{|1100100\ldots0\rangle, |1011000\ldots0\rangle\}$ is $\frac{\kappa}{\kappa+1}|\lambda|^2 + \mathcal{O}(|\lambda|^4)$.

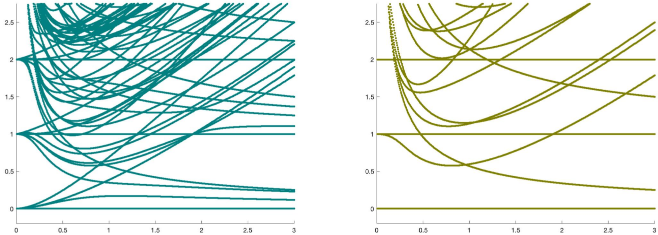


Figure: Plot $(\lambda, E(\lambda))$ of $\operatorname{spec} H_{[1,9]}(\lambda)$ and $\operatorname{spec} H^{\mathrm{per}}_{[1,9]}(\lambda)$ for $\kappa = (3^{3/4}/4)\lambda^{-3/4}$.

Uniform Spectral Gap

$$\gamma(\kappa,\lambda) := \inf_{|\Lambda|\geq L} \mathrm{gap}(H_\Lambda).$$

Nachtergaele-Warzel-Y. '21 (Bulk Gap) There is a constant $C(\lambda,\kappa) = \mathcal{O}(1)$ so that for all $0 < |\lambda| < 5.3$ and $L \geq 11$

$$\mathrm{gap}(H_{[1,L]}^{\mathrm{per}}) \geq C(\lambda,\kappa)\left(\min_{k=7,8,9} \mathrm{gap}(H_{[1,k]}) - \frac{1}{L}\right) \propto |\lambda|^2$$

Moreover, for $\lambda = 0$: $\mathrm{gap}(H_\Lambda) = \min\{1,\kappa\}$.

Edge Modes for $0 < |\lambda| << 1$: Lowest eigenvalue of H_Λ is $\frac{\kappa}{\kappa+1}|\lambda|^2 + \mathcal{O}(|\lambda|^4)$ in invariant subspace $\mathrm{span}\{|110010\ldots0\rangle, |101100\ldots0\rangle\}$.

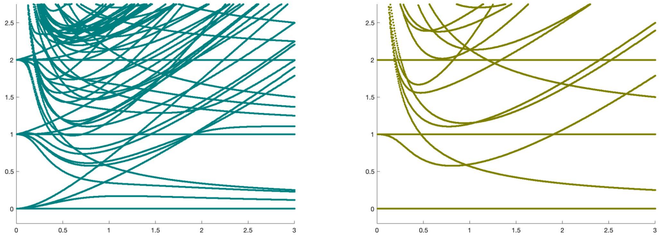
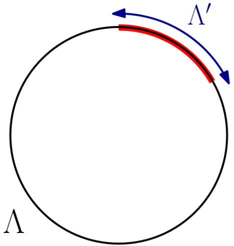


Figure: Plot $(\lambda, E(\lambda))$ of $\mathrm{spec}\, H_{[1,9]}(\lambda)$ and $\mathrm{spec}\, H_{[1,9]}^{\mathrm{per}}(\lambda)$ for $\kappa = (3^{3/4}/4)\lambda^{-3/4}$.

Edge States vs. Uniform Bulk Gaps Estimates

It is an interesting question in its own right to consider a bulk spectral gap in the presence of edges states, e.g. for studying topological insulators Loring '19, Hege-Moscolari-Teufel '22.

The main difficulty in producing uniform estimates on the bulk gap comes from the general approach based on localizing excitations:



$$\Rightarrow \qquad \sup_{|\Lambda|>>|\Lambda'|} \mathrm{gap}(H_\Lambda^{\mathrm{per}}) \geq C \cdot \mathrm{gap}(H_{\Lambda'})$$

Main Result: Bulk Spectral Gap

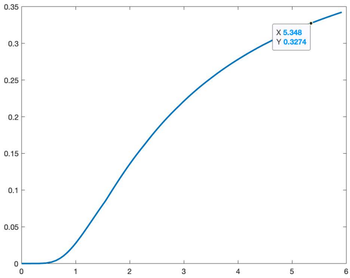


Figure: Plot of $(\lambda, f(|\lambda|^2))$.

