Universal and Exact Aspects of Ideal Flatbands

—— quantum geometry and momentum space curved quantum Hall effect

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Outline

- Fractional Chern insulators
  - motivation, stability (Girvin-MacDonald-Platzmann or W-∞ algebra)
- Common lore: no exact GMP algebra in flat band (lattice) systems
  - Concrete models that violates it
- **Ideal flatbands:**
  - Quantum geometry
  - Universal wavefunction
  - Emergent exact GMP algebra
- Implications to real materials
  - moire graphene, moire TMD

Key References:

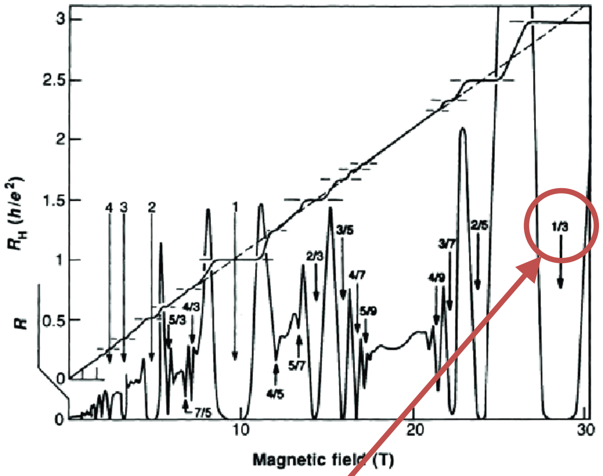
- Exact Landau level description of geometry and interaction in a flatband — **Jie Wang**, Cano, Millis, Liu and Yang (PRL, 2021);
- Hierarchy of ideal flatband in chiral twisted multilayer graphene models — **Jie Wang** and Zhao Liu (PRL, 2022);
- Origin of model fractional Chern insulators in all topological ideal flatbands — **Jie Wang**, Klevtsov and Zhao Liu (PRR, 2022).

Related on similar topics (partial list): Bergholtz, Cano, Crepel, Estienne, Fu, Kruchkov, Ozawa, Mera, Mora, Repellin, Vishwanath.

Introduction to Chern bands and fractional Chern insulators

Fractional quantum Hall effect

Fractionally quantized Hall plateau



Laughlin state

$$\sigma_{xy} = -\sigma_{yx} = \frac{1}{3}\frac{e^2}{h}$$

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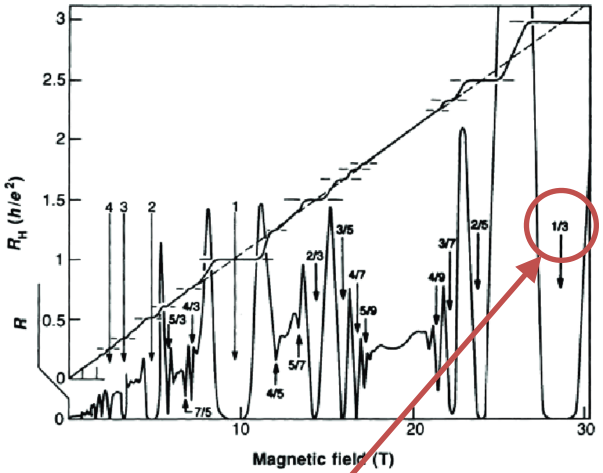

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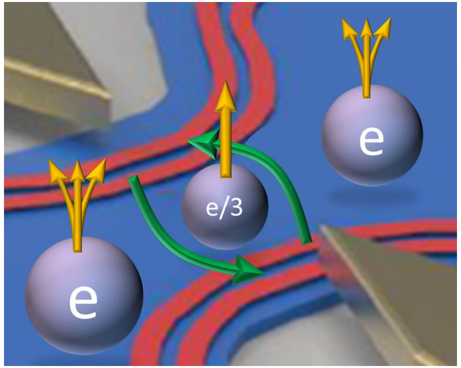
Chiral edge modes
Anyon excitation



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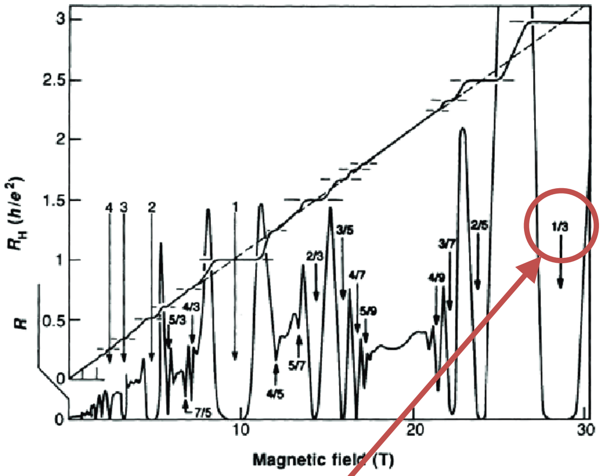

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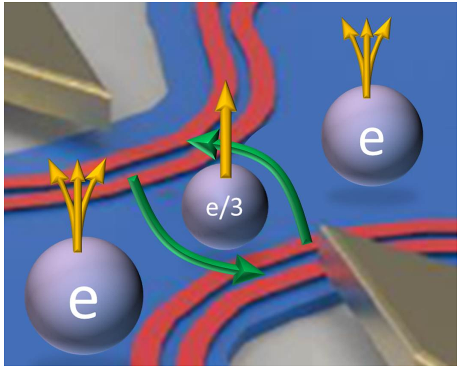
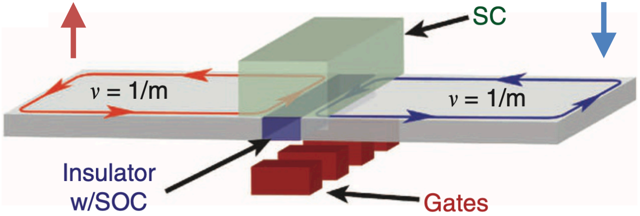
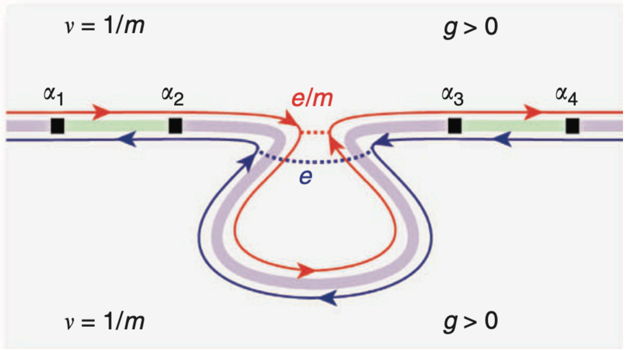


Applications:
Topological quantum computation

Clarke, Alicea, Stengel (2012)
See also A. Stern et al; M. Cheng; others





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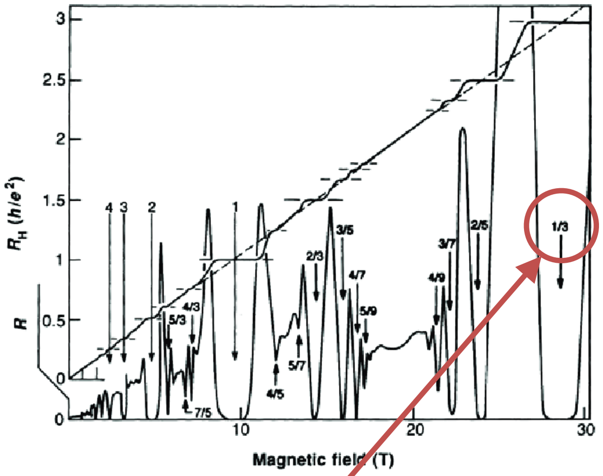

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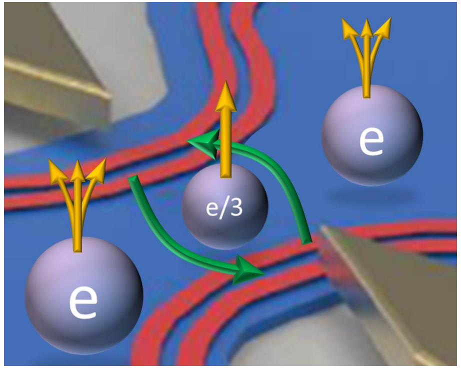

Superconductivity is not compatible with **magnetic field.**

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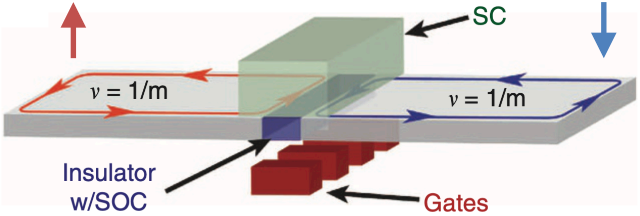
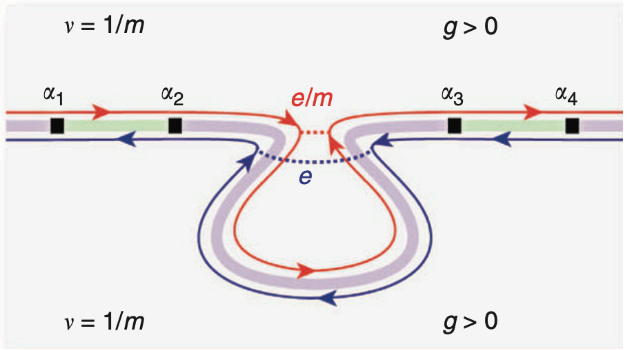
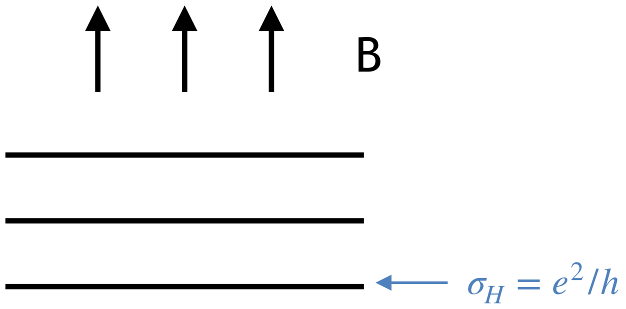




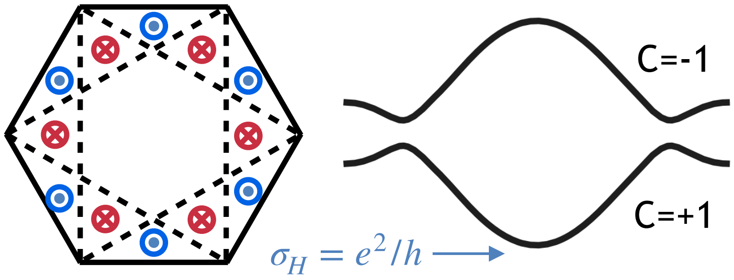
Fractional Chern insulator in Chern band

Landau level



$\sigma_H = e^2/h$

Chern band



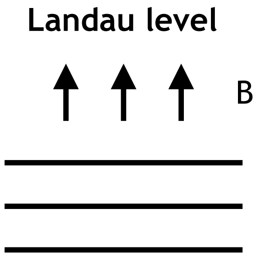
$\sigma_H = e^2/h$

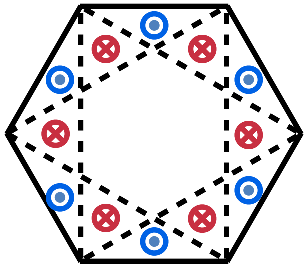
FCI = partial filling + interaction ?

Not guaranteed! Competing phases (Fermi liquid, CDW), geometry important!

Haldane (88)

Comparing Landau level and flatband





Landau levels:

- Zero dispersion
- Uniform, continuous translation symmetry
- Chern number C=1
- Holomorphic LLL wave function

Flat Chern bands:

- Dispersive
- Lattice translation symmetry
- Chern number C=0,1,2,…,
- Non-holomorphic wave function

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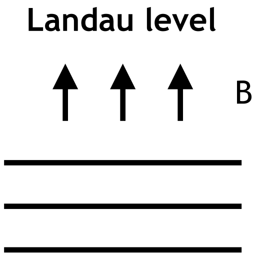

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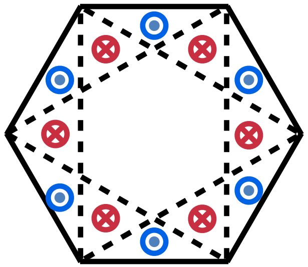
Exact zero energy ground state for model interactions (W-∞ algebra, Haldane-pseudopotential)



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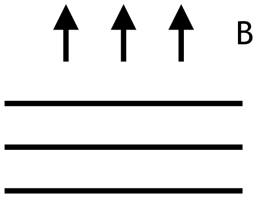

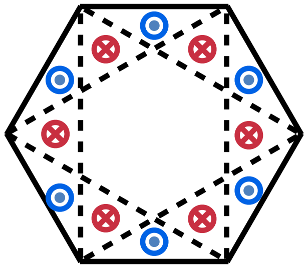

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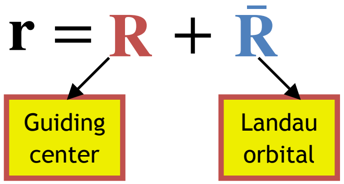
Exact zero energy ground state for model interactions
(W-∞ algebra, Haldane-pseudopotential)

Usually interacting ground state is not **exact**.
(Ideal flatbands are exceptions)

Review of Landau level: guiding center

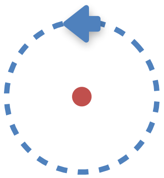
Haldane (83)
Girvin, MacDonald, Platzmann (86)
A. Cappelli, Trugenberger, Zemba (92)



$$\mathbf{r} = \mathbf{R} + \bar{\mathbf{R}}$$

Guiding center

Landau orbital



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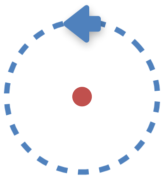

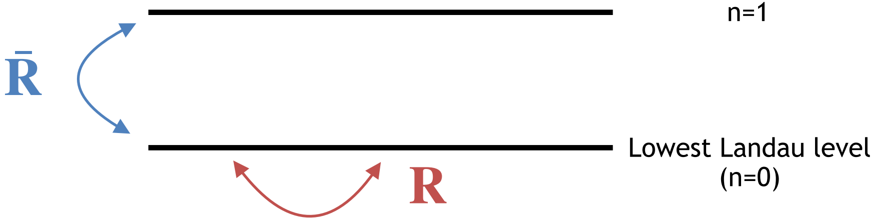

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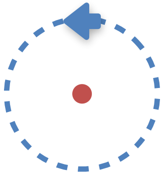

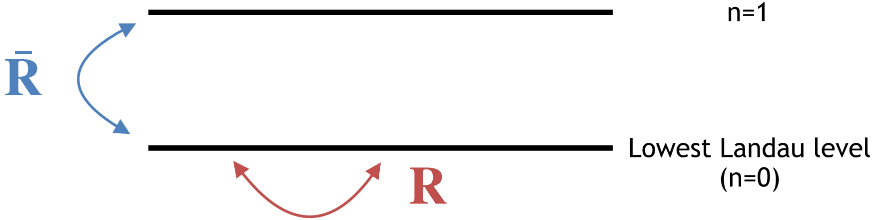

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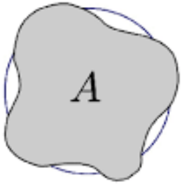
- Girvin-MacDonald-Plazman ($W - \infty$) algebra

$$H = \sum_q v_q \rho_q \rho_q^\dagger \qquad [\rho_q, \rho_{q'}] = 2i\sin(\frac{q \times q'}{2} l_B^2) \cdot \rho_{q+q'}$$



