Chiral anomaly in fluid dynamics: coupling axial gauge field to Euler fluid

Alexander Abanov

Stony Brook University
The Rosi and Max Varon Visiting Professor, Weizmann Institute of Science
in collaboration with Paul Wiegmann

Geometric and analytic aspects of the QHE, May 8, 2023 Les Diablerets, Switzerland

Motivation

- Quantum anomalies play important role in QFT with Dirac fermions
- Sometimes, Dirac fermions behave as fluids
- What are the effects of the anomalies on hydrodynamics?
- Pioneered by:
  Alekseev, Cheianov, Fröhlich; Son, Surowka; Haehl, Loganayagam, Rangamani; ...
- Hydrodynamics (bosonization) captures anomalies in 1D. What are anomalies in conventional 3D (Euler) fluids?
- Geometrical fluid dynamics:
  Lichnerowitz, Carter, Arnold, Marsden, Holm, ...
- Generalization of some of QHE physics to 4+1 dimensions with 3+1 dimensional boundary

References

1. A. G. Abanov and P. B. Wiegmann, Phys. Rev. Lett. **128**, 054501 (2022)
   Axial-Current Anomaly in Euler Fluids.
   arXiv:2110.11480

2. P. B. Wiegmann and A. G. Abanov, J. High Energ. Phys. 2022, **38** (2022)
   Chiral anomaly in Euler fluid and Beltrami flow.
   arXiv:2202.12437

3. A. G. Abanov and P. B. Wiegmann, J. Phys. A: Math. Theor. **55** 414001 (2022)
   Anomalies in fluid dynamics: flows in a chiral background via variational principle.
   arXiv:2207.10195

Warm-up: 1+1 classical anomaly

1+1 free boson

Classical 1d action with $U(1)$ global symmetry $\phi \to \phi + const$:

$$S = \int d^2x \, \frac{1}{2} (\partial_\mu \phi)^2 \, .$$

The model has two conserved currents

$$\partial_\mu j^\mu = 0 \, , \qquad j^\mu = \partial_\mu \phi \, , \qquad \text{by Noether theorem}$$
$$\partial_\mu j_A^\mu = 0 \, , \qquad j_A^\mu = \epsilon^{\mu\nu} \partial_\nu \phi, \, , \qquad \text{emergent}$$

Let us introduce background gauge fields coupled to conserved currents

1+1 classical anomaly

Adding gauge background for $U(1)$ global symmetry:

$$S = \int d^2x \, \frac{1}{2}(\partial_\mu \phi - A_\mu)^2 \, .$$

The model has two currents

$$\partial_\mu j^\mu = 0 \, , \qquad j^\mu = \partial_\mu \phi - A_\mu \, , \qquad \text{conserved}$$

$$\partial_\mu j_A^\mu = 0 \, , \qquad j_A^\mu = \epsilon^{\mu\nu}\partial_\nu \phi, \, , \qquad \text{conserved, but NOT gauge invariant} \quad \partial_\mu j_A^\mu = -\epsilon^{\mu\nu}\partial_\mu A_\nu \, ,$$

Similar to 1+1 Dirac fermion QFT

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/_\mu - A\!\!\!/_\mu)\psi \, , \qquad j^\mu = \langle \bar{\psi}\gamma^\mu \psi \rangle \, , \qquad j_A^\mu = \langle \bar{\psi}\gamma^\mu \gamma^5 \psi \rangle \, .$$

Anomaly inflow

Adding gauge background for $U(1)$ global symmetry:

$$S = \int d^2x \, \frac{1}{2}(\partial_\mu \phi - A_\mu)^2 - \tilde{A}_\mu \epsilon^{\mu\nu}(\partial_\nu \phi - A_\nu) + \int_{M_3} d^3x \, \epsilon^{\mu\nu\lambda} \tilde{A}_\mu \partial_\nu A_\lambda \, .$$