Bulk Gap Result: Warzel-Y. '22 *For all $\lambda \neq 0$ with $f(|\lambda|^2) < 1/3$ (i.e. $|\lambda| < 5.3\ldots$)*

$$\liminf_{|\Lambda|\to\infty} \mathrm{gap}(H_\Lambda^{\mathrm{per}}) \geq \min\left\{\gamma^{\mathrm{per}}, \frac{\kappa}{6(1+2|\lambda|^2)}\left(1-\sqrt{3f(|\lambda|^2)}\right)^2\right\}$$

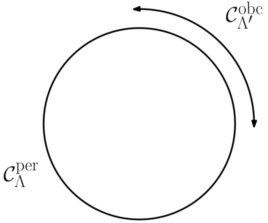
$$\gamma^{\mathrm{per}} = \frac{1}{3}\min\left\{1, \frac{\kappa}{\kappa+1}, \frac{\kappa}{2+2\kappa|\lambda|^2}\right\}$$

Remarks:

- Bulk gap stays open despite edge states for OBC.
- Analogous result for the $\nu = 1/2$ (bosonic) truncated model Warzel-Y. '22.

Block-Diagonalization Strategy

$$H_\Lambda^\# = \begin{bmatrix} \mathcal{C}_\Lambda^\# \;\; \mathcal{G}_\Lambda^\# & 0 \\ 0 & (\mathcal{C}_\Lambda^\#)^\perp \;\; \mathcal{E}_{\Lambda'}^{\mathrm{obc}} \end{bmatrix}$$



For both $\# \in \{\mathrm{obc}, \mathrm{per}\}$ decompose $\mathcal{H}_\Lambda = \mathcal{C}_\Lambda^\# \oplus (\mathcal{C}_\Lambda^\#)^\perp$ so that:

- Invariant under $H_\Lambda^\#$: $H_\Lambda^\# \mathcal{C}_\Lambda^\# \subseteq \mathcal{C}_\Lambda^\#$
- Contains ground state space: $\mathcal{G}_\Lambda^\# := \ker(H_\Lambda^\#) \subseteq \mathcal{C}_\Lambda^\#$
- Separates edge states: Edge states satisfy $\mathcal{E}_{\Lambda'}^{\mathrm{obc}} \subseteq (\mathcal{C}_\Lambda^{\mathrm{obc}})^\perp \subseteq (\mathcal{C}_\Lambda^{\mathrm{per}})^\perp$ for all $\Lambda' \subseteq \Lambda$.

As a consequence:

$$\mathrm{gap}(H_\Lambda^\#) = \min\left\{E_1(\mathcal{C}_\Lambda^\#),\ E_0\big((\mathcal{C}_\Lambda^\#)^\perp\big)\right\} \quad \text{where}$$

$$E_1(\mathcal{C}_\Lambda^\#) := \inf_{\psi \in \mathcal{C}_\Lambda^\# \cap (\mathcal{G}_\Lambda^\#)^\perp} \frac{\langle \psi | H_\Lambda^\# \psi \rangle}{\|\psi\|^2}, \qquad E_0\big((\mathcal{C}_\Lambda^\#)^\perp\big) := \inf_{\varphi \in (\mathcal{C}_\Lambda^\#)^\perp} \frac{\langle \varphi | H_\Lambda^\# \varphi \rangle}{\|\varphi\|^2}.$$

Constructing Invariant Subspaces with Ground States

Since the interaction terms are all nonnegative:

$$\ker(H_\Lambda) = \bigcap_{k=a}^{b-2} \ker(n_k n_{k+2}) \cap \bigcap_{k=a+1}^{b-2} \ker(q_k).$$

Observations:

- $|\mu\rangle$ is a ground state of the electrostatic terms iff $\mu_k \mu_{k+2} = 0$ for all k.
- $q_k = c_k c_{k+1} - \lambda c_{k-1} c_{k+2}$ acts nontrivially on the sites $[k-1, k+2]$:

$$q_k \left(|1001\rangle + \lambda|0110\rangle\right) = 0$$

Moreover, $\mathrm{span}\{|1001\rangle, |0110\rangle\}$ is invariant under the hopping term $q_k^* q_k$.