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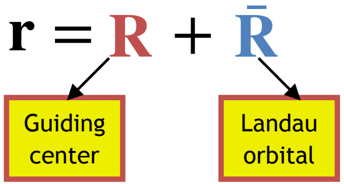

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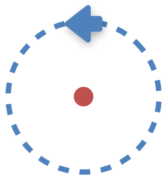

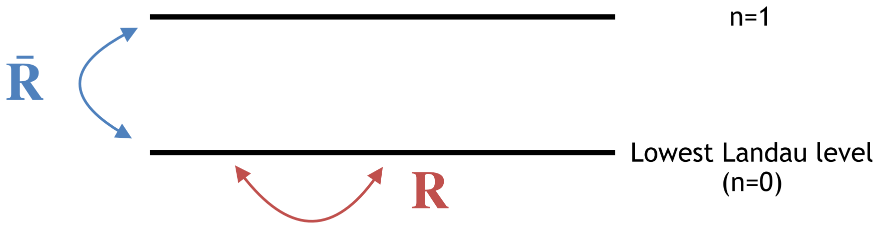

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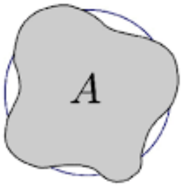
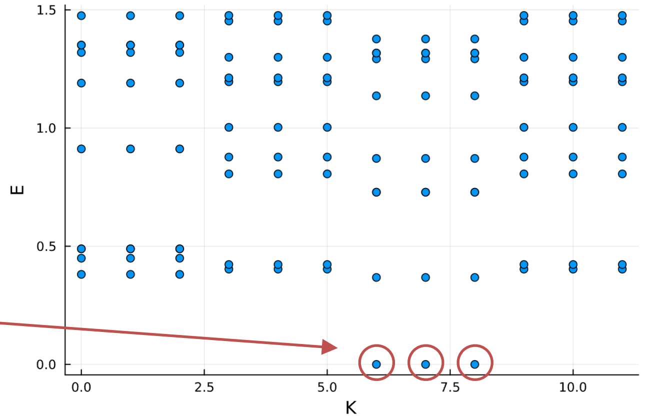


- Pseudopotential projectors

$$P_m^{ij} = 2\int \frac{dq^2 l_B^2}{2\pi} L_m(q^2) e^{-\frac{1}{2}q^2} e^{iq(R_i - R_j)}$$



Exact ground state
(crucial for stability)

Common lore for Chern bands

Common lore:

- Only Landau level has exact GMP algebra
- It's destroyed by Berry curvature fluctuation

Formalism of Band Theory. E. Blount (62)

Common lore for Chern bands

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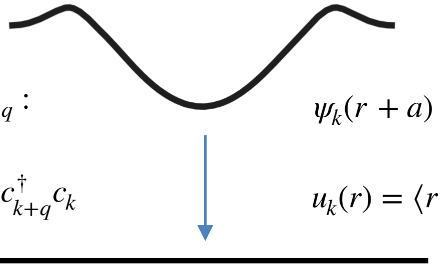
- Only Landau level has exact GMP algebra
- It's destroyed by Berry curvature fluctuation

Even if we ignore one-body dispersion, and consider projected short-ranged interaction:

$$H = \sum_q V_q : \rho_q \rho_{-q} :$$

$$\rho_q = \sum_k \langle u_{k+q} | u_k \rangle c^\dagger_{k+q} c_k$$

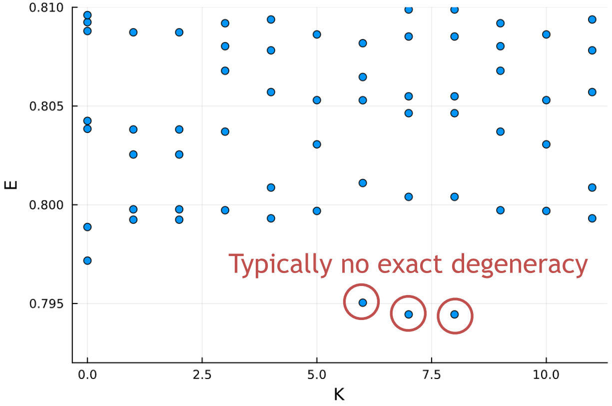
Quantum geometry



$$\psi_k(r+a) = e^{ika}\psi_k(r)$$

$$u_k(r) = \langle r | u_k \rangle = e^{-ikr}\psi_k(r)$$

Spectrum



Typically no exact degeneracy

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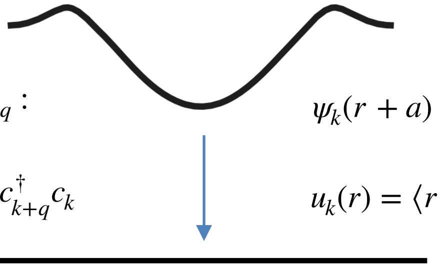

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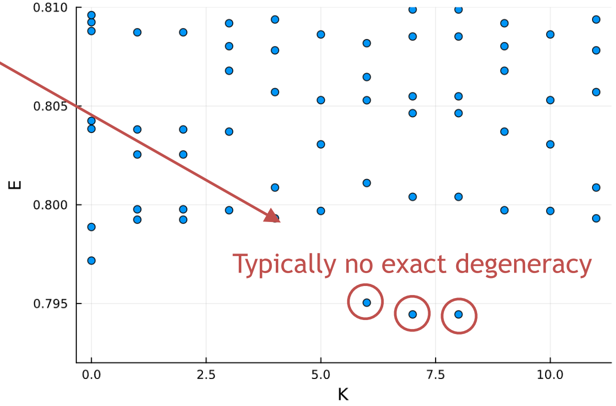
Quantum geometry

Spectrum

Reduced stability:

- Due to wavefunction's inhomogeneity (lack of continuous translation)
- Can lead to phase transitions (to CDW)



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Reasoning is in below:

- Can define projected coordinates

$$R_k^a = -i\partial_k^a + A_k^a$$

- However, algebra no longer closes:
  - $[R_k^a, R_k^b] = -i\Omega_k \epsilon^{ab}$
  - $[R_k^a, [R_k^b, R_k^c]] \sim \partial_k \Omega_k + \ldots$
- No closed density algebra, because of the lack of continuous translation invariance.

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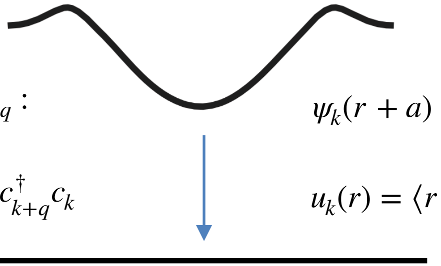

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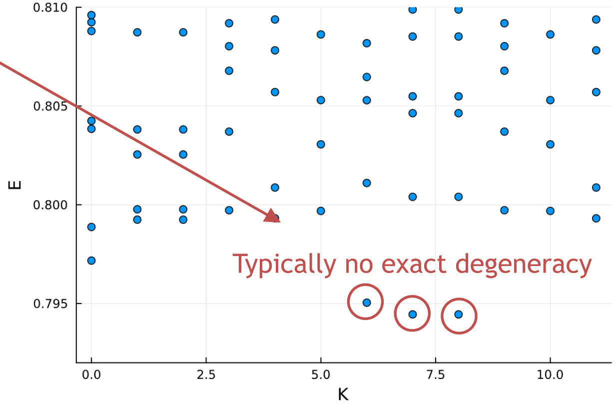

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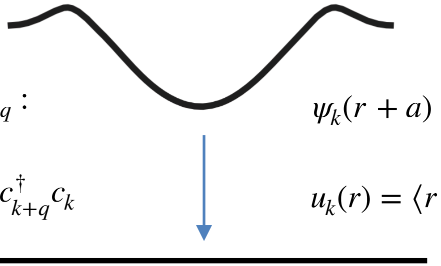

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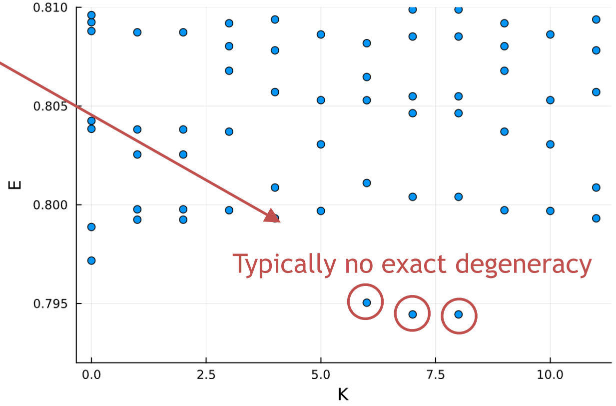
There are a large class of flat bands (ideal flatbands) that **disprove** this common lore, allowing **exact** interacting ground states with **arbitrary amount** of $\Omega_{\mathbf{k}}$ fluctuation.

Spectrum



Typically no exact degeneracy

Quantum geometry

Berry connection: $A^a(k) \equiv -\langle u_k | i\partial^a_k u_k \rangle$

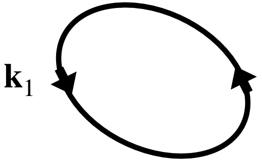
Covariant derivative: $|D^a_{\mathbf{k}} u_{\mathbf{k}}\rangle = \left(\partial^a_{\mathbf{k}} - iA^a_{\mathbf{k}}\right)|u_{\mathbf{k}}\rangle, \quad \langle u_{\mathbf{k}} | D^a_{\mathbf{k}} u_{\mathbf{k}}\rangle = 0.$

Quantum geometric tensor: $\mathcal{Q}^{ab}_{\mathbf{k}} \equiv \langle D^a_{\mathbf{k}} u_{\mathbf{k}} | D^b_{\mathbf{k}} u_{\mathbf{k}}\rangle$

Quantum geometric tensor:

$$\mathcal{Q}^{ab}_{\mathbf{k}} = \underbrace{g^{ab}_{\mathbf{k}}}_{\text{Fubini-Study metric}} + \frac{i}{2}\epsilon^{ab}\underbrace{\Omega_{\mathbf{k}}}_{\text{Berry curvature}}$$

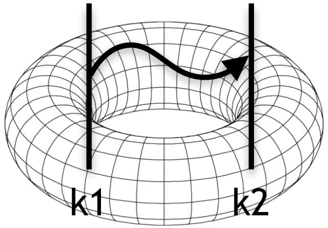
Berry curvature (phase)



$\mathbf{k}_2 = \exp(i\iint \Omega_{\mathbf{k}} d^2\mathbf{k})$

Fubini-Study metric (amplitude)

$|\langle u_{\mathbf{k}} | u_{\mathbf{k}+\delta\mathbf{k}}\rangle| \approx 1 - g^{ab}_{\mathbf{k}}\delta\mathbf{k}_a \delta k_b$



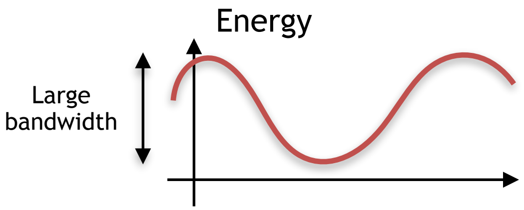
$\mathcal{Q}^{ab}_{\mathbf{k}}$ can be expressed by band projectors: $\mathcal{Q}^{ab} = 2\langle \partial^a_{\mathbf{k}} u | (\mathbb{1} - P_{\mathbf{k}}) | \partial^b_{\mathbf{k}} u\rangle$ where $P_{\mathbf{k}} = |u_{\mathbf{k}}\rangle\langle u_{\mathbf{k}}|$.

So it is positive semi-definite with two eigenvalues $\lambda'_{\mathbf{k}} \geq \lambda_{\mathbf{k}} \geq 0$.

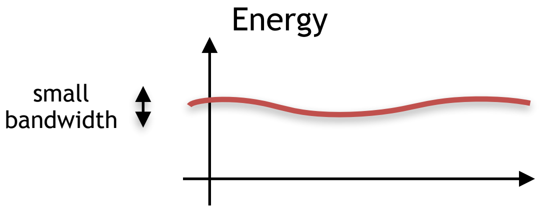
Exceptions: concrete models and common feature

General introduction to moire system

Criteria for “strong correlation”: $|t/U| \ll 1$



Large kinetic energy



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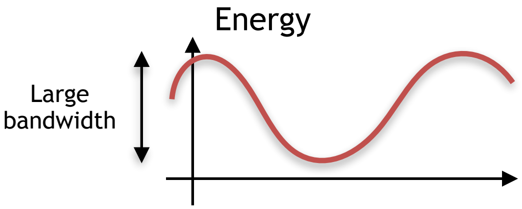
General introduction to moire system

Criteria for “strong correlation”: $|t/U| \ll 1$

A novel route (2018) to create and control “strong correlation”



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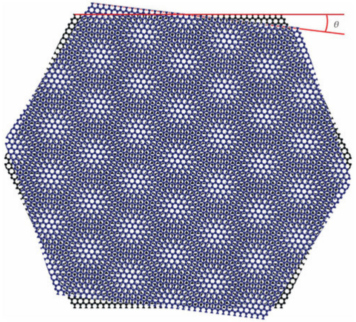
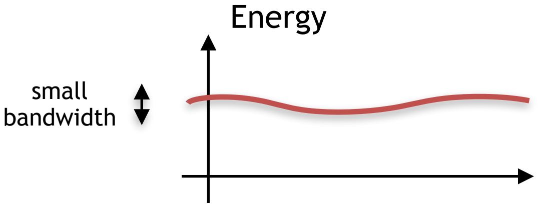


Moire pattern of twisted bilayer graphene



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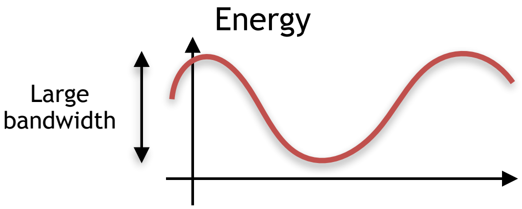

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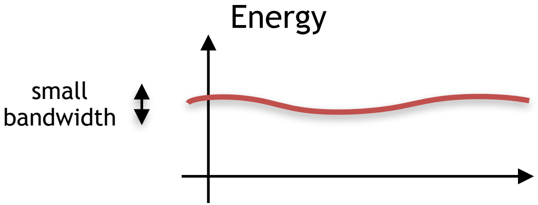

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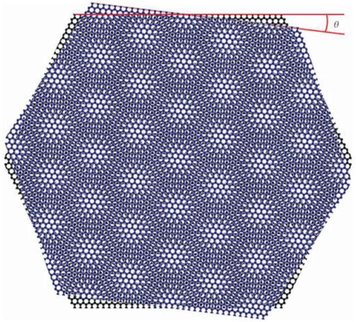

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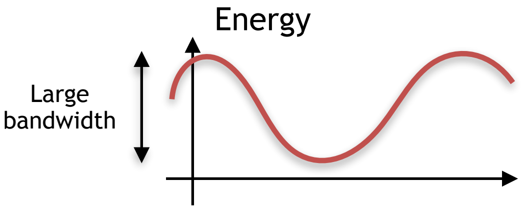