The model has two consistent currents

$\partial_\mu j^\mu = 0 \, ,$	$j^\mu = \partial_\mu \phi - A_\mu + \epsilon^{\mu\nu} \tilde{A}_\nu \, ,$	conserved, NOT axial gauge invariant
$\partial_\mu j^\mu_A = -\epsilon^{\mu\nu} \partial_\mu A_\nu \, ,$	$j^\mu_A = \epsilon^{\mu\nu}(\partial_\nu \phi - A_\nu) \, ,$	gauge invariant, but NOT conserved

or two covariant currents

$\partial_\mu j^\mu_{cov} = -\epsilon^{\mu\nu} \partial_\mu \tilde{A}_\nu \, ,$	$j^\mu = \partial_\mu \phi - A_\mu \, ,$	gauge invariant, but NOT conserved
$\partial_\mu j^\mu_{A\,cov} = -\epsilon^{\mu\nu} \partial_\mu A_\nu \, ,$	$j^\mu_A = \epsilon^{\mu\nu}(\partial_\nu \phi - A_\nu) \, ,$	gauge invariant, but NOT conserved

Mixed $U(1)_V \times U(1)_A$ anomaly

- 3+1 massless Dirac fermion has $G = U(1)_V \times U(1)_A$ global symmetry
- Gauging this symmetry, i.e., introducing dynamic gauge fields coupled to G-currents makes theory inconsistent - anomaly
- In the presence of background gauge field the theory is consistent but charges do not conserve - mixed t'Hooft anomaly
- This anomaly can be interpreted as a system's coupling to a charge reservoir (5d topological insulator, 4d spectators ...)
- For any system with global symmetry one can ask whether there is an anomaly

Barotropic fluid: axial anomaly

Barotropic fluid dynamics

- Barotropic fluid dynamics: pressure is a function of density only $P = P(\rho)$
- Equations of motion: continuity and Euler

$$\partial_t \rho + \boldsymbol{\nabla}(\rho \boldsymbol{v}) = 0\,,$$
$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P\,.$$

- There is a conservation of charge/mass

$$\frac{dQ}{dt} = 0\,, \qquad Q = \int d^3x\, \rho\,.$$

- There is an additional conservation! Fluid helicity is conserved. If $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$

$$\frac{dQ_A}{dt} = 0\,, \qquad Q_A = \int d^3x\, \boldsymbol{v}\cdot\boldsymbol{\omega}\,.$$

- Is there a mixed anomaly between the corresponding symmetries?

Main observation

Couple fluid density to external E/M field (charged fluid):

$$\partial_t \rho + \boldsymbol{\nabla}(\rho \boldsymbol{v}) = 0\,,$$
$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\,.$$

The fluid helicity density $\boldsymbol{v}\cdot\boldsymbol{\omega}$ is not conserved!
However the corrected helicity is "almost" conserved (AGA and Wiegmann, '22)

$$\partial_t\Big(\boldsymbol{v}\cdot(\boldsymbol{\omega}+2\boldsymbol{B})\Big) + \boldsymbol{\nabla}\cdot\boldsymbol{j}_A = 2\boldsymbol{E}\cdot\boldsymbol{B}\,.$$

We have an anomalous equation

$$\boxed{\partial_\mu j_A^\mu = 2\boldsymbol{E}\cdot\boldsymbol{B}\,.}$$

Main observation. Details

Couple fluid density to external E/M field (charged fluid):

$$\partial_t \rho + \boldsymbol{\nabla}(\rho \boldsymbol{v}) = 0\,,$$
$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P + \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\,.$$

Derive

$$\partial_t(v\omega) + \nabla\left[v(v\omega) + \omega\left(\mu - \frac{v^2}{2}\right) + v\times(E + v\times B)\right] - 2\omega(E + v\times B) = 0\,,$$
$$\partial_t(vB) + \nabla\left[v(vB) + B\left(\mu - \frac{v^2}{2}\right) - v\times(E + v\times B)\right] + \omega(E + v\times B) = EB\,.$$

and combine

$$\partial_t\left(v(\omega + 2B)\right) + \nabla\left[v\left(v(\omega + 2B)\right) + (\omega + 2B)\left(\mu - \frac{v^2}{2}\right) - v\times(E + v\times B)\right] = 2EB\,.$$

