Symmetry-resolved entanglement entropy in quantum Hall states

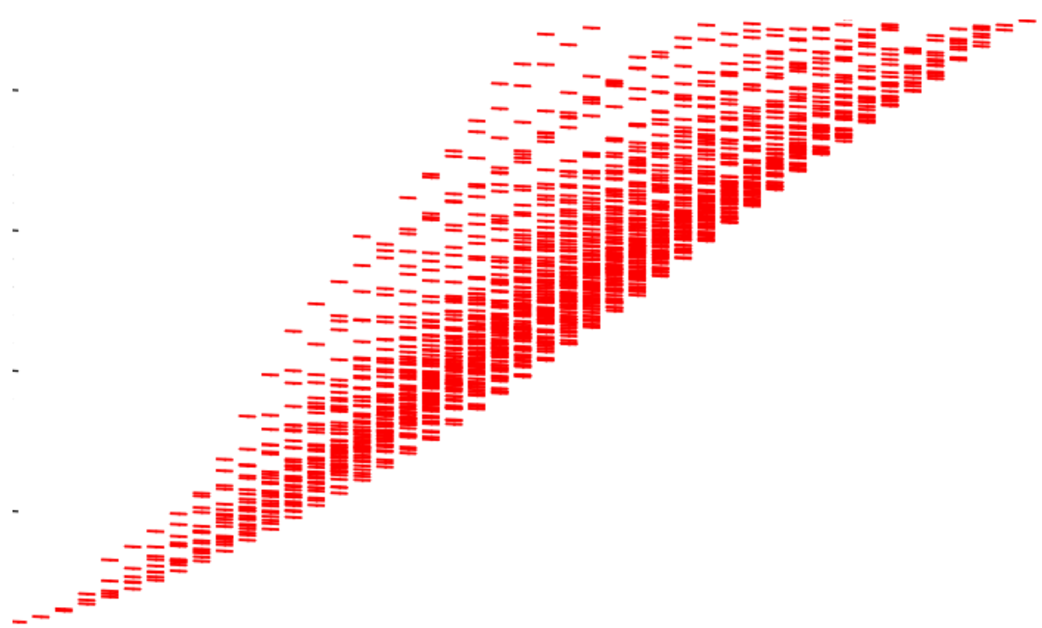
with B.Oblak and N. Regnault
(related works with L. Charles and J.M. Stéphan)

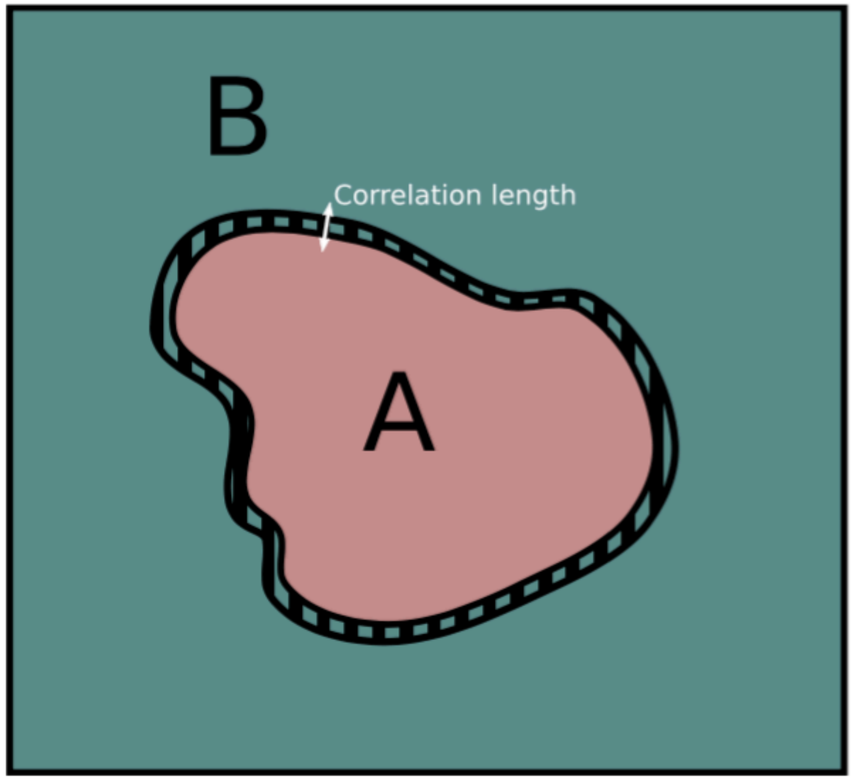
Geometric and analytic aspects of the Quantum Hall effect