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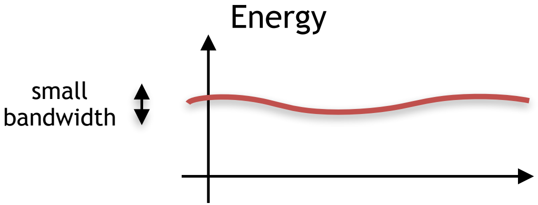

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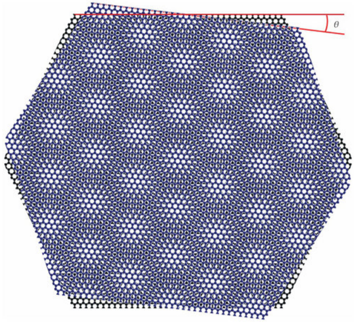

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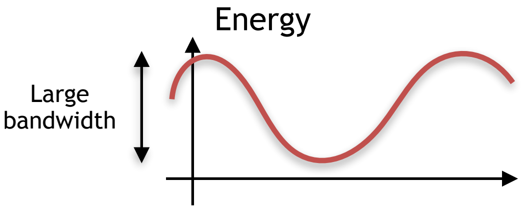

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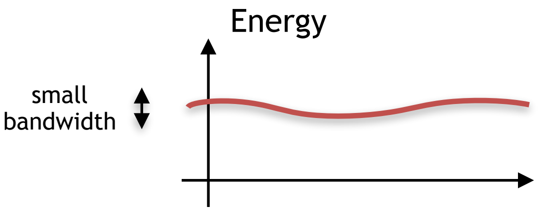

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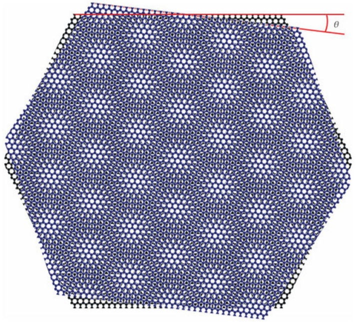

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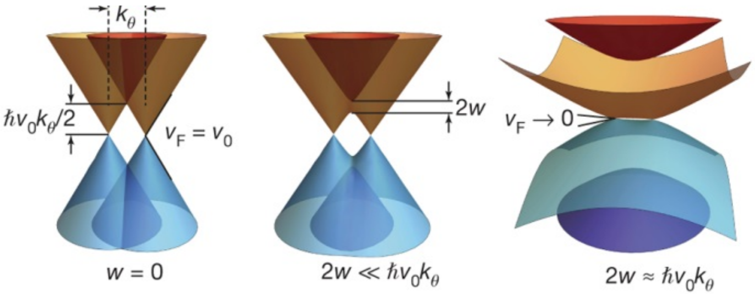

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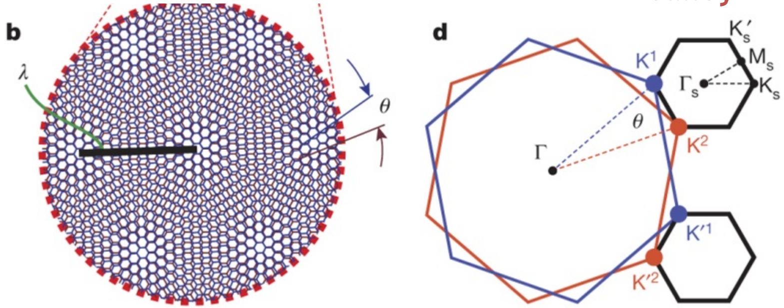
Band structure of twisted bilayer graphene

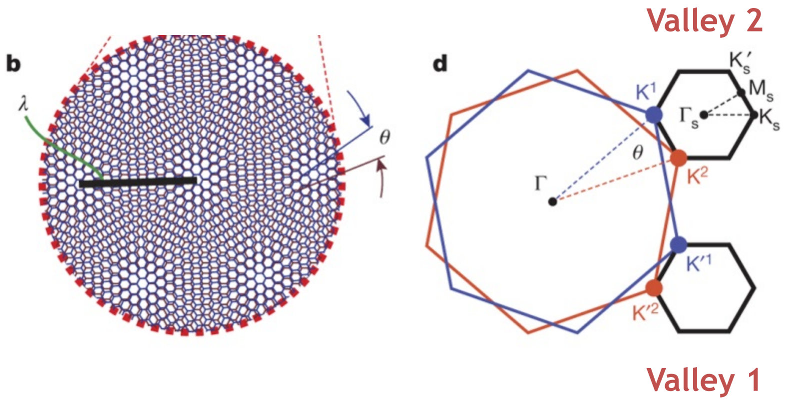
Valley 2



Valley 1

Moire (mini) Brillouin zone

Band structure of twisted bilayer graphene



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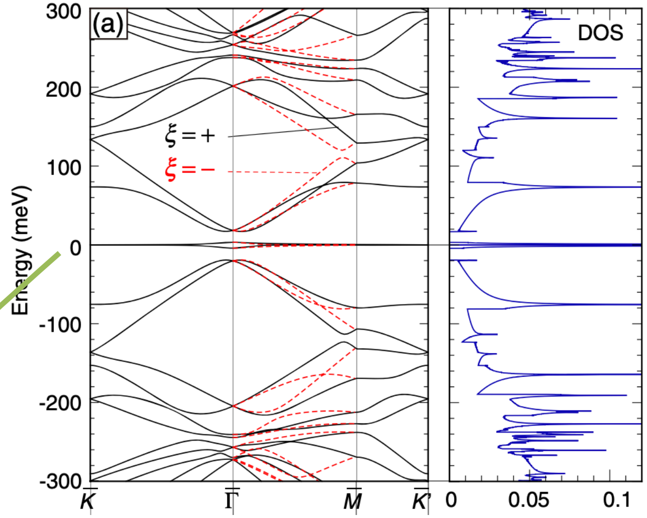
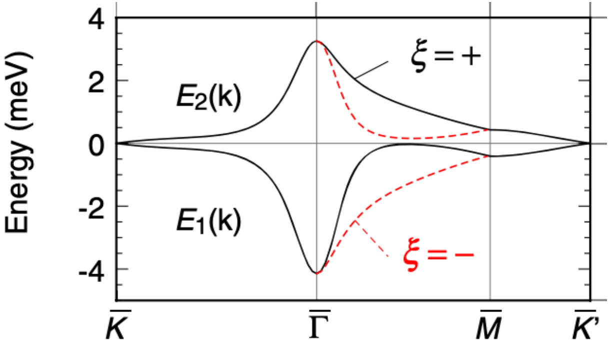


Important bands



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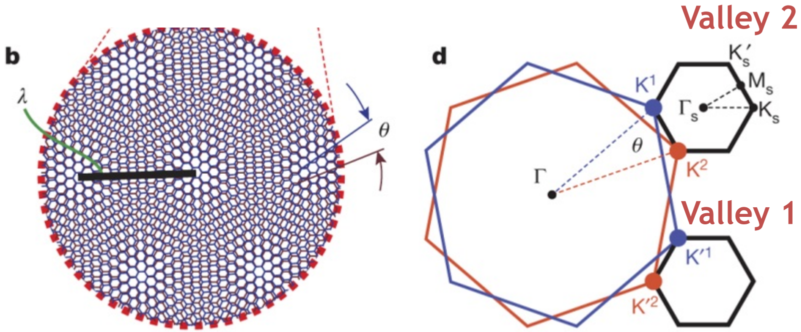

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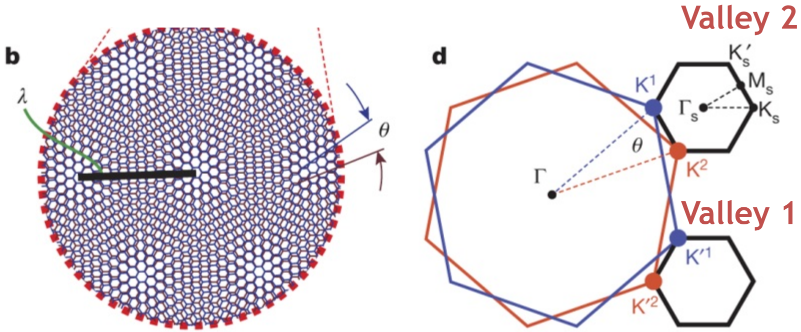

~~A/A~~ A/B

B/A ~~B/B~~

$$H = \begin{pmatrix} \overset{\text{Bottom layer}}{\boxed{-iv_0\sigma_{+\theta/2}\cdot\nabla}} & T(\mathbf{r}) \\ T^{\dagger}(\mathbf{r}) & \underset{\text{Top layer}}{\boxed{-iv_0\sigma_{-\theta/2}\cdot\nabla}} \end{pmatrix}$$

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Dirac cone (rotated frame)

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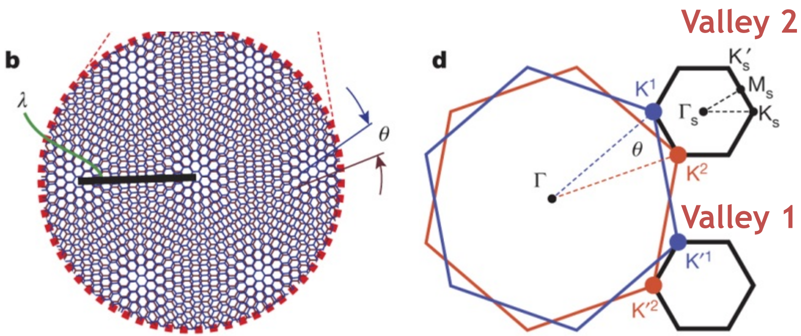

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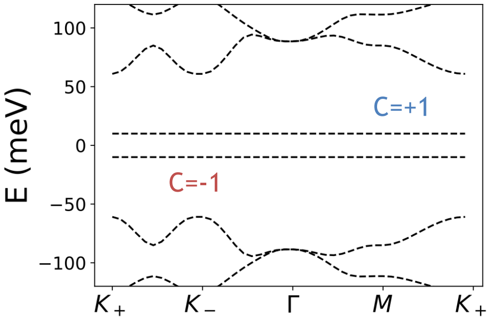
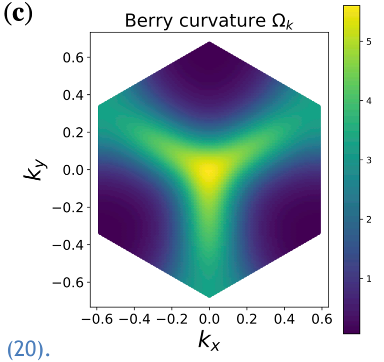


Geometry of Bloch wavefunction



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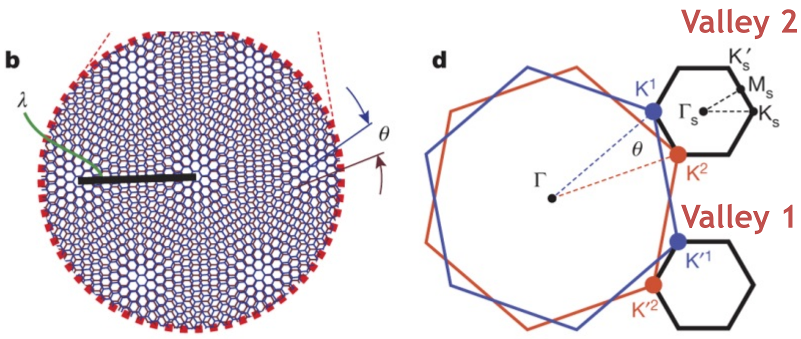

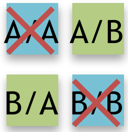

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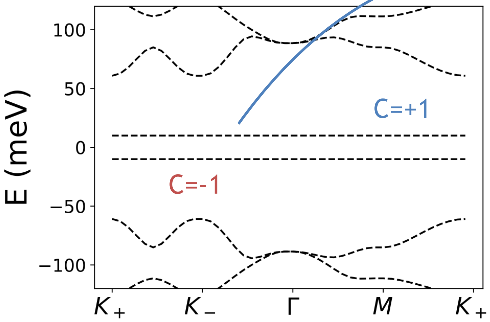
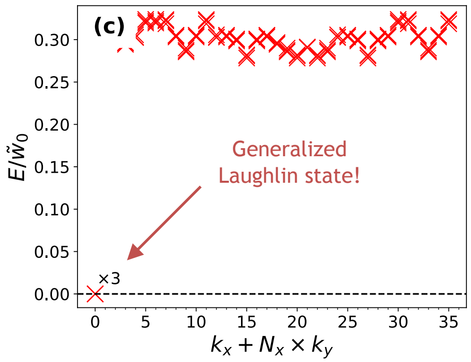


Fill $\nu = 1/3$ and turn on short-ranged interaction



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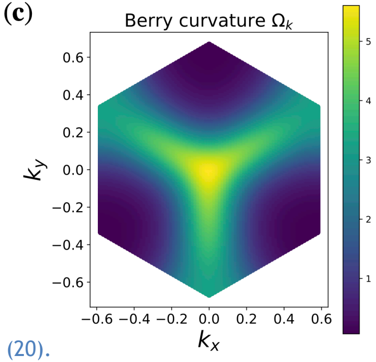

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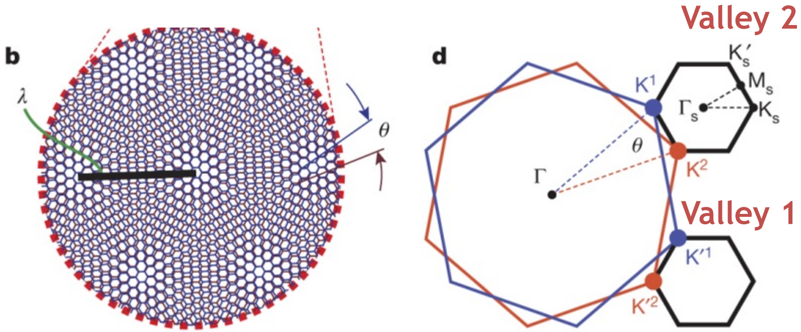

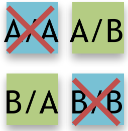

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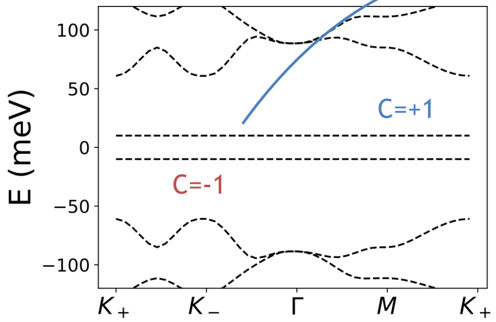

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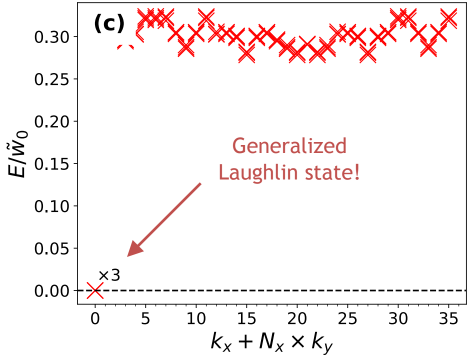
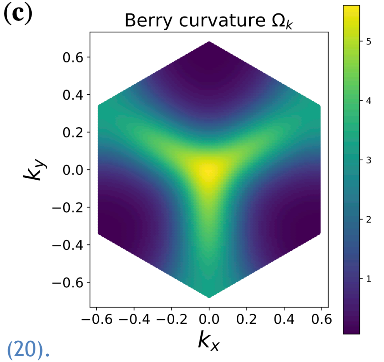


Generalized Laughlin state!

