Full counting statistics of charge fluctuations

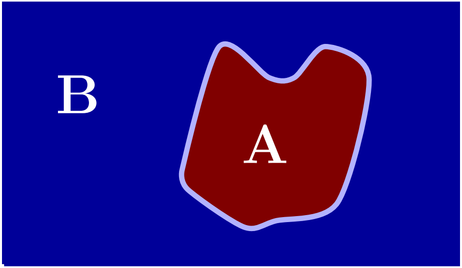
Jean-Marie Stéphan

Camille Jordan Institute, University of Lyon 1, France

Geometric and analytic aspects of the Quantum Hall effect,
Les Diablerets

Based on [Estienne, JMS, Witczak-Krempa arXiv:2102.06223]
[Berthière, Estienne, JMS, Witczak-Krempa arXiv:2211.05159]

Motivation



- Space $\mathbb{R}^2$. Take $A \subset \mathbb{R}^2$, $B = \mathbb{R}^2 \backslash A$ its complement.
- Wrt the bipartition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, compute the entanglement entropy $S_A = S_B$ of some pure state $|\psi\rangle$.
- Scaling: how does this behave as region A gets larger? Or, dilate A by a factor L. Behavior of S_{LA} as $L \to \infty$?

- Extract useful physical information from entanglement scaling.

- Successes in the context of topological phases and quantum Hall effect [Kitaev & Preskill 2006; Levin & Wen 2006]: $S_{LA} = \alpha L + \gamma + \ldots$ where γ is universal.

- Other systems, e. g. quantum criticality [Holzhey, Larsen & Wilczek 1994; Vidal, Latorre, Rico & Kitaev 2003; Calabrese & Cardy 2004]

- Many choices for subsystem A, smooth regions, regions with corner, cones, etc. Disentangling geometry of A from physics information.

This talk: simpler observable based on charge fluctuations, but exact same logic as for entanglement.

Some charge $Q = \int \rho(\mathbf{r}) d\mathbf{r}$ which is conserved. Subsystem charge $Q_A = \int_A \rho(\mathbf{r}) d\mathbf{r}$ can fluctuate.

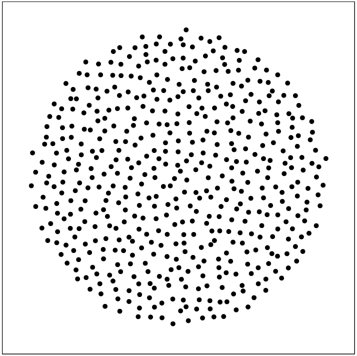
Charge variance/second cumulant

$$C_2(A) = \left\langle (Q_A)^2 \right\rangle - \left\langle Q_A \right\rangle^2 = \int_{A^2} \left\langle \rho(\mathbf{r}) \rho(\mathbf{r}') \right\rangle_c d\mathbf{r} d\mathbf{r}' \quad (1)$$

[Martin & Yalcin 1980; Klich & Levitov 2006; Song, Rachel, Flindt, Klich, Laflorencie & Le Hur 2012; Leblé 2021] and more generally higher cumulants.

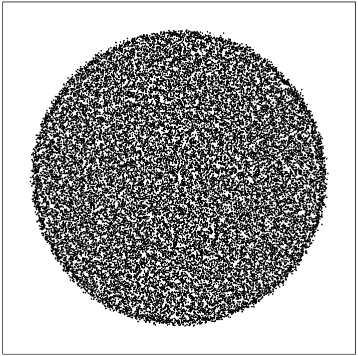
Example: Laughlin state

$$\psi(z_1, \ldots, z_N) = \prod_{j<k}(z_j - z_k)^m e^{-\frac{1}{2}\sum_{j=1}^{N}|z_j|^2}$$



Example: Laughlin state

$$\psi(z_1, \ldots, z_N) = \prod_{j<k}(z_j - z_k)^m e^{-\frac{1}{2}\sum_{j=1}^{N}|z_j|^2}$$



Do particle statistics in some large region in the bulk, where translation invariance, rotational invariance should hold.

What we do

Compute the expansion of the cumulants $C_m(LA)$ as $L \to \infty$ with the following assumptions

- Translation invariance
- Invariance wrt rotations and inversion.
- Fast enough decay of the connected correlation function $f(\mathbf{r}_1, \ldots, \mathbf{r}_{m-1}) = \langle \rho(\mathbf{r}_1), \ldots, \rho(\mathbf{r}_{m-1})\rho(0)\rangle$. For this talk we take this decay to be exponentially fast (correlation length ξ).