Benoit Estienne **May 7 - 12, 2023**

Probing the bulk-edge correspondence

Li-Haldane : upon partitioning the system in two regions A and B, the entanglement Hamiltonian is conjectured to be in the same universality class as the effective edge Hamiltonian.





Entanglement spectrum of 1/3 Laughlin on the sphere [from N.Regnault]

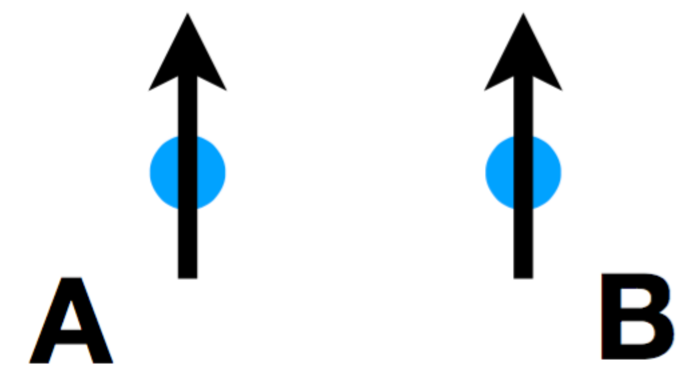
Symmetry-resolved entanglement entropy probes this conjecture.

Entanglement in a nutshell

Given a **bipartition** of the Hilbert space $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$, and a quantum state $|\Psi\rangle \in \mathscr{H}$

- if $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$ (**product-state**) **:** there is **no entanglement**

- otherwise the degrees of freedom in A and B are said to be **entangled** (in the state $|\Psi\rangle$).



Example : two spin 1/2 for which $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$

with $\mathscr{H}_A = \mathscr{H}_B = \mathbb{C}|\uparrow\rangle \oplus \mathbb{C}|\downarrow\rangle \simeq \mathbb{C}^2$

Then $|\uparrow\uparrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$ is a pure state, while $\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$ is entangled.

But it's not always this easy :

$$\frac{|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle}{2} = \left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}\right)$$

Entanglement in a nutshell

Given a **bipartition** of the Hilbert space $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$, and a quantum state $|\Psi\rangle \in \mathscr{H}$

$$|\Psi\rangle = \sum_{i,j} A_{ij} |e_i\rangle \otimes |f_j\rangle$$

Singular value decomposition of the matrix (A_{ij}) yields the

Schmidt decomposition of $|\Psi\rangle$ **w.r.t. the bipartition** $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$ **:**

$$|\Psi\rangle = \sum_{j=1}^{r} \sqrt{p_j}\, |u_j\rangle \otimes |v_j\rangle, \qquad p_j > 0\,, \quad \sum_{j=1}^{r} p_j = 1\,.$$

where $\{|u_j\rangle\}$ and $\{|v_j\rangle\}$ are **orthonormal** vectors in $\mathscr{H}_A$ and $\mathscr{H}_B$, respectively.

Facts :

- the positive integer r is well-defined (*i.e.* independent of any possible choice made when performing the Schmidt decomposition). It is called the **Schmidt rank.**

r =1 : $|\Psi\rangle$ is a product state OR **r > 1 :** $|\Psi\rangle$ is entangled

- the **Schmidt coefficients** p_j (counted with multiplicities) are also well-defined

Reduced density matrix ρ_A

- A quantum system can