Geometry of Bloch wavefunction



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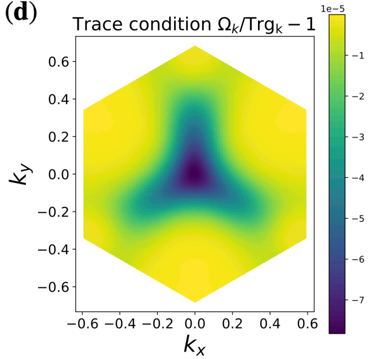

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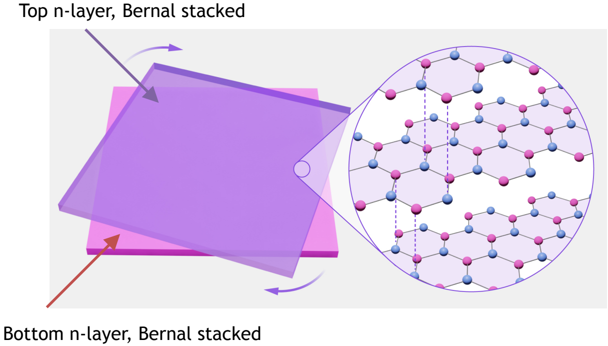

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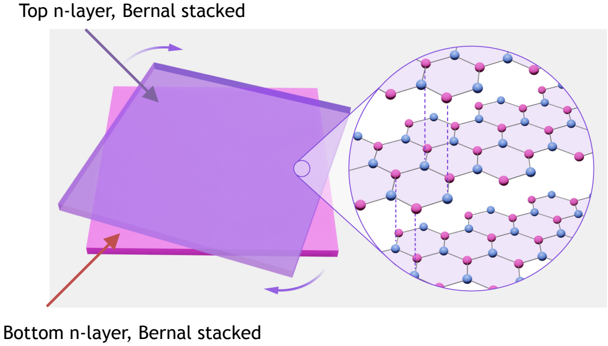
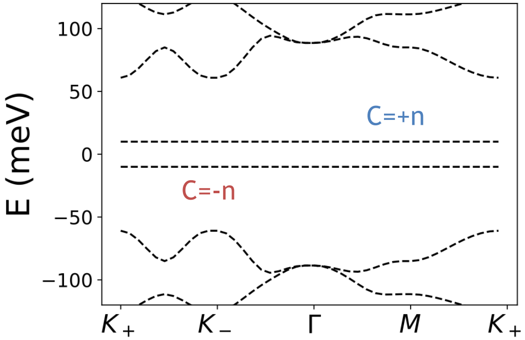


<u>Band structure @ chiral limit, magic angle</u>



Exact flat band exhibits nonuniform $\Omega_{\mathbf{k}}$,
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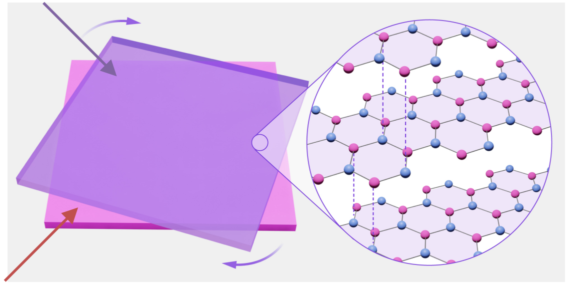
Top n-layer, Bernal stacked



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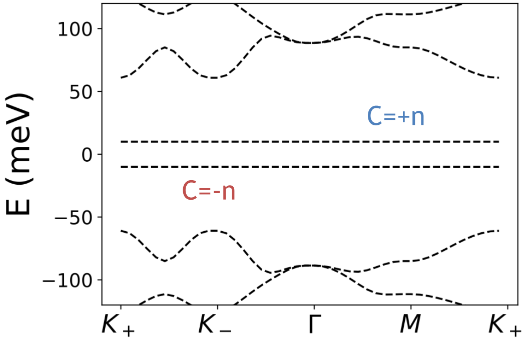

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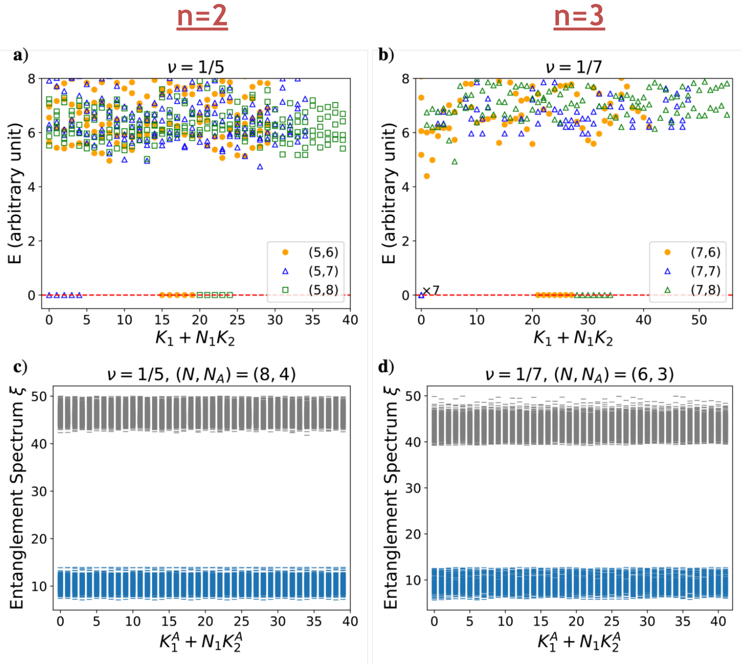

- Short range interaction exact zero modes start to occur at $\nu = 1/(2C + 1)$.
- Exactly Halperin state's filling fraction (C-layered Landau level).
- Entanglement properties all agree with model Halperin state (infinity entanglement gap).

JW and Z. Liu; PRL (22); Ledwith et al; PRL (22).

Exception III: many other models

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(A variant of Hofstadter model with fine tuned toppings)

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(2) Dirac fermion in periodic magnetic field

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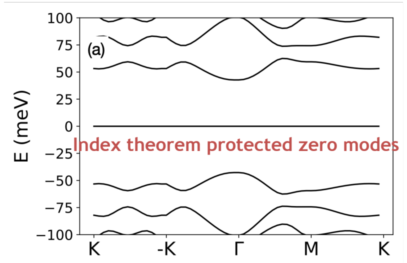

Index theorem protected zero modes

Kapit, Mueller PRL (10); J Dong, **JW**, L Fu (22).

Exception III: many other models

$\mathrm{Tr}_\omega g_{\mathbf{k}} \equiv \omega_{ab} g_{\mathbf{k}}^{ab}$

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All common features:

- Exact flat single-particle dispersion.
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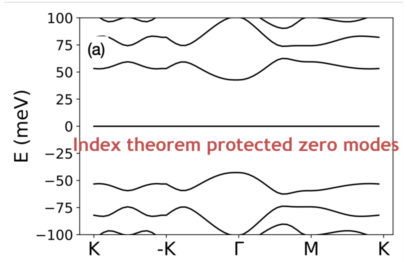

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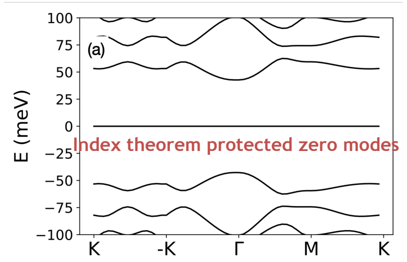

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↓

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JW, J. Cano, A.J. Millis, Z. Liu, B. Yang; PRL (2021).
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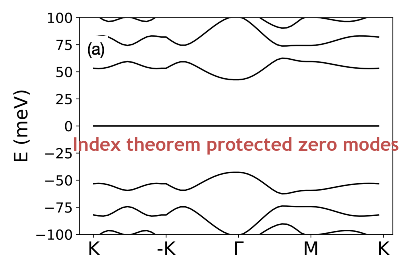

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All these examples strongly suggest:

- Universal properties implied from quantum geometry.
- Emergent projective properties (hidden GMP algebra)
- High Chern number corresponds to multiple Landau-levels.

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Ideal flatbands

Quantum geometry and momentum space holomorphicity

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Quantum geometric tensor: $\mathcal{Q}^{ab}_{\mathbf{k}} = g^{ab}_{\mathbf{k}} + \frac{i}{2}\epsilon^{ab}\Omega_{\mathbf{k}}$.

Denote its two eigenvalues $\lambda'_{\mathbf{k}} \geq \lambda_{\mathbf{k}} \geq 0$ and eigenvectors $\omega'_{\mathbf{k},a}, \omega_{\mathbf{k},a}$.

$$\mathcal{Q}^{ab}_{\mathbf{k}}\omega'_{\mathbf{k},b} = \lambda'_{\mathbf{k}}\omega'^{a}_{\mathbf{k}} \qquad \mathcal{Q}^{ab}_{\mathbf{k}}\omega_{\mathbf{k},b} = \lambda_{\mathbf{k}}\omega^{a}_{\mathbf{k}}$$

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From its positive semi-definite property: $\mathrm{Tr}_{\omega}(g_{\mathbf{k}}) \geq |\det g_{\mathbf{k}}| \geq |\Omega_{\mathbf{k}}|$.

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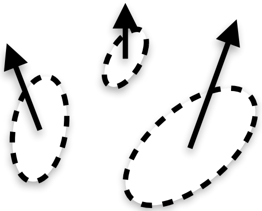
Physical intuition (make analogous to Landau level physics)

$$|\det g_{\mathbf{k}}| = |\Omega_{\mathbf{k}}|$$

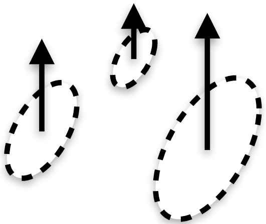


Local unit flux, but shape unfixed

Non-constant null vector

$$\mathcal{Q}^{ab}_{\mathbf{k}}\omega_{\mathbf{k},b} = 0$$

$$\mathrm{Tr}_{\omega}(g_{\mathbf{k}}) = |\Omega_{\mathbf{k}}|$$



Local unit flux, but shape unfixed

Constant null vector (complex structure $k \equiv \omega^{a}\mathbf{k}_{a}$)

$$\mathcal{Q}^{ab}_{\mathbf{k}}\omega_{b} = 0 \qquad \omega_{ab} = \omega_{a}\omega^{*}_{b} + \omega^{*}_{a}\omega_{b}$$

Quantum geometry and momentum space holomorphicity

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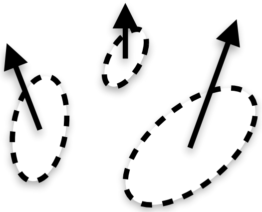

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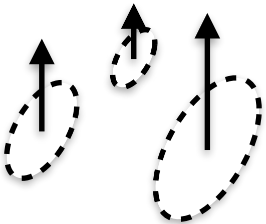

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Saturation of the trace bound ($\mathrm{Tr}_{\omega}g_{\mathbf{k}} = \Omega_{\mathbf{k}}$) is fully equivalent to momentum space holomorphilicity.

The cell-periodic part of Bloch wavefunction $u_{\mathbf{k}}(\mathbf{r}) \equiv e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{\mathbf{k}}(\mathbf{r})$ is holomorphic in $k \equiv \omega^{a}\mathbf{k}_{a}$ up to a norm:

$$u_{\mathbf{k}}(\mathbf{r}) = N_{\mathbf{k}}\; u^{\mathrm{holo}}_{k=\omega^{a}k_{a}}(\mathbf{r})$$

Bruno Mera, T. Ozawa; PRB (2021).
M. Claassen et al PRL (2015).
Ledwith et al PRR (2020).
JW et al PRL (2021)

Consequence from holomorphicity - I: universal wavefunction form (C=1)

Most general form of C=1 ideal flatband wavefunction:

$$\psi_{\mathbf{k}}(\mathbf{r}) = N_{\mathbf{k}}\ \mathscr{B}(\mathbf{r})\ \Phi_{\mathbf{k}}(\mathbf{r})$$

Normalization

k-independent function

LLL wavefunction

Torus lowest Landau level wavefunction in terms of Weierstrass Sigma function $\sigma(z + a_{1,2}) = -e^{a^*(z+a/2)}\sigma(z)$

$$\Phi_{\mathbf{k}}(\mathbf{r}) = \sigma(z - z_k)e^{z_k^* z}e^{-\frac{1}{2}|z|^2}e^{-\frac{1}{2}|z_k|^2} \qquad z_k \equiv -ik = -i\omega^a \mathbf{k}_a$$

Jie Wang, J. Cano, A.J. Millis, Z. Liu, B. Yang; PRL (2021).

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To check: $u_{\mathbf{k}}(\mathbf{r}) = e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{\mathbf{k}}(\mathbf{r}) = N_{\mathbf{k}}e^{-\frac{1}{2}|z_k|^2} \times \sigma(z - z_k)e^{z^* z_k}e^{-\frac{1}{2}|z|^2}$

Non-holomorphic part (normalization)

Holomorphic in k

Jie Wang, J. Cano, A.J. Millis, Z. Liu, B. Yang; PRL (2021).

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Generalized Laughlin wavefunction (exact zero modes):

$$\Psi_{\mathrm{FCI}} = \prod_i \mathscr{B}(r_i) \times \Psi_{\mathrm{Laughlin}}\,.$$

$$\Psi_{\mathrm{Laughlin}} = \prod_{i<j}\sigma^3(z_i - z_j)\prod_{i=1}^{3}\sigma(\sum_i z_i - \alpha_i)\,.$$

Jie Wang, J. Cano, A.J. Millis, Z. Liu, B. Yang; PRL (2021).

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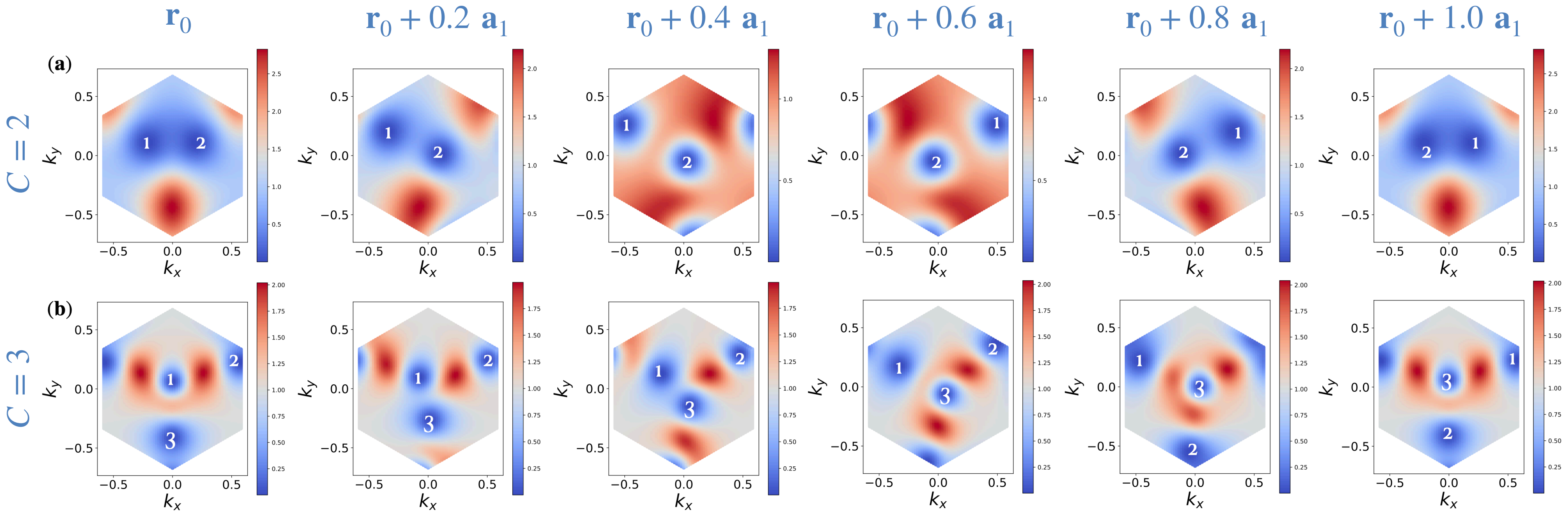
$$\Psi_{\text{Laughlin}} = \prod_{i<j} \sigma^3(z_i - z_j) \prod_{i=1}^{3} \sigma\Big(\sum_i z_i - \alpha_i\Big) \, .$$

Factor $\mathscr{B}(\mathbf{r})$ satisfies $\mathscr{B}(\mathbf{r} + \mathbf{a}) = e^{-\frac{i}{2}\mathbf{a}\times\mathbf{r}}\mathscr{B}(r)$:

- k-independent, quasi-periodic (s.t. $\psi_{\mathbf{k}}$ is Bloch).
- breaks translation symmetry from continuous to lattice
- Determines Berry curvature distribution (next slides)

Jie Wang, J. Cano, A.J. Millis, Z. Liu, B. Yang; PRL (2021).