Barotropic fluid: variational principle

Variational principle for perfect fluid

Variational principle for the dynamics of the perfect fluid

- has a long history
- is nontrivial
- has many versions

The geometric version we adopt here

- is close to the one used by Carter and by Arnold's school
- involves the use of "restricted variations"
- suitable for both relativistic and nonrelativistic hydro

.

Action for perfect barotropic fluid

The action of a barotropic fluid:

$$\mathcal{S}[\pi] \;=\; -\int d^4x\; P(\pi)$$

- P is a function of π_μ

Only restricted variations of π_μ are allowed

1) Diffeomorphisms $x^\mu \to x^\mu + \epsilon^\mu$

$$\delta_\epsilon \pi_\nu = \epsilon^\mu \partial_\mu \pi_\nu + \pi_\mu \partial_\nu \epsilon^\mu \,,$$

- π is a 1-form

2) Gauge transformations (not needed)

$$\delta_\lambda \pi_\nu = \partial_\nu \lambda \,,$$

- π is charged

Equations of motion

Variational principle gives equations of motion

$$\begin{aligned} \partial_\mu T^\mu_\nu &= 0\,, \qquad & T^\mu_\nu = j^\mu \pi_\nu + \delta^\mu_\nu P\,, \\ \partial_\mu j^\mu &= 0\,, \qquad & j^\mu = -\frac{\partial P}{\partial \pi_\mu}\,, \end{aligned}$$

Identify:

- T^μ_ν - energy-momentum tensor
- j^μ - charge/number current
- π_μ - specific momentum
- P - pressure

Relativistic vs. non-relativistic

Consider pressure $P(\mu)$ as a function of the chemical potential, so that $dP = \rho\, d\mu$.

1) $\mu = -\pi_0 - \frac{\pi_i^2}{2}$ describes non-relativistic (Galilean) fluid

$$\pi_0 = -\mu - \frac{v^2}{2}\,, \qquad \pi_i = v_i\,, \qquad j^\mu = (\rho, \rho v)\,,$$
$$v^i - \text{3-velocity}\,, \quad \rho - \text{number density}\,, \quad \mu(\rho) - \text{chemical potential}$$

2) $\mu = \sqrt{-\pi_\mu \pi^\mu}$ describes relativistic fluid

$$\pi_\mu = \mu u_\mu\,, \qquad j^\mu = n u^\mu\,, \qquad u^\mu u_\mu = -1\,,$$
$$u^\mu - \text{4-velocity}\,, \quad n - \text{number density}\,, \quad \mu(n) - \text{chemical potential}$$

Admissible variations

Vary the action $S[\pi]$ over admissible variations

$$\delta\pi_\nu = \epsilon^\mu \partial_\mu \pi_\nu + \pi_\mu \partial_\nu \epsilon^\mu \,, \qquad \delta\pi = \mathcal{L}_\epsilon \pi$$

and obtain

$$\delta S = \int d^4x \, \frac{\delta S}{\delta \pi_\nu} (\epsilon^\mu \partial_\mu \pi_\nu + \pi_\mu \partial_\nu \epsilon^\mu) = 0 \,.$$

Introducing "particle current" $\mathcal{J}^\nu \equiv -\frac{\delta S}{\delta \pi_\nu}$

$$\mathcal{J}^\nu (\partial_\nu \pi_\mu - \partial_\mu \pi_\nu) + \pi_\mu \partial_\nu \mathcal{J}^\nu = 0 \,.$$

Multiply by $\mathcal{J}^\mu$

$$\mathcal{J}^\nu (\partial_\nu \pi_\mu - \partial_\mu \pi_\nu) = 0 \,, \qquad \partial_\nu \mathcal{J}^\nu = 0 \,.$$

Notice, that the conservation of $\partial_\nu \mathcal{J}^\nu = 0$ follows from diffeos.