For the example of FQH states in the bulk: physically reasonable, mathematically highly non-trivial.

Inspiration

- Smooth regions: geometric expansion for the entanglement entropy [Grover, Turner & Vishwanath 2011] with translation invariance (TI) and rotation invariance (RI). Exploit symmetry $S_A = S_B$ to predict which powers of L appear in the expansion.

- Asymptotics of determinants [Kaufman & Onsager 1948; Szego 1952; Kac 1954; Widom 1960; Roccaforte 1984] related to counting statistics for free fermions with TI kernel.

Smooth regions

$$C_2(LA) = -L\mathrm{vol}(\partial A)\int_0^\infty 2r^2 f(r)dr - \frac{1}{L}\int \kappa^2 d\sigma \int_0^\infty \frac{r^4}{12} f(r)dr + \dots$$

Leading term is called boundary law or “area” law.

All terms $L^1, L^{-1}, L^{-3}, \dots$ with corresponding powers of $r^2, r^4, r^6, \dots$ appear in the expansion.

Polygons

Different expansion:

$$C_2(LA) = -L\mathrm{vol}(\partial A) \int_0^\infty dr\, 2r^2 f(r) \;+ \sum_{\text{corners } i} b(\theta_i, f) + O(L^{-\infty})$$

Explicit corner contribution

$$b(\theta; f) = \Big(1 + (\pi - \theta)\cot\theta\Big) \int_0^\infty dr\, \frac{r^3}{2} f(r)$$

Polygons

Explicit corner contribution

$$b(\theta; f) = \Big(1 + (\pi - \theta)\cot\theta\Big)\int_0^\infty dr\,\frac{r^3}{2} f(r)$$

Angular dependence is independent from the physics (function f), except for a (physically relevant) prefactor.

This angular dependence was encountered in several previous papers, which all correspond to a particular choice of f.

[Brandt, Neri & Sato 1981; Korchemsky & Radyushkin 1987; Casini, Fosco & Huerta 2005; Swingle 2010; Herviou, Le Hur & Mora 2019; Estienne & JMS 2019]

Sum rules

Liquid in the bulk. $\int r^3 f(r) dr$ gives exactly the k^2 coefficient of the static structure factor and is known to be universal [Stillinger & Lovett 1968]

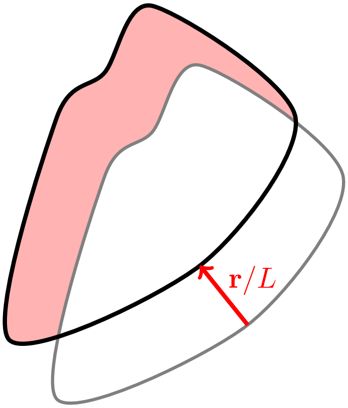
$$b(\theta; \mathrm{Lauglin}) = \frac{1}{4\pi^2 m} \left(1 + (\pi - \theta) \cot \theta\right)$$

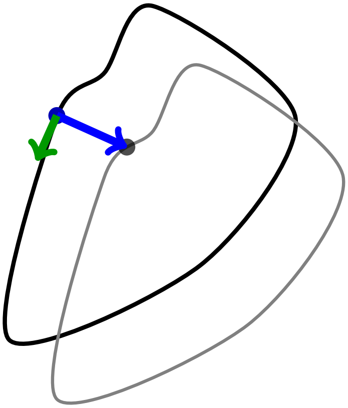
Some higher moments r^5, r^7 "sum rules" are known for Laughlin and other FQH states [Kalinay, Markos, Samaj, Travenec 2000; Can, Laskin, Wiegmann 2015; Dwidevi & Klevtsov 2019].
Can presumably be probed too, but not clear how to do it in a not too unnatural way.

Where do these results come from?

Just compute volumes (variance C_2)

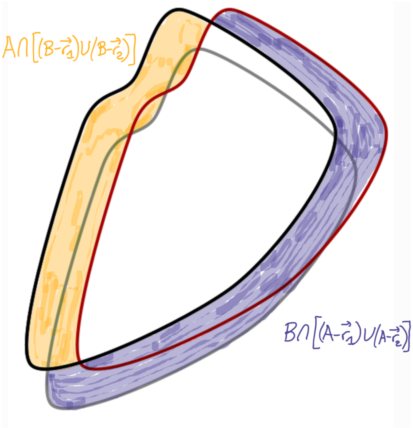
Need to evaluate $\mathrm{vol}\,[A \cap (B - \mathbf{r}/L)]$ for large L.