Consequence from holomorphicity - I: universal wavefunction form (C>1)



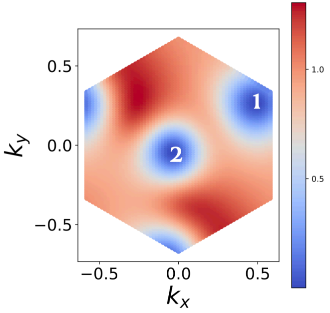
Momentum-space plots of $|\Psi_k(r)|$ for fixed r

$$C = C(\mathbf{r}) = \frac{1}{2\pi i}\oint dk\ \ \partial_k \ln u_k(\mathbf{r}) = \frac{1}{2\pi i}\oint dk\ \ \partial_k \ln | u_k\rangle$$

- Topology = zeros (Riemann-Roch Theorem)
- Dual Thouless pumping: $\mathbf{r}$ is a “boundary condition”
- Color-entangled wavefunction.

Jie Wang, S. Klevtsov, Z. Liu (2022).

Consequence from holomorphicity - I: universal wavefunction form (C>1)



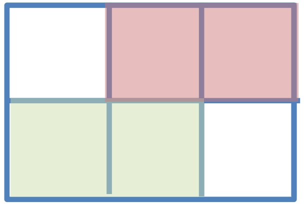
At fixed r, wavefunction are classified following classification of holomorphic line bundles (theta/Weierstrass functions).

Color-entangled wavefunction (C>1):

$$u_{\mathbf{k}}(\mathbf{r}) = \mathscr{B}(\mathbf{r})\ v_{\mathbf{k}}(\mathbf{r}) + \mathscr{B}(\mathbf{r}+\mathbf{a}_1)\ v_{\mathbf{k}}(\mathbf{r}+\mathbf{a}_1)$$

Two (C) inequivalent sets of LLL wave functions: $v_{\mathbf{k}}(\mathbf{r})$ and $v_{\mathbf{k}}(\mathbf{r}+\mathbf{a}_1)$.

Two inequivalent mapping to LLL, C=2

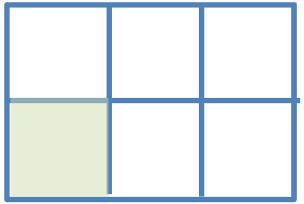


Generalized LLL wavefunction (C=1):

$$u_{\mathbf{k}}(\mathbf{r}) = \mathscr{B}(\mathbf{r})\ v_{\mathbf{k}}(\mathbf{r})$$

One set of LLL wave functions: $v_{\mathbf{k}}(\mathbf{r})$.

For comparison, C=1.



Here $v_{\mathbf{k}}(\mathbf{r}) = \sigma(z - z_k)e^{\frac{1}{C}z^* z_k}e^{-\frac{1}{2C}|z|^2}e^{-\frac{1}{2C}|z_k|^2}$

JW, Semyon Klevtsov, Zhao Liu. (2022)

Consequence from holomorphicity - II: Kahler potential & GMP algebra

Wavefunction: $u_{\mathbf{k}}(\mathbf{r}) = N_{\mathbf{k}} \times u_k^{\text{holo}}(\mathbf{r})$

Anti-holomorphic Berry connection: $\bar{A}_k \equiv \langle u_{\mathbf{k}} | \bar{\partial}_k u_{\mathbf{k}} \rangle = i \left(i N_{\mathbf{k}}^{-1} \bar{\partial}_k N_{\mathbf{k}} \right) \int_{\mathbf{r}} N_{\mathbf{k}}^2 u_k^{\text{holo}*}(\mathbf{r}) u_k^{\text{holo}}(\mathbf{r}) = -\bar{\partial}_k \log N_{\mathbf{k}}$

Berry curvature: $\Omega_{\mathbf{k}} = -i\partial_k \bar{A}_k + i\bar{\partial}_k A_k = \nabla_{\mathbf{k}}^2 \log N_{\mathbf{k}}$

Important: this relates the Berry curvature to normalization factors

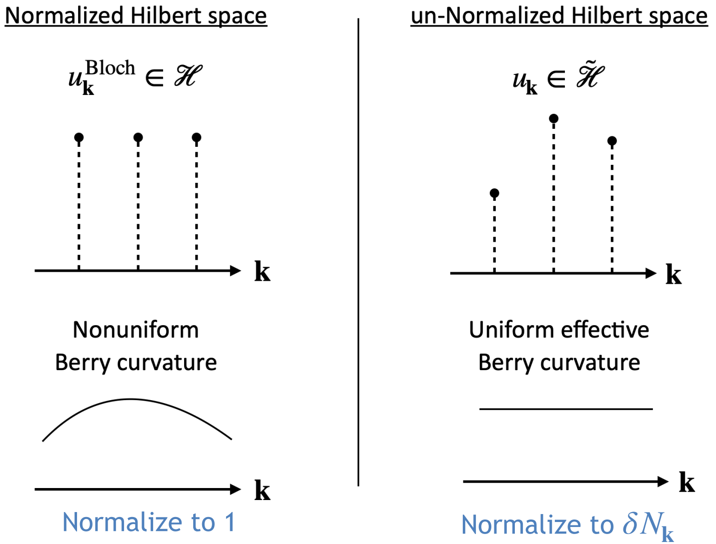
JW, Jen Cano, Andrew Millis, Zhao Liu, Bo Yang. arXiv 2105.07491 (PRL, 21)

Consequence from holomorphicity - II: Kahler potential & GMP algebra

Key intuition: $\Omega_{\mathbf{k}} = \nabla^2_{\mathbf{k}} \log N_{\mathbf{k}}$

$$\Omega_{\mathbf{k}} = C + \delta\Omega_{\mathbf{k}}$$

$$N_{\mathbf{k}} = e^{-\frac{C}{2}\mathbf{k}^2} \times \delta N_{\mathbf{k}}$$



<u>Normalized Hilbert space</u>

$u^{\text{Bloch}}_{\mathbf{k}} \in \mathscr{H}$

Nonuniform Berry curvature

Normalize to 1

<u>un-Normalized Hilbert space</u>

$u_{\mathbf{k}} \in \tilde{\mathscr{H}}$

Uniform effective Berry curvature

Normalize to $\delta N_{\mathbf{k}}$

$$u^{\text{Bloch}}_{\mathbf{k}} = \delta N_k \cdot u^{\text{holo}}_k e^{-\frac{C}{2}\mathbf{k}^2}$$

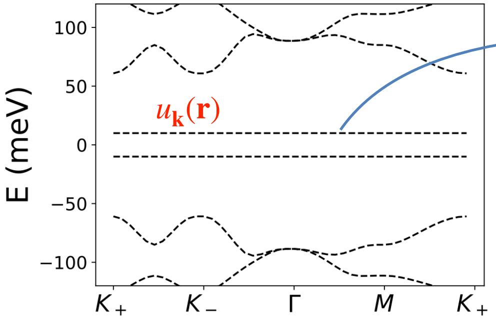
$$u_{\mathbf{k}} = u^{\text{holo}}_k e^{-\frac{C}{2}\mathbf{k}^2}$$

JW, Semyon Klevtsov, Zhao Liu. (2022)

Emergent exact new "guiding center" and "Landau orbital" operators

$$Q^a = -i\partial^a_{\mathbf{k}} - C\epsilon^{ab}k_b/2 \qquad \bar{Q}^a = -i\partial^a_{\mathbf{k}} + C\epsilon^{ab}k_b/2$$

$$[Q^a, \bar{Q}^b] = 0 \qquad [Q^a, Q^b] = -i\epsilon^{ab}C \qquad [\bar{Q}^a, \bar{Q}^b] = +i\epsilon^{ab}C$$



Exact dual LLL condition

$$\bar{a}u_{\mathbf{k}}(\mathbf{r}) = \omega_a Q^a u_{\mathbf{k}}(\mathbf{r}) = 0$$

Exact dual magnetic translation

$$e^{i\mathbf{q}_1\cdot\mathbf{Q}} e^{i\mathbf{q}_2\cdot\mathbf{Q}} = e^{iC\mathbf{q}_1\times\mathbf{q}_2/2} e^{i(\mathbf{q}_1+\mathbf{q}_2)\cdot\mathbf{Q}}$$

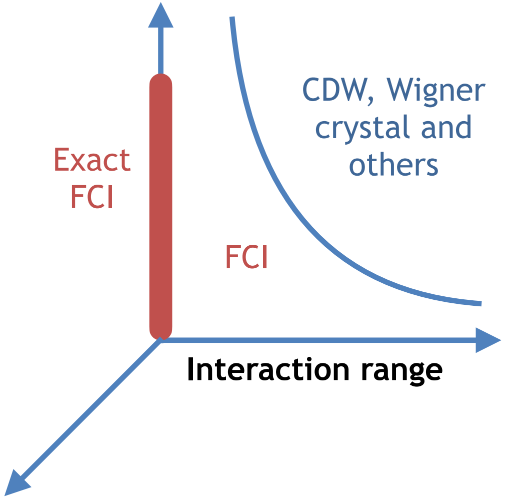
$$e^{i\mathbf{q}\cdot\mathbf{Q}} u_{\mathbf{k}}(\mathbf{r}) = e^{\frac{i}{2}\mathbf{q}\cdot\mathbf{r}} u_{\mathbf{k}}(\mathbf{r} + C\epsilon^{ab}q_b)$$

<u>Key points:</u>

- Exact closed density algebra for unnormalized states (emergent, hidden).
- Normalization does not affect interacting zero modes.
- Position-momentum duality is important.

Implication to real materials

Phase diagram for ideal flatband



Berry curvature Fluctuation

CDW, Wigner crystal and others

Exact FCI

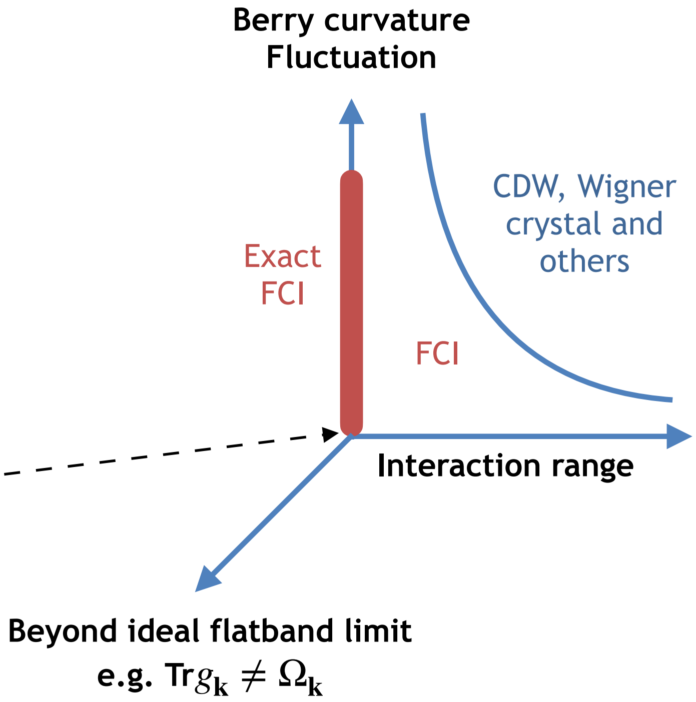
FCI

Interaction range

Beyond ideal flatband limit

e.g. Tr$g_{\mathbf{k}} \neq \Omega_{\mathbf{k}}$

Phase diagram for ideal flatband



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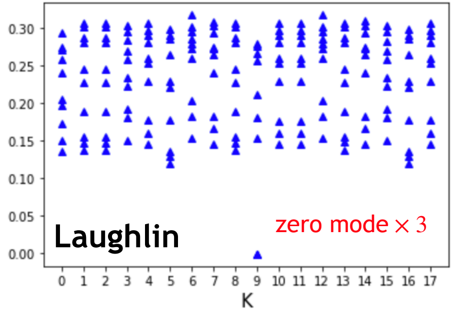

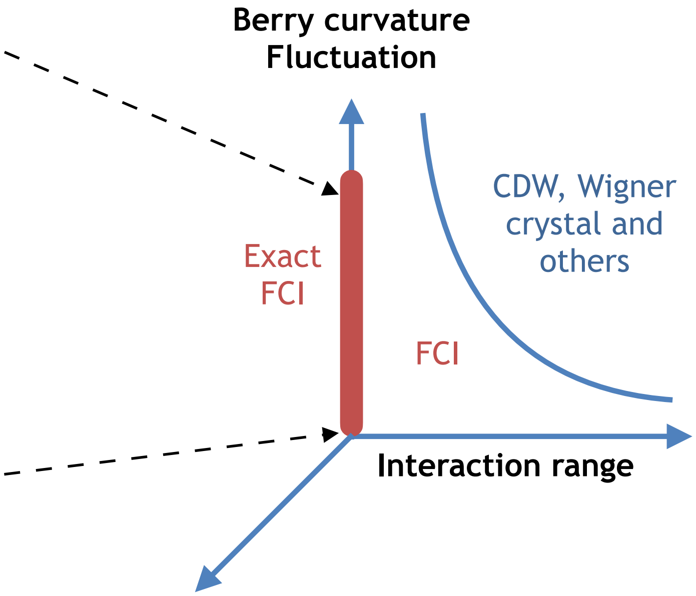

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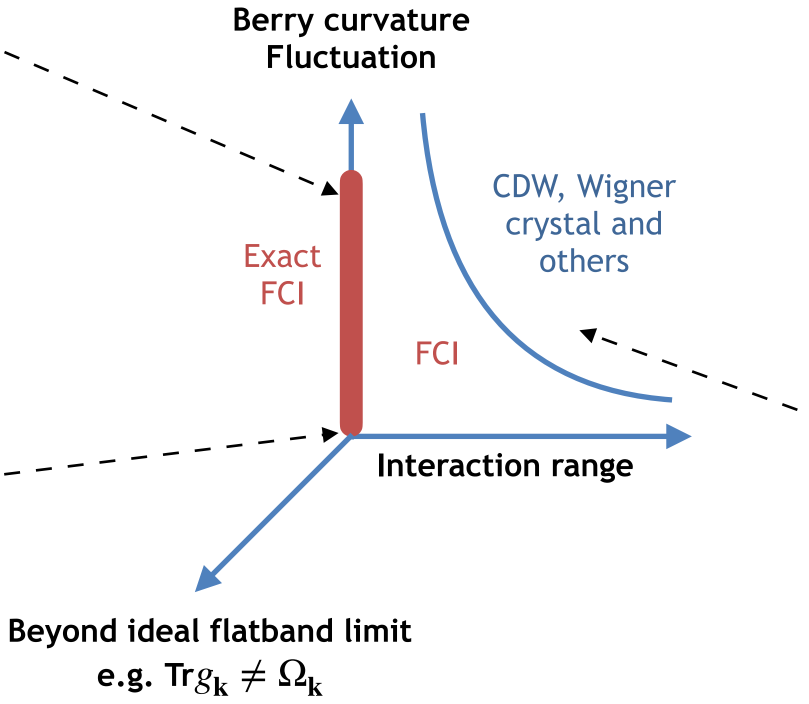