Geometric transport

The Lichnerowicz-Carter equation:

$$\mathcal{J}^\nu(\partial_\nu\pi_\mu - \partial_\mu\pi_\nu) = 0\,, \quad \text{or in form notation} \quad i_{\mathcal{J}} d\pi = 0\,.$$

Then

$$i_{\mathcal{J}}(d\pi\wedge d\pi) = 2(i_{\mathcal{J}} d\pi)\wedge d\pi = 0\,.$$

But $d\pi\wedge d\pi$ is a 4-form and is proportional to the volume form and should be zero. Or we can conclude that $d\pi$ has a rank 2 and, we have a fundamental property of barotropic fluid

$$d\pi\wedge d\pi = 0\,.$$

In components:

$$\partial_\mu(\epsilon^{\mu\nu\lambda\rho}\pi_\nu\partial_\lambda\pi_\rho) = 0\,.$$

Chiral current

In external e/m field $S[\pi] \to S[\pi - A]$ but still

$$\partial_\mu(\epsilon^{\mu\nu\lambda\rho}\pi_\nu\partial_\lambda\pi_\rho) = 0\,.$$

However, the helicity density is NOT gauge invariant. Introduce the chiral current:

$$J_A^\mu = \epsilon^{\mu\nu\lambda\rho}(\pi_\nu - A_\nu)\partial_\lambda(\pi_\rho + A_\rho) = \epsilon^{\mu\nu\lambda\rho}p_\nu(\partial_\nu p_\rho + F_{\nu\rho})\,, \qquad p_\mu \equiv \pi_\mu - A_\mu\,.$$

The chiral current is gauge invariant but it is NOT conserved!

$$\boxed{\partial_\mu J_A^\mu = 2\boldsymbol{E}\cdot\boldsymbol{B}\,,} \qquad d(\pi - A)\wedge d(\pi + A) = -dA\wedge dA\,.$$

Axial/chiral anomaly!

AGA, P.B. Wiegmann, PRL 128, 054501 (2022) [arXiv:2110.11480]

Axial gauge field background

Coupling to axial gauge field

The action of a charged barotropic fluid with higher gradients:

$$\mathcal{S}[\pi, A, \tilde{A}] = -\int d^4x\, P(\pi - A) + \ldots + \int A^A \wedge (\pi - A) \wedge d(\pi + A)$$

The extra term is

- Topological $\rightarrow$ does not contribute to energy-momentum tensor
- Linear in A^A, in fact, equal to $A^A_\mu J^\mu_A$
- Gauge invariant w.r.t. A

Compare to: G. Monteiro, AGA, and V. Nair, Phys. Rev. D 91 (2015) 125033.

Advection flux and axial current

Variational principle gives:

$$\boxed{\mathcal{J}^\mu(\partial_\mu\pi_\nu - \partial_\nu\pi_\mu) = 0\,,} \qquad \partial_\mu\mathcal{J}^\mu = 0\,, \qquad \mathcal{J}^\mu = -\frac{\delta\mathcal{S}}{\delta\pi_\mu}\,.$$

$\mathcal{J}^\mu$ is a particle current. Explicitly

$$\mathcal{J} = j + \star\Big(2A^A \wedge d\pi - (\pi - A)\wedge dA^A\Big)\,, \qquad j^\mu = -\frac{\partial P}{\partial\pi_\mu}\,.$$

Immediate consequence:

$$\partial_\mu J^\mu_A = 2\boldsymbol{E}\cdot\boldsymbol{B}\,.$$

Energy-momentum tensor and vector current

Equation for energy-momentum tensor

$$\boxed{\partial_\mu T^\mu_\nu = F_{\nu\mu} J^\mu + F^A_{\nu\mu} J^\mu_A \,,} \qquad T^\mu_\nu = j^\mu(\pi_\nu - A_\nu) + \delta^\mu_\nu P \,, \qquad j^\mu = -\frac{\partial P}{\partial \pi_\mu} \,.$$