Starting from $|100100...\rangle$, can construct a set of occupation states that span an invariant subspace of H_Λ by replacing '1001' with '0110':

1	0	0	1	0	0	1	0	0	1	0	0
0	1	1	0	0	0	1	0	0	1	0	0
1	0	0	0	1	1	0	0	0	1	0	0
1	0	0	1	0	0	0	1	1	0	0	0
0	1	1	0	0	0	0	1	1	0	0	0

$$\psi_\Lambda(R) = \sum_{T \leftrightarrow R} \lambda^{d(T)} |T\rangle$$

Jansen-Lieb-Seiler '08, Jansen '11, Nakamura-Wang-Bergholtz '12

BVMD Tiling Spaces (OBC)

More generally, an invariant subspace of H_Λ is generated by any root tiling R of Λ consisting of **void**, **monomer**, and **boundary tiles**:

[0] [1 0 0] Left: [1 1 0 0 0] Right: [1] [1 0] [0 1 1]

Example:

[1 1 0 0 0][1 0 0][0][1 0 0][1 0 0][0][1 0 0][1]

[1 1 0 0 0][1 0 0][0][0 1 1 0 0 0][0][1 0 0][1]
[1 1 0 0 0][1 0 0][0][1 0 0][1 0 0][0][0 1 1 0]
[1 1 0 0 0][1 0 0][0][0 1 1 0 0 0][0][0 1 1 0]

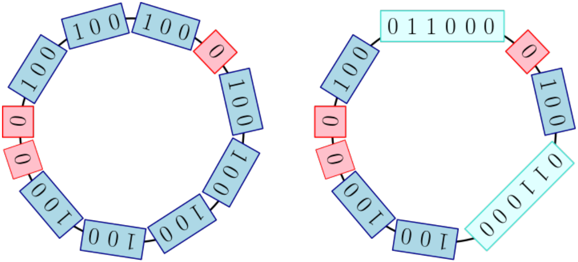
The Boundary-Void-Monomer-Dimer (BVMD) space generated by a root tiling R is

$$\mathcal{C}_\Lambda(R) = \operatorname{span}\{|T\rangle : T \leftrightarrow R\}.$$

Lemma: Nachtergaele-Warzel-Y. '21 $\mathcal{G}_\Lambda^{\mathrm{obc}} \subseteq \mathcal{C}_\Lambda^{\mathrm{obc}} := \bigoplus_R \mathcal{C}_\Lambda(R)$. BVMD spaces generated by different roots are orthogonal, and each contains a unique ground state:

$$\psi_\Lambda(R) = \sum_{T \leftrightarrow R} \lambda^{d(T)} |T\rangle.$$

VMD Tiling Spaces (PBC)



Analogous construction in the case of periodic boundary conditions:

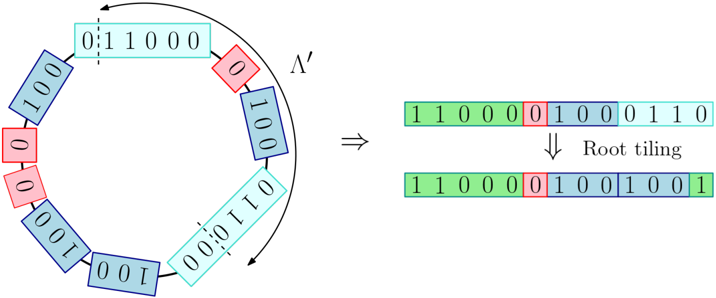
$$\mathcal{G}_\Lambda^{\mathrm{per}} \subseteq \mathcal{C}_\Lambda^{\mathrm{per}} := \bigoplus_R \mathcal{C}_\Lambda^{\mathrm{per}}(R), \quad \psi_\Lambda^{\mathrm{per}} = \sum_{T \leftrightarrow R} \lambda^{d(T)} |T\rangle,$$

where root tilings of the ring only use monomers and voids. Note: $\mathcal{C}_\Lambda^{\mathrm{per}} \subseteq \mathcal{C}_\Lambda^{\mathrm{obc}}$