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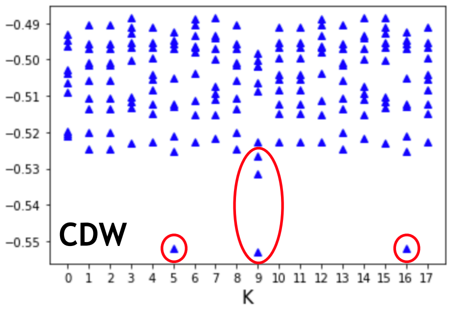


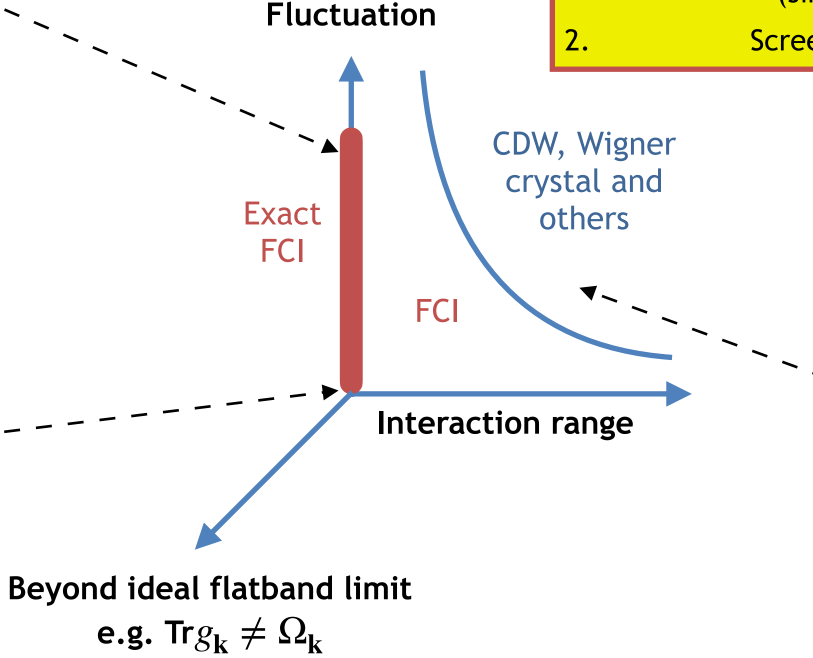
Phase diagram for ideal flatband

Message to experimentalists (to search for FCI):

1. Design/engineer material's **quantum geometry** (single-particle property)
2. Screen the **interaction range**

Laughlin

zero mode × 3



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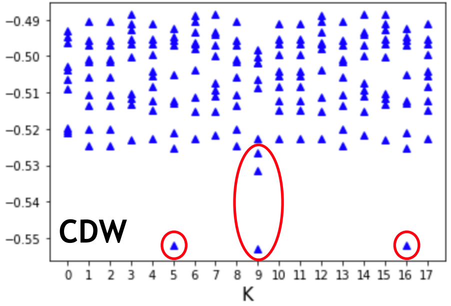
all × 3

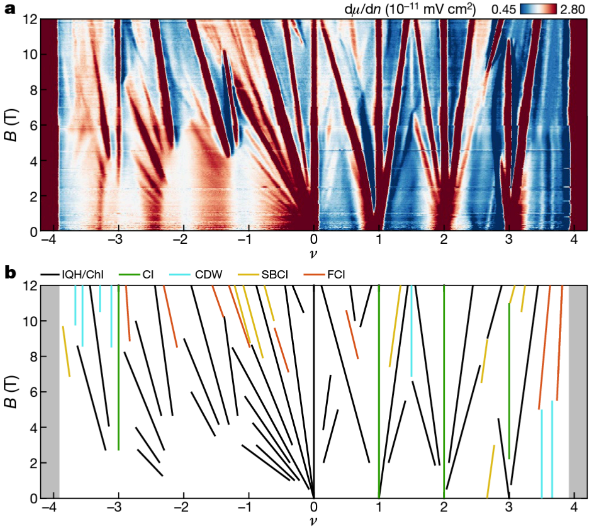


CDW

Application of ideal flatband theory - I: twisted bilayer graphene

Single-electron transistor microscopy

——— local inverse compressibility $d\mu/dn$



Yonglong Xie et, al; Nature (21)

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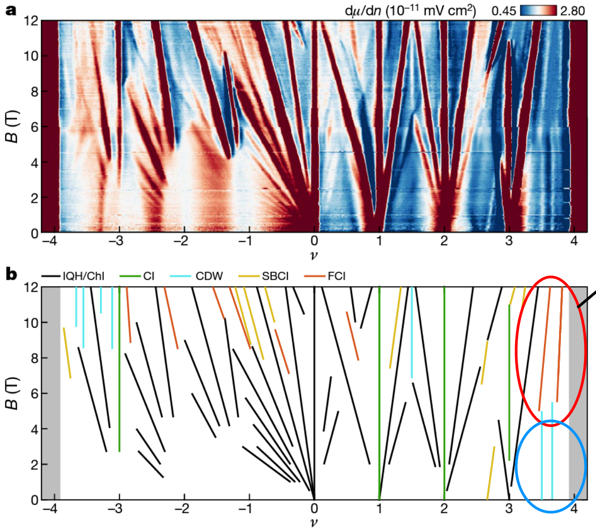
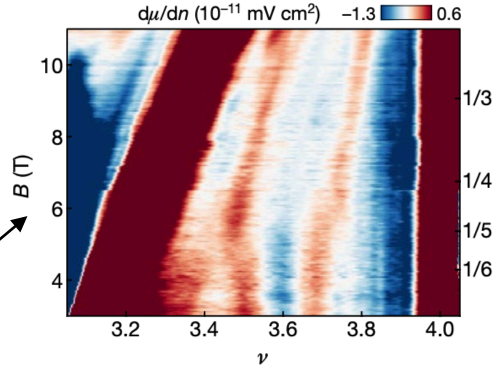


FCIs at high field

CDW at low field

Zoom in



Quantum phases identified from the Streda formula

$$\nu = t\left(\Phi/\Phi_0\right) + s$$

Chern number

Zero-field filling

CDW: t = 0, s = fractional

FCI: t = fractional, s = fractional

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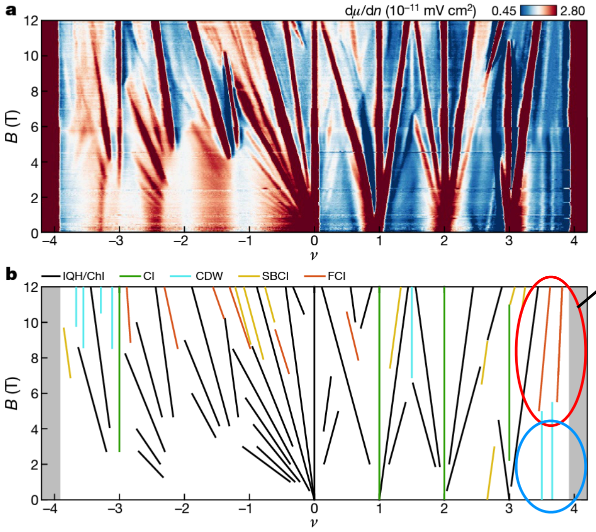

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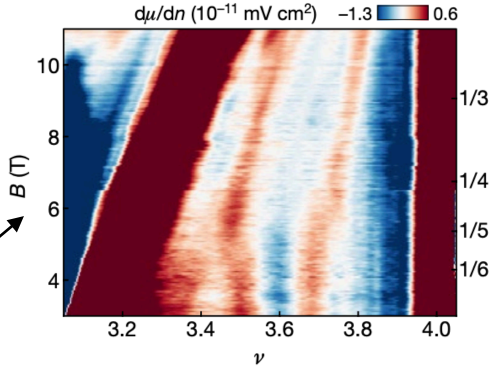

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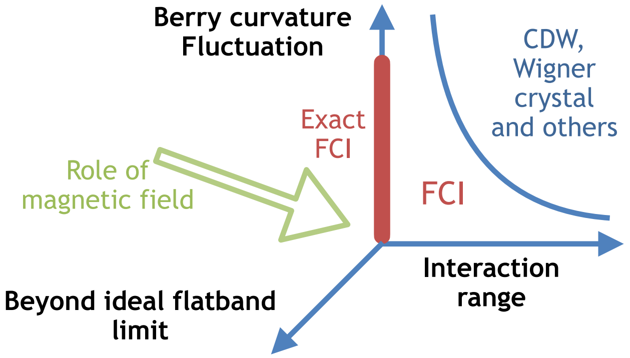
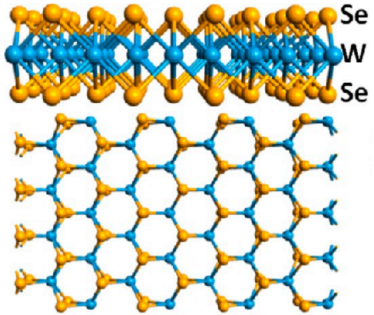
Calculation shows small B-field tunes quantum geometries



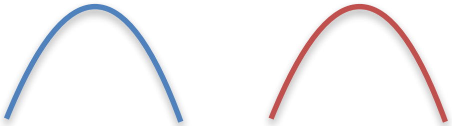
Yonglong Xie et, al; Nature (21)

Application of ideal flatband theory - II: twisted TMD



Strong spin-orbit coupling

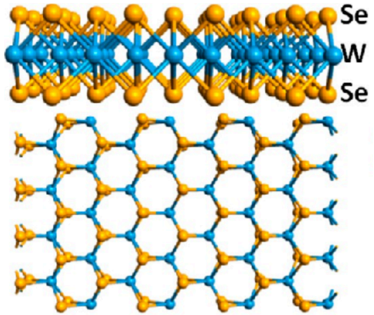
Monolayer hole-band



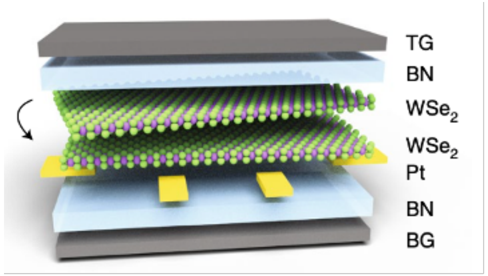
K, spin-up

K’, spin-down

Application of ideal flatband theory - II: twisted TMD

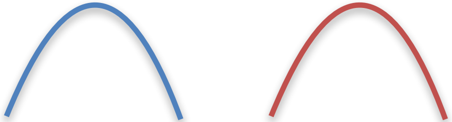


AA-stacked homo-bilayer TMD



Strong spin-orbit coupling

Monolayer hole-band

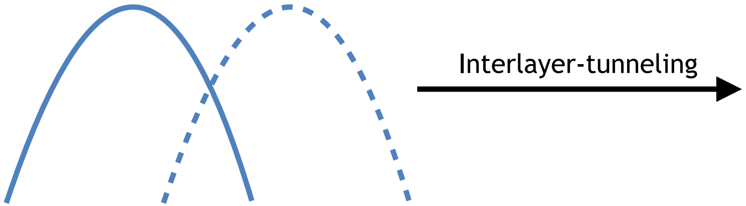


K, spin-up

K’, spin-down

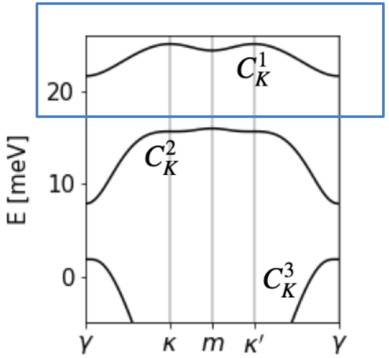
Top, K

Bottom, K



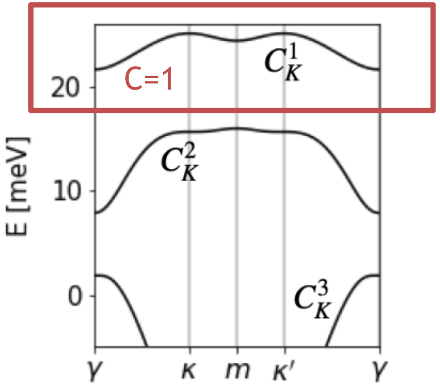
Interlayer-tunneling

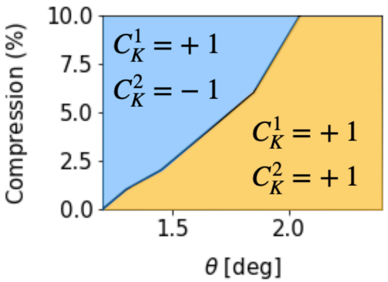
Flat moire band



Application of ideal flatband theory - II: twisted TMD

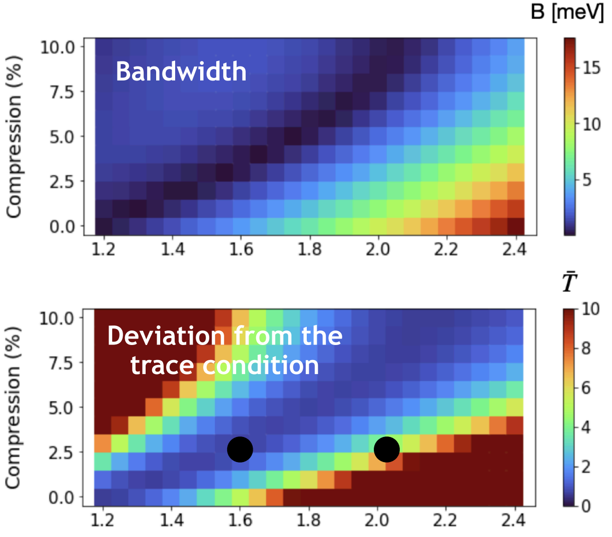
Moire TMD band structure





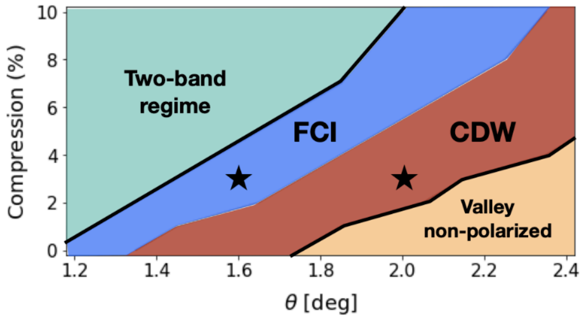
Magic line:

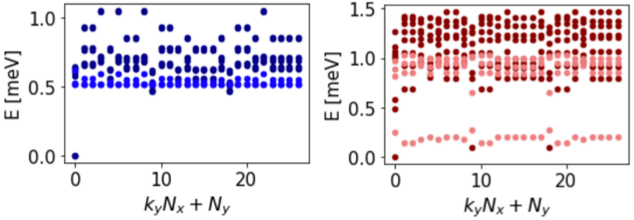
simultaneous optimization of bandwidth and trace condition



$$\bar{T} = \int d^2\mathbf{k}\,\left[\mathrm{Tr}\,g_{\mathbf{k}} - \Omega_{\mathbf{k}}\right]$$

Phase diagram





Guidance to material engineering:

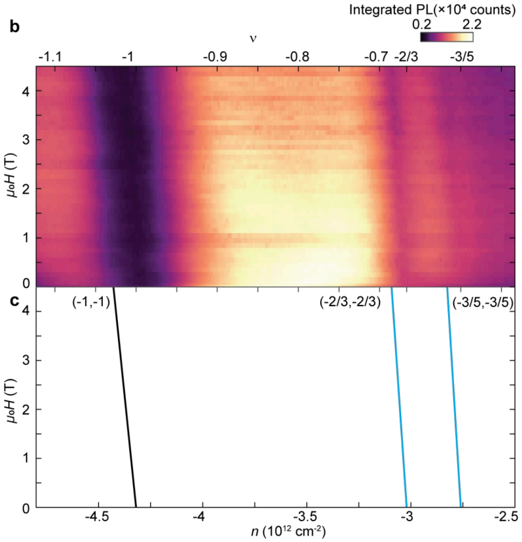
pressure and **angle** tuned band properties!