Electric current:

$$J = \mathcal{J} + \star\Big(2d\Big[A^A(\pi - A)\Big] - A^A \wedge F\Big) = j + \star(\pi - A) \wedge dA^A$$

$$\boxed{\partial_\mu J^\mu = 2\boldsymbol{E}^A \cdot \boldsymbol{B} + 2\boldsymbol{E} \cdot \boldsymbol{B}^A \,.}$$

Equations of motion

(Non)conservation laws following from the constructed action:

$$\partial_\mu T^\mu_\nu = F_{\nu\mu} J^\mu + F^A_{\nu\mu} J^\mu_A \,, \qquad T^\mu_\nu = j^\mu(\pi_\nu - A_\nu) + \delta^\mu_\nu P \,.$$
$$\partial_\mu J^\mu = 2\boldsymbol{E}^A \cdot \boldsymbol{B} + 2\boldsymbol{E} \cdot \boldsymbol{B}^A = -2 \star dA \wedge dA^A \,,$$
$$\partial_\mu J^\mu_A = 2\boldsymbol{E} \cdot \boldsymbol{B} = - \star dA \wedge dA \,.$$

Only 4 of these 6 equations are independent!

The equations exhibit the covariant mixed anomaly between charge and helicity conservations.

$$\mathcal{J}^\mu(\partial_\mu \pi_\nu - \partial_\nu \pi_\mu) = 0 \,.$$

Comment on three vector fields

There are three important vector fields

$$\begin{aligned} \pi_\mu - A_\mu &\quad \text{- specific kinematic momentum}\,, \\ J^\mu &\quad \text{- electric current}\,, \\ \mathcal{J}^\mu &\quad \text{- particle current}\,. \end{aligned}$$

In Galilean fluid mechanics all three vector fields are proportional to the fluid velocity $\boldsymbol{v}$.
They are all different in the presence of axial and vector backgrounds.

Stationary flows

Static equilibrium (stationary flow)

Assume time-independent external fields and put for simplicity $\boldsymbol{B}^A = 0$. Equilibrium conditions $\boldsymbol{\mathcal{J}} = 0,\ \pi_0 = 0$ give (for Galilean fluid):

$$\rho \boldsymbol{v} = 2\mu^A(\boldsymbol{\omega} + \boldsymbol{B}) + \boldsymbol{E}^A \times m\boldsymbol{v} \qquad \text{– generalized Beltrami flow}$$

We find currents in equilibrium:

$$\boldsymbol{J} = \overbrace{2\mu^A \boldsymbol{B}}^{transport} + \overbrace{2\boldsymbol{\nabla} \times (\mu^A m\boldsymbol{v})}^{micro},$$
$$\boldsymbol{J}^A = 2\mu \boldsymbol{B} + 2\boldsymbol{\nabla} \times (\mu m\boldsymbol{v}).$$

$$\boldsymbol{J} = \overbrace{2\mu^A \boldsymbol{B}}^{CME} + \overbrace{2\mu^A \boldsymbol{\omega}}^{CVE},$$
$$\boldsymbol{J}^A = \underbrace{2\mu \boldsymbol{B}}_{CSE} + \underbrace{\mu \boldsymbol{\omega}}_{CVSE}.$$

for constant chemical potentials

Conclusions

- Background gauge and axial gauge fields are introduced for perfect barotropic fluid
- There is a mixed anomaly between the charge and the helicity conservations
- A variational (and Hamiltonian) formulation of anomalous fluid dynamics is given
- Stationary solutions are given by generalized Beltrami flows

A few open questions

- Realization of external gauge fields in real systems
- Explicit demonstration of the mixed anomaly for underlying gauge and relabeling symmetries
- Analogous phenomena for baroclinic flows and hydro with many species