Properties: 1. $\dim \mathcal{G}_\Lambda^{\mathrm{per}} \propto \left(\frac{1+\sqrt{5}}{2}\right)^{|\Lambda|}$ 2. Maximum filling: $N_\Lambda(R)/|\Lambda| \leq 1/3$

Isospectral relationship

If $\Lambda' \subset \Lambda$, then $T_\Lambda \restriction_{\Lambda'} = T_{\Lambda'}$:



As a consequence, if $|\Lambda| \geq |\Lambda'| + 4$, then for either $\# \in \{\mathrm{obc}, \mathrm{per}\}$,

$$\mathrm{spec}(H_{\Lambda'}^{\mathrm{obc}} \restriction_{\mathcal{C}_\Lambda^\#}) = \mathrm{spec}(H_{\Lambda'}^{\mathrm{obc}} \restriction_{\mathcal{C}_{\Lambda'}^{\mathrm{obc}}}).$$

This is the key relationship for successfully applying spectral gap techniques (martingale method and finite size criterion) in the tiling space.

Edge Tiling Spaces (OBC)

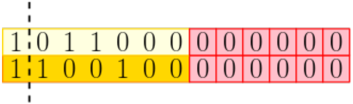
For open boundary conditions, every state with energy $\mathcal{O}(|\lambda|^2)$ belongs to an invariant subspace generated from a root tiling consisting of the BVMD-tiles

0   1 0 0   Left: 1 1 0 0 0   Right: 1   1 0   0 1 1

and at least one edge boundary tile:

Left: 1 1 0 0 1 0 0   Right: 1 0 0 1 1

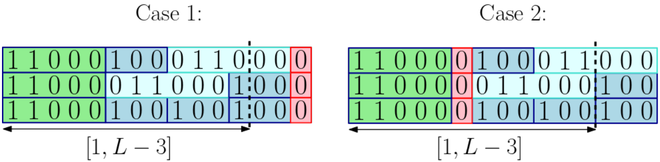
The edge tiling spaces require several other new tiles and replacement rules. Regardless, these tilings only differ from BVMD tilings at the first and/or last site of Λ.



Lower Bound Strategy:

1. Lower bound on $E_1(\mathcal{C}_\Lambda^{\mathrm{obc}})$: Apply martingale method from Nachtergaele-Sims-Y. '18. Result follows from showing that for $\Lambda_L = [1, L]$:

$$\|G_{[L-8,L]}(\mathbb{1} - G_{[1,L]})G_{[1,L-3]}\|_{\mathcal{C}_{\Lambda_L}^{\mathrm{obc}}} \leq \sup_{n\geq 4} f_n(|\lambda|^2) =: f(|\lambda|^2).$$



- Case 1: $(\mathbb{1} - G_{[1,L]})G_{[1,L-3]}\mathcal{C}_{\Lambda_L}(R) = \{0\}$
- Case 2 $(\mathbb{1} - G_{[1,L]})G_{[1,L-3]}\mathcal{C}_{\Lambda_L}(R) = \mathrm{span}\{\eta_{\Lambda_L}(R)\}$.

The function

$$f_n(r) = r\alpha_n\alpha_{n-2}\left(\frac{[1-\alpha_{n-1}(1+r)]}{1+2r} + \alpha_{n-3}\frac{r(1-\alpha_{n-1})^2}{1+r}\right)$$

depends on $\alpha_m = \|\varphi_{m-1}\|^2/\|\varphi_m\|^2$, where φ_m is the ground state on $3m$ sites associated generated by m-monomers.