N. Duran, **Jie Wang**, Kaxiras group, Repellin, Cano (2023).

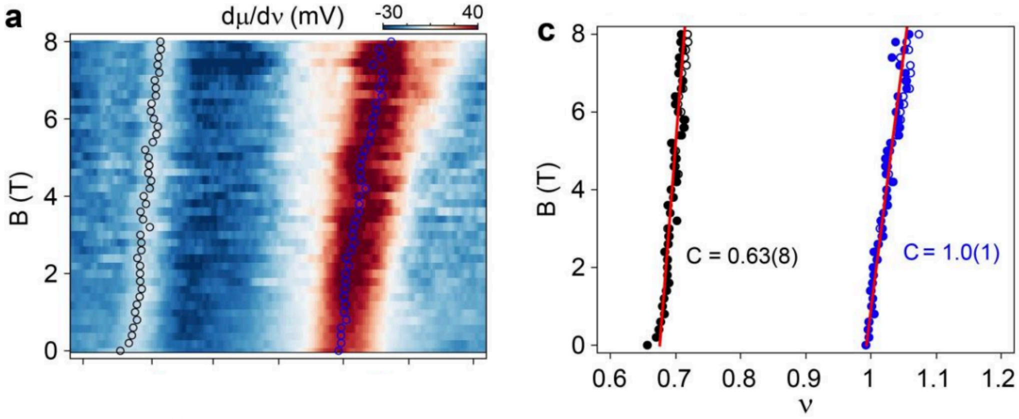
Application of ideal flatband theory - II: twisted TMD (twisted MoTe2)

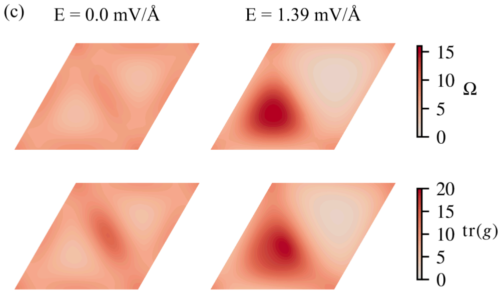
Two independent experiments show evidence of zero field fractional Chern insulator.

University of Washington (Xiao-dong Xu group; arXiv 2304.08470)



Cornell (Jie Shan, Kinfai Mak; arXiv 2305.00973)





Quantum geometry of twisted MoTe2 is nearly ideal.

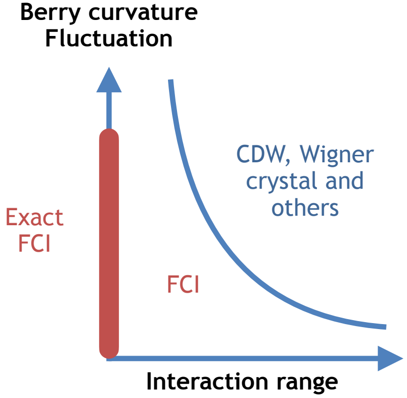
Theory: Di Xiao group, Liang Fu group.

Summary:

Take home message:

- Common wisdom that only Landau level has GMP algebra is now **revisited**.
- Ideal flatbands are ideal platform to study quantum Hall effect on **curved manifold**.
- Quantum geometry (especially overlooked Fubini-Study metric) are important indicator for **real experiment**.

Theory



Berry curvature Fluctuation

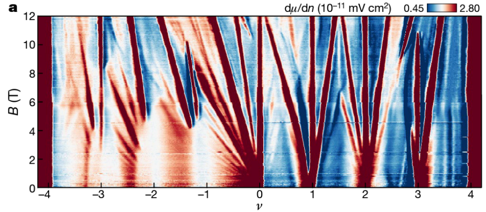
Exact FCI

FCI

CDW, Wigner crystal and others

Interaction range

Experiment & material design



Jie Wang, J. Cano, A.J. Millis, Z. Liu, B. Yang; PRL (2021).
Jie Wang, Zhao Liu; PRL (2022).
Jie Wang, Y. Zheng, A.J. Millis, J. Cano; PRR (2021).

Jie Wang, S. Klevtsov, Z. Liu; arXiv (2022).
J. Dong, **Jie Wang**, Liang Fu; arXiv (2022).
N. Duran, **Jie Wang**, Kaxiras group, Repellin, Cano (2023).

Backup slides

Fractional Chern insulator: to be or not to be?

$$\Delta \to \infty,\ D \to 0 \qquad \nu = \frac{1}{3}$$

Is fractional Chern insulator guaranteed?

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$$\Delta \to \infty, \, D \to 0 \qquad \nu = \frac{1}{3}$$

Is fractional Chern insulator guaranteed?

No!

Wavefunctions can still be highly nonuniform!

Flatband-projected interacting Hamiltonian:

$$H = \sum_{\mathbf{q}} v_{\mathbf{q}} : \rho_{\mathbf{q}} \rho_{-\mathbf{q}} : , \quad \rho_{\mathbf{q}} = \sum_{\mathbf{k}} \langle u_{\mathbf{k}+\mathbf{q}} | u_{\mathbf{k}} \rangle c^{\dagger}_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}}$$

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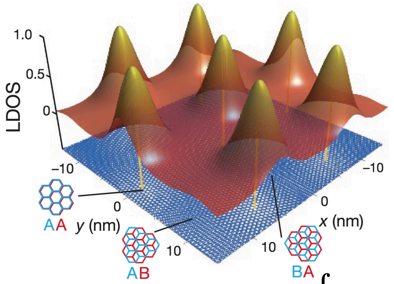
Bloch wavefunction (lattice translational invariant)



Plot: local density of states $n(\mathbf{r}) = \int d^2\mathbf{k} |\psi_{\mathbf{k}}(\mathbf{r})|^2$

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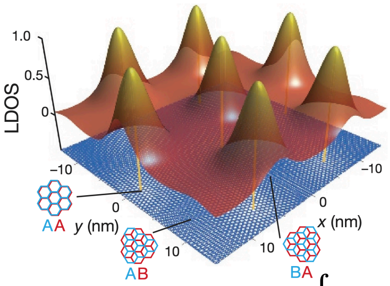
Bloch wavefunction (lattice translational invariant)

If wavefunction varies strongly ⇒ Charge density wave/ Wigner crystal is favored



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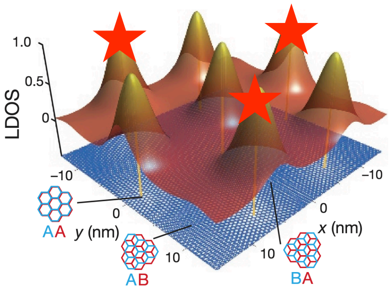

Uncorrelated state (single slater determinant)

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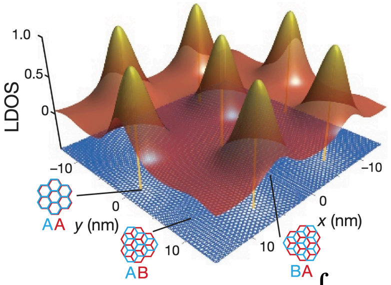
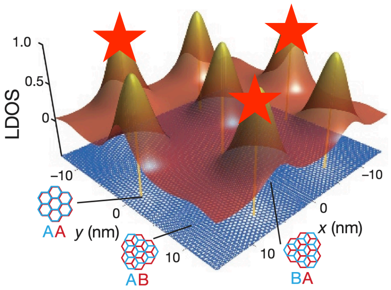


Plot: local density of states $n(\mathbf{r}) = \int d^2\mathbf{k} |\psi_{\mathbf{k}}(\mathbf{r})|^2$

Key question of this talk:

How to characterize wavefunction's role in interacting physics?



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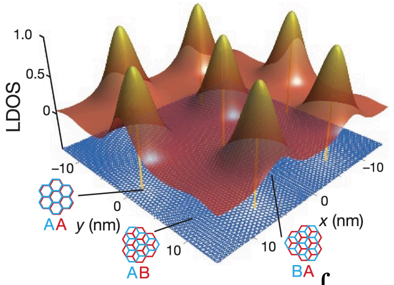
Bloch wavefunction (lattice translational invariant)

Important geometric information!

If wavefunction varies strongly ⇒ Charge density wave/ Wigner crystal is favored



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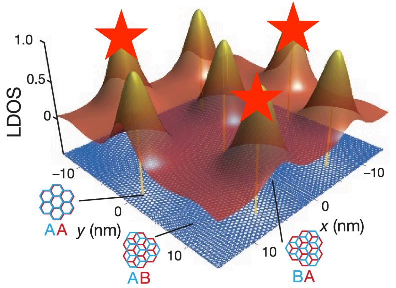
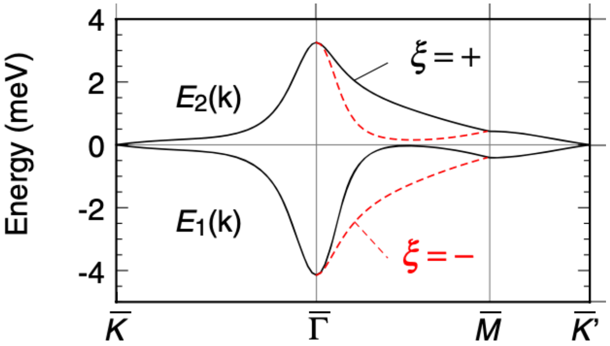


Uncorrelated state (single slater determinant)

Quantum anomalous Hall effect in twisted bilayer graphene (TBG)

Small spin-orbit coupling: spin degenerate

8 flatbands = 2 spin, 2 valley, 2 sub-lattice



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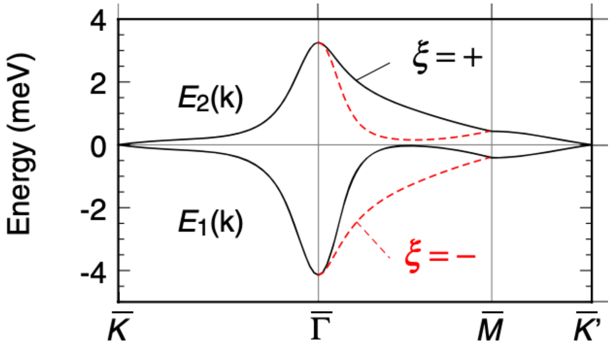
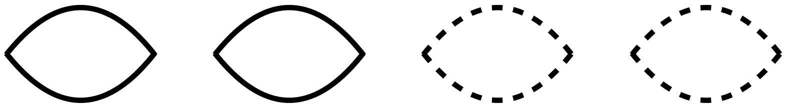


8 flatbands = 2 spin, 2 valley, 2 sub-lattice

Schematic illustration of the 8 flatbands of TBG



Valley $\zeta = +1$

Valley $\zeta = -1$

Without hBN

C_2T symmetric

Quantum anomalous Hall effect in twisted bilayer graphene (TBG)

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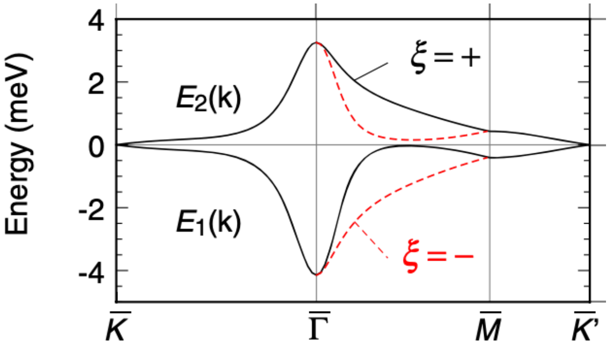
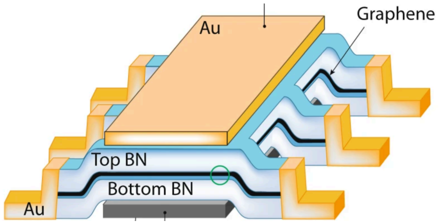


Hexagonal boron nitride (hBN) encapsulation



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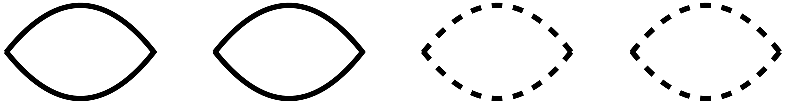

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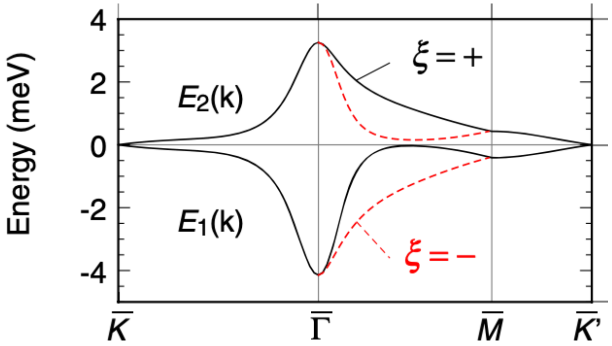