Just compute volumes (third cumulant C_3)

Need to evaluate $\text{vol} A \cap [(B - \mathbf{r}_1/L) \cup (B - \mathbf{r}_2/L)]$ for large L.



Third cumulant

Final result

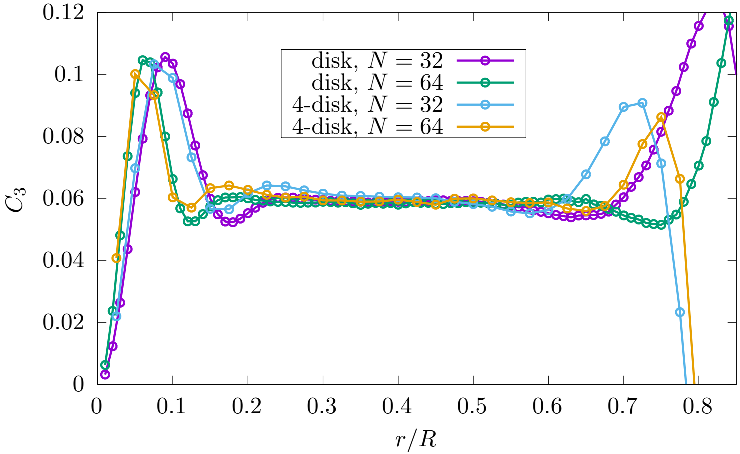
$$C_3(LA) = L^0 \times \int_{\partial A} \kappa \, d\sigma \times \int \frac{|\mathbf{r}_1 \wedge \mathbf{r}_2|}{4\pi} f(\mathbf{r}_1, \mathbf{r}_2) \; + \; \ldots$$

All other odd cumulants are similar.

The geometric integral is nothing but 2π times the Euler characteristic of A.

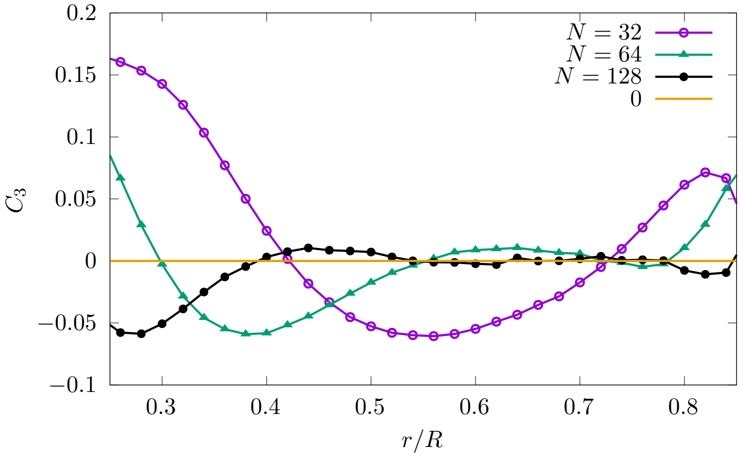
Numerical illustrations (Laughlin $\nu = 1/2$)

Disk $x^2 + y^2 < r^2$ and 4-disk $x^4 + y^4 < R^4$



Numerical illustrations (Laughlin $\nu = 1/2$)

Annulus ($\chi(A) = 0$)



Higher even cumulants

The $1/L$ constribution to $C_m/(m-1)$ is given by

$$-\int_{\partial A} \kappa^2 d\sigma \times \int' \left(\frac{x_1}{8} \sum_{j,k=1}^{m-1} \frac{y_j^2 y_k^2 \partial^2 f}{\partial x_j \partial x_k} + \frac{x_1^2}{4} \sum_{j=1}^{m-1} \frac{y_j^2 \partial f}{\partial x_j} \right)$$

where $f = f(x_1, y_1, \ldots, x_{m-1}, y_{m-1})$, and the integration $\int' = \int_{x_1 \geq \max_k(0, x_k)} dx_1 \ldots dx_{m-1} dy_1 \ldots dy_{m-1}$.

Generalizations to higher dimension $\mathbb{R}^d$

- Smooth case: $\kappa \to$ extrinsic curvature tensor K. Similar but more complicated results, e.g. C_3 proportional to $\int_{\partial A} K_a^a d\sigma$. Expansion for any cumulant to any order in any dimension using the same method.

- Cubes, cones, etc. Case by case results.

Conclusion

- Systematic geometric expansion: extraction of moments of connected correlation functions by counting particles.
- The case of slow decay is also interesting.
- Relation to entanglement entropy.
- Finite-size effects: smooth vs polygons.
- More general manifolds?

Thank you!