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$$\|G_{[L-8,L]}(\mathbb{1} - G_{[1,L]})G_{[1,L-3]}\|_{\mathcal{C}_{\Lambda_L}^{\mathrm{obc}}} \leq \sup_{n\geq 4} f_n(|\lambda|^2) =: f(|\lambda|^2).$$

2. Lower bound on $E_1(\mathcal{C}_\Lambda^{\mathrm{per}})$: Apply finite size criterion from Knabe '89 with isospectral relation.

$$\mathrm{gap}(H_{[1,L]}^{\mathrm{per}} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}}) \geq C\left(\mathrm{gap}(H_{[1,L/3]}^{\mathrm{obc}} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}}) - \frac{1}{L}\right) = C\left(E_1(\mathcal{C}_{[1,L/3]}^{\mathrm{obc}}) - \frac{1}{L}\right)$$

3. Lower bound on $E_0((\mathcal{C}_\Lambda^{\mathrm{per}})^\perp)$: Electrostatic estimates via Cauchy-Schwarz. Does not use OBC Hamiltonian! Setting $e_\Lambda(\mu) = \sum_{k=a}^{b} \mu_k \mu_{k+2}$, we produce a bound of the form

$$\langle\psi|H_\Lambda^{\mathrm{per}}\psi\rangle = \sum_{|\mu\rangle\notin\mathcal{C}_\Lambda^{\mathrm{per}}} e_\Lambda(\mu)|\psi(\mu)|^2 + \sum_{\nu\in\{0,1\}^\Lambda}\sum_{k=a}^{b}\kappa|\langle\nu|q_k\psi\rangle|^2 \geq \gamma^{\mathrm{per}} \sum_{|\mu\rangle\notin\mathcal{C}_\Lambda^{\mathrm{per}}} |\psi(\mu)|^2.$$

where γ^{per} is independent of $\psi = \sum_{|\mu\rangle\notin\mathcal{C}_\Lambda^{\mathrm{per}}} \psi(\mu)|\mu\rangle \in (\mathcal{C}_\Lambda^{\mathrm{per}})^\perp$.

Conclusion:

Last remarks:

1. Improved spectral gap bounds from Nachtergaele-Warzel-Y. '20 for the truncated $\nu = 1/3$ Haldane pseudopotential to avoid edge excitations.
2. Same approach works to prove bulk gap of truncated $\nu = 1/2$ pseudopotential (bosonic model).
3. Incompressibility is a consequence of the uniform spectral gap and maximal $\nu = 1/3$ filling of the ground state.
4. Nachtergaele-Warzel-Y. '20 Modified tilings used to identify invariant subspaces we conjecture to contain first and second excited energy states for $|\lambda| << 1$.

Thank you for your attention!

Incompressibility

Recall that the truncated FQHE model is particle preserving and set

$$E^0_\Lambda(N) = \min \mathrm{spec}(H^{\mathrm{per}}_\Lambda \restriction_{\mathcal{H}_\Lambda(N)}), \quad \mathcal{H}_\Lambda(N) = \{\psi : N_\Lambda \psi = N\psi\}$$

The maximal ground state filling satisfies $N^{\max}_\Lambda \leq |\Lambda|/3$. Thus, given uniform spectral gap γ,

$$E^0_\Lambda(N) \begin{cases} = 0, & N \leq N^{\max}_\Lambda \\ > \gamma, & N > N^{\max}_\Lambda \end{cases}$$

Thus, at zero temperature and critical filling, i.e. $N = N^{\max}_\Lambda$, the compressibility $\kappa_\Lambda(N)$ vanishes as $|\Lambda| \to \infty$:

$$\kappa_\Lambda(N)^{-1} := |\Lambda| \cdot \frac{E^0_{\Lambda_+}(N) + E^0_{\Lambda_-}(N) - 2E^0_\Lambda(N)}{(2\pi\ell^2)^2} \geq \frac{\gamma|\Lambda|}{(2\pi\ell^2)^2}$$

where $\Lambda_\pm$ is the volume obtained by increasing/decreasing Λ by a single site.