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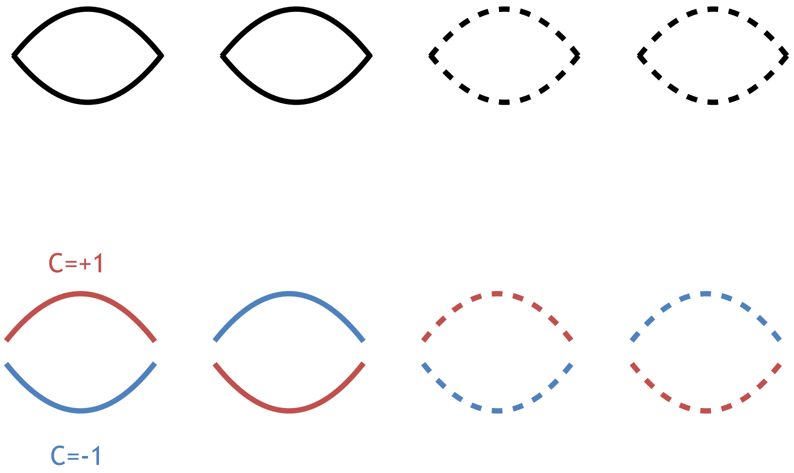
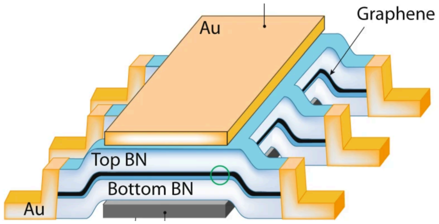


Without hBN
C_2T symmetric

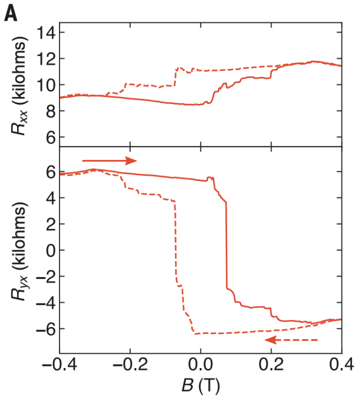
With hBN
C_2 breaking

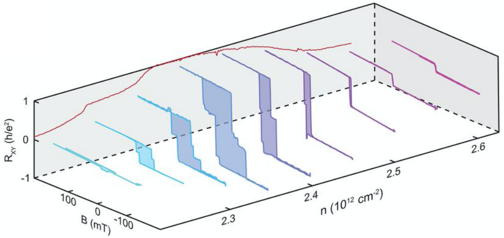
Hexagonal boron nitride (hBN) encapsulation



Quantum anomalous Hall effect in twisted bilayer graphene

Observation of anomalous Hall effect @ $\nu = 3/4$





Stanford, D. Goldhaber-Gordon group; Science (19)
UCSB, A. Young group; Science (19)

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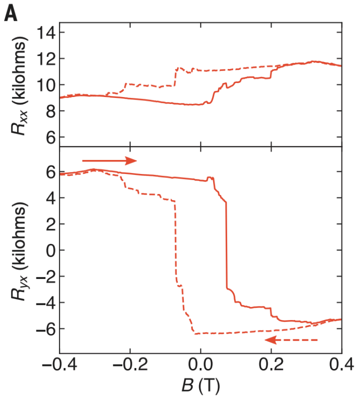

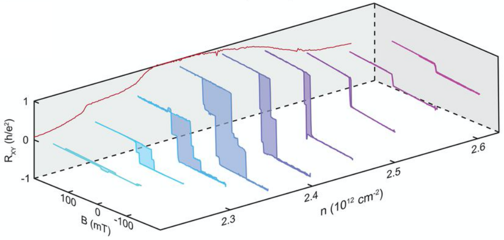

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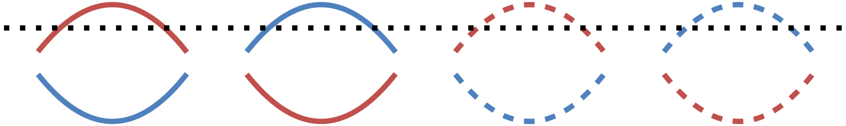
Non-interacting



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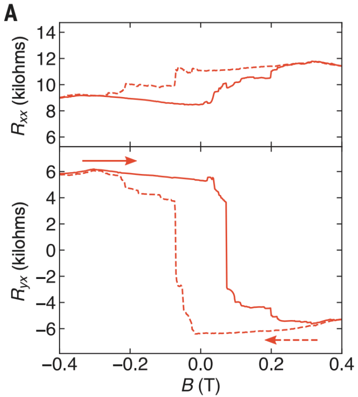

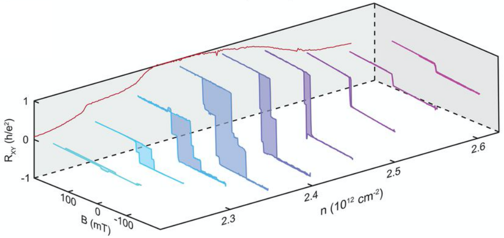

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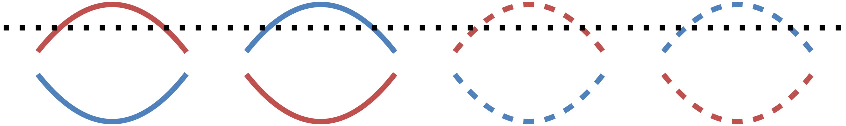
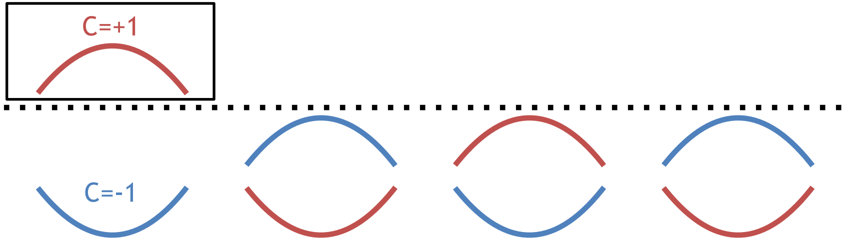


Interaction driven spontaneous time-reversal breaking



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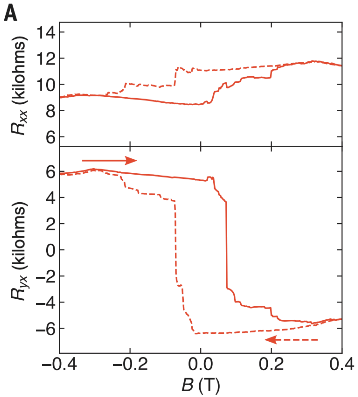

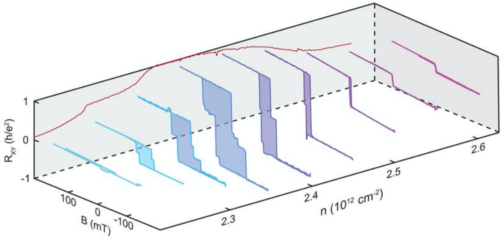

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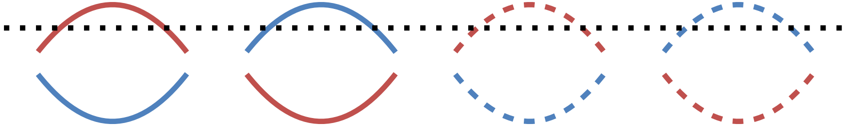

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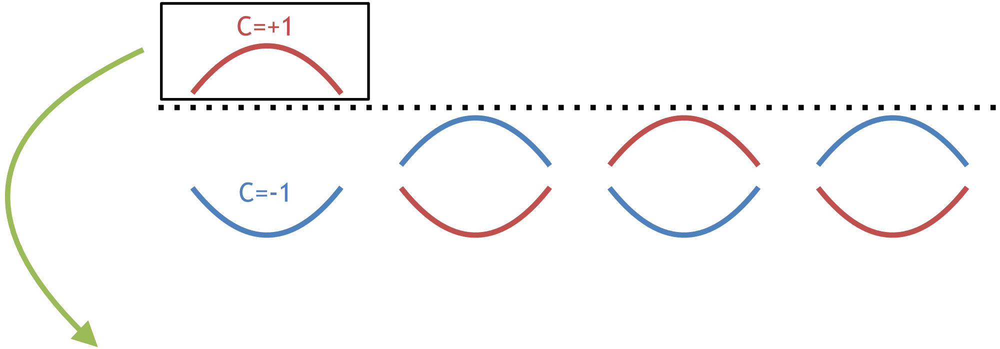
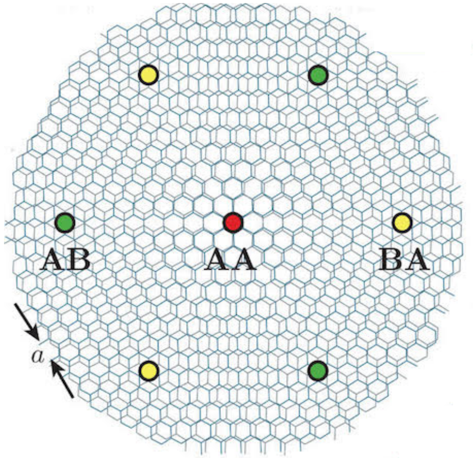
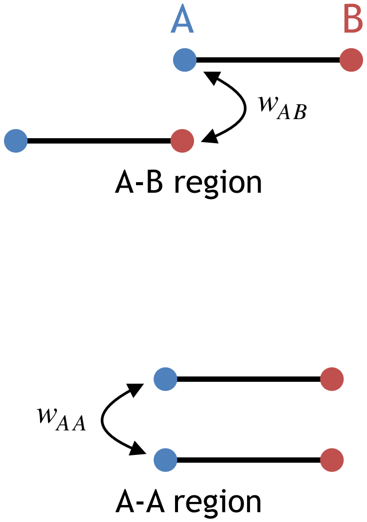


Q: Is fractional Chern insulators **possible** by partially fill the band ?

Stanford, D. Goldhaber-Gordon group; Science (19)
UCSB, A. Young group; Science (19)

Lattice relaxation and the chiral model of twisted bilayer graphene





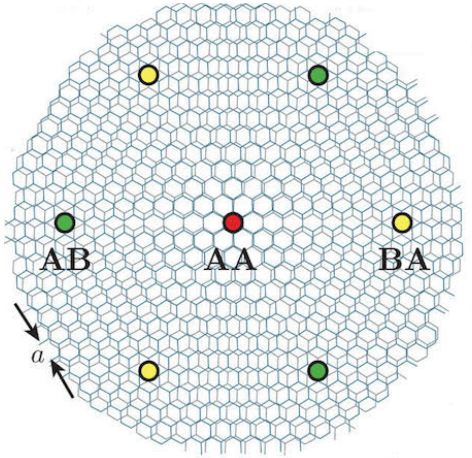
Lattice relaxation: $w_{AA} < w_{AB}$

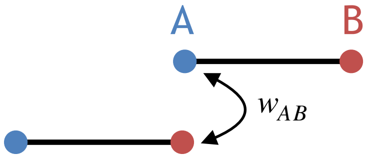
90 meV

110 meV

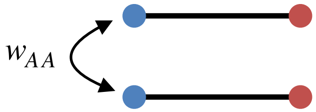
Bistritzer, MacDonald (PNAS 11)
Tarnopolsky, Kruchkov, Vishwanath (PRL 19)

Lattice relaxation and the chiral model of twisted bilayer graphene



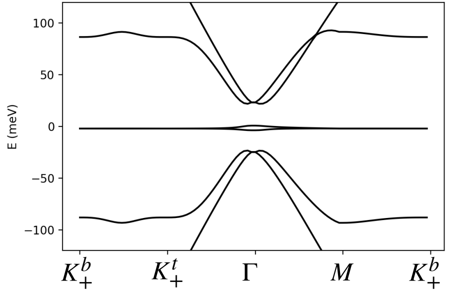


A-B region



A-A region

Bistritzer-MacDonald: $w_{AA} = w_{AB} = 100$ meV



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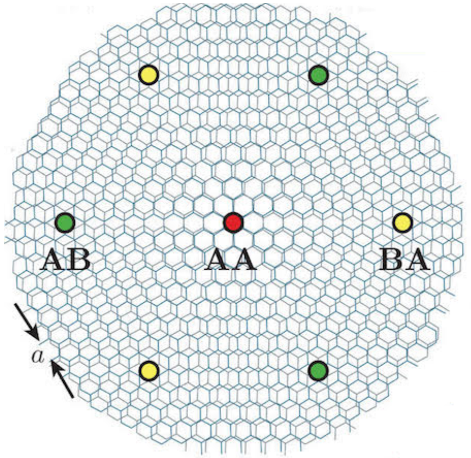
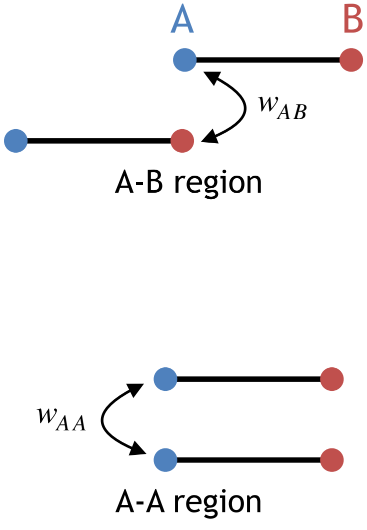
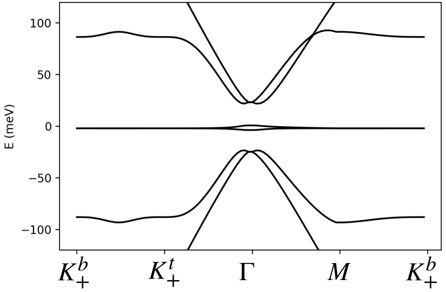
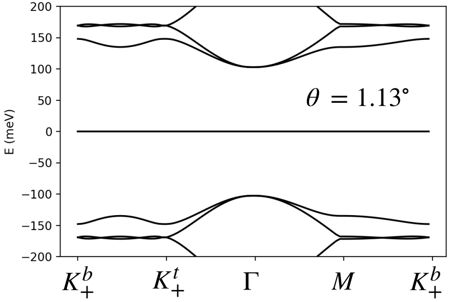




Bistritzer-MacDonald: $w_{AA} = w_{AB} = 100$ meV



Chiral model: $w_{AA} = 0$, $w_{AB} = 110$ meV



Lattice relaxation: $w_{AA} < w_{AB}$

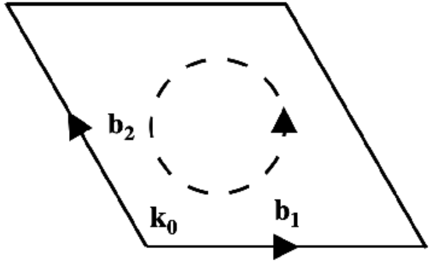
90 meV   110 meV

Bistritzer, MacDonald (PNAS 11)
Tarnopolsky, Kruchkov, Vishwanath (PRL 19)

Universality of the flatband wavefunction

Start from k-space holomorphic function

$$u_{k+b}(\mathbf{r}) = e^{i\phi_{k,b}} e^{-i\mathbf{b}\cdot\mathbf{r}} u_k(\mathbf{r})$$



Boundary condition is constrained by the Chern number

$$C = -\frac{1}{2\pi}\left(\phi_{k_0+b_1,b_2} - \phi_{k_0,b_2} + \phi_{k_0,b_1} - \phi_{k_0+b_2,b_1}\right)$$

Trick: origin of Brillouin zone is a 'gauge choice', so C is independent on it:

$\phi_{k,b}$ Is constrained to be linear, holomorphic function of k (fixed form up to gauge transformations)

Therefore, the most general form of C=1 ideal flatband wavefunction is chiral TBG's:

$$\Psi_{\mathbf{k}}(\mathbf{r}) = N_{\mathbf{k}}\ \mathscr{B}(\mathbf{r})\ \Phi_{\mathbf{k}}(\mathbf{r})$$

JW, Jen Cano, Andrew Millis, Zhao Liu, Bo Yang. PRL (21)

Thank you for attention