Tiling Spaces and Excited States

Invariant subspaces with bulk excitations can also be identified by adding new tiles and replacement rules. Conjecture: For $|\lambda|$ small, low-lying excitations belong to tiling spaces with 'quasi-hole/quasi-particle' roots:

1st Excited Root: 1 0 0 | 1 0 0 | 0 | 1 0 0 | 1 0 | 1 0 0

2nd Excited Root: 1 0 0 | 1 0 0 | 0 | 1 0 | 1 0 0 | 1 0 0

Energy Bounds: Nachteragele-Warzel-Y. '21 The minimal energy in these subspaces is approximated via:

- Upper bound: variational state calculation.
- Lower bound: applying finite volume Hamiltonian associated to region effected by quasi-hole/quasi-particle.

$$E^{(1)}_{\mathrm{approx}} = 1 - \frac{2\kappa}{\kappa - 1}|\lambda|^2 + \mathcal{O}(|\lambda|^4)$$

$$E^{(2)}_{\mathrm{approx}} = 1 - \frac{\kappa}{\kappa - 1}|\lambda|^2 + \mathcal{O}(|\lambda|^4)$$

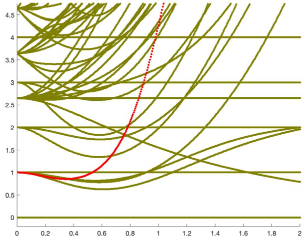

Figure: Plot of $(\lambda, \mathrm{spec}\, H^{\mathrm{per}}_{[1,12]})$ for $\kappa = 2.648$ with $E^{(1)}_{\mathrm{approx}}$.

Tiling Spaces and Excited States

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1st Excited Root:	1 0 0	1 0 0	0	1 0 0	1 0	1 0 0

2nd Excited Root:	1 0 0	1 0 0	0	1 0	1 0 0	1 0 0

$$E^{(1)}_{\text{approx}} = 1 - \frac{2\kappa}{\kappa - 1}|\lambda|^2 + \mathcal{O}(|\lambda|^4)$$

$$E^{(2)}_{\text{approx}} = 1 - \frac{\kappa}{\kappa - 1}|\lambda|^2 + \mathcal{O}(|\lambda|^4)$$

Many-body Scars: Exact mid and high energy states with low Schmidt rank created using voids as domain walls. E.g.:

ψ =	φ_L	0	1 0	1 0 0	0	φ_R

Figure: Exact energy state with $E = 1$.

Bounding $E_0\big((\mathcal{C}_\Lambda^{\mathrm{per}})^\perp\big)$

Partition non-BVMD tiling configurations $\mathcal{S}_\Lambda = \{\mu : |\mu\rangle \notin \mathcal{C}_\Lambda^{\mathrm{per}}\}$ as

$$\mathcal{S}_\Lambda = \mathcal{S}_E \,\dot\cup\, \mathcal{S}_D, \quad \mathcal{S}_E = \left\{\mu \in \mathbb{N}_0^\Lambda : e_\Lambda(\mu) > 0\right\}$$

where we introduce the electrostatic energy $e_\Lambda(\mu) = \sum_{k=a}^{b} \mu_k \mu_{k+2}$.

Goal: For any $\psi = \sum_{\mu\in\mathcal{S}_\Lambda} \psi(\mu)|\mu\rangle \in (\mathcal{C}_\Lambda^{\mathrm{per}})^\perp$, bound

$$\langle\psi|H_\Lambda^{\mathrm{per}}\psi\rangle = \sum_{\mu\in\mathcal{S}_E} e_\Lambda(\mu)|\psi(\mu)|^2 + \sum_{\nu\in\{0,1\}^\Lambda} \sum_{k=a}^{b} \kappa|\langle\nu|q_k\psi\rangle|^2 \geq \gamma^{\mathrm{per}} \sum_{\mu\in\mathcal{S}_\Lambda} |\psi(\mu)|^2.$$

Strategy: Clearly, $\langle\psi|H_\Lambda^{\mathrm{per}}\psi\rangle \geq \sum_{\mu\in\mathcal{S}_E} |\psi(\mu)|^2$ since $e_\Lambda(\mu) \geq 1$.

Choose individual (ν_μ, k_μ) for each $\mu \in \mathcal{S}_D$ and apply CS to show that a similar lower bound holds for sum over $\mu \in \mathcal{S}_D$. Combining the two estimates produces the desired result.

Bounding $E_0\big((\mathcal{C}_\Lambda^{\mathrm{per}})^\perp\big)$

Example: $\mu = (11001000\ldots0) \in \mathcal{S}_D$ (edge state configuration).

For PBC: Choosing $k_\mu = a$ with $\nu_\mu = (000010\ldots0)$ produces:

$$|\langle \nu_\mu | q_{k_\mu}\psi\rangle|^2 = |\psi(\mu) - \lambda\psi(\eta)|^2 \geq (1-\delta)|\psi(\mu)|^2 - \frac{1-\delta}{\delta}|\lambda|^2|\psi(\eta)|^2$$

for any $0 < \delta < 1$ where $\eta = (001010\ldots01)$. Picking $\delta = \frac{\kappa|\lambda|^2}{1+\kappa|\lambda|^2}$ yields,

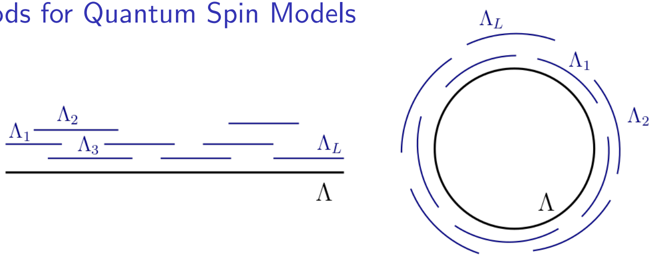
$$e_\Lambda(\eta)|\psi(\eta)|^2 + \kappa|\langle \nu_\mu | q_{k_\mu}\psi\rangle|^2 \geq \frac{\kappa}{1+\kappa|\lambda|^2}|\psi(\mu)|^2$$

Partitioning $\mathcal{S}_D$ appropriately and similarly estimating shows:

$$2\langle\psi|H_\Lambda^{\mathrm{per}}\psi\rangle \geq \min\left\{\frac{\kappa}{2+2\kappa|\lambda|^2}, \frac{\kappa}{\kappa+1}\right\}\sum_{\mu\in\mathcal{S}_D}|\psi(\mu)|^2$$

from which the result follows.

Gap Methods for Quantum Spin Models



- For $\dim(\mathcal{H}_\Lambda) < \infty$ and frustration-free model: $\mathcal{G}_{\Lambda'} \equiv \ker(H_{\Lambda'} \otimes \mathbb{1}_{\Lambda \setminus \Lambda'}) \ \ \forall \Lambda' \subseteq \Lambda$.
- Requires a covering of Λ by a sequence of smaller intervals with OBC.

$$\gamma = \inf_i \operatorname{gap}(H_{\Lambda_i}), \qquad \Gamma = \sup_i \|H_{\Lambda_i}\|, \qquad \Lambda_{n,k} = \bigcup_{i=k}^{n-k+1} \Lambda_i$$

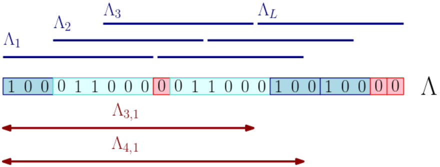
- Martingale Method: If $\epsilon := \sup_n \|G_{\Lambda_{n+1}}(\mathbb{1} - G_{\Lambda_{n+1,1}})G_{\Lambda_{n,1}}\| < 1/\sqrt{\ell}$, then

$$\operatorname{gap}(H_\Lambda) \geq \frac{\gamma}{\ell}(1 - \epsilon\sqrt{\ell})^2.$$

- Finite Size Criterion: For any n such that $|\Lambda_{n,k}| < |\Lambda|$ for all k:

$$\operatorname{gap}(H_\Lambda^{\mathrm{per}}) \geq \frac{\gamma n}{\ell \Gamma (n-1)} \left[\inf_k \operatorname{gap}(H_{\Lambda_{n,k}}) - \frac{\Gamma}{n} \right]$$

Bounding $E_1(\mathcal{C}_\Lambda^{\mathrm{obc}})$



We apply the martingale method to the Hilbert space $\mathcal{C}_\Lambda^\infty$ and Hamiltonians

$$H_{\Lambda'}^{\mathrm{BVMD}} := (H_{\Lambda'} \otimes \mathbb{1}_{\Lambda \setminus \Lambda'}) \restriction_{\mathcal{C}_\Lambda^{\mathrm{obc}}} \quad \forall \Lambda' \subseteq \Lambda$$

where the overlap intervals are chosen so that $\ell = 3$ and $|\Lambda_n| = 9$ for all n.

Calculations using isospectrality and orthogonality of the BVMD states/spaces gives

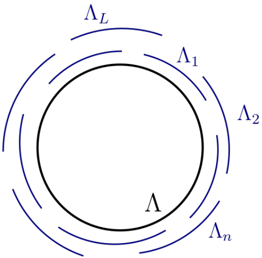
$$\gamma = \mathrm{gap}(H_{[1,9]}^{\mathrm{BVMD}}) = \mathrm{gap}(H_{[1,9]} \restriction_{\mathcal{C}_{[1,9]}^{\mathrm{obc}}}) = \kappa$$

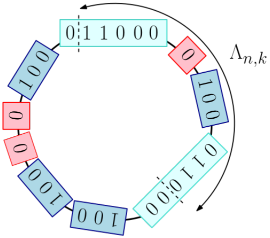
$$\epsilon = \sup_n \| G_{\Lambda_n}^{\mathrm{BVMD}} (\mathbb{1} - G_{\Lambda_{n,1}}^{\mathrm{BVMD}}) G_{\Lambda_{n-1,1}}^{\mathrm{BVMD}} \| \leq \sqrt{f(|\lambda|^2)}$$

where $\mathcal{G}_{\Lambda'}^{\mathrm{BVMD}} = \ker(H_{\Lambda'}^{\mathrm{BVMD}}) \subseteq \mathcal{C}_\Lambda^{\mathrm{obc}}$. This produces the final estimate:

$$E_1(\mathcal{C}_\Lambda^{\mathrm{obc}}) \geq \frac{\kappa}{3} \left(1 - \sqrt{3 f(|\lambda|^2)}\right)^2 .$$

Bounding $E_1(\mathcal{C}_\Lambda^{\mathrm{per}})$





We apply Knabe's finite size criteria to the Hilbert space $\mathcal{C}_\Lambda^{\mathrm{per}}$ and Hamiltonians

$$H_\Lambda^{\mathrm{per}} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}} \quad \text{and} \quad H_{\Lambda_{n,k}} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}}$$

where $\Lambda_{n,k} = \bigcup_{i=k}^{n+1-k} \Lambda_i$ chosen so $|\Lambda_i| = 5, 6$. Another isospectral argument shows

$$\| H_{\Lambda_i} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}} \| = \kappa(1 + 2|\lambda|^2) \quad \text{and} \quad \mathrm{gap}(H_{\Lambda_{n,k}} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}}) = E_1(\mathcal{C}_{\Lambda_{n,k}}^{\mathrm{obc}})$$

This yields:

$$E_1(\mathcal{C}_\Lambda^{\mathrm{per}}) \geq \frac{n}{2(n-1)(1+2|\lambda|^2)} \left(E_1(\mathcal{C}_{\Lambda_{n,k}}^{\mathrm{obc}}) - \frac{\kappa(1+2|\lambda|^2)}{n} \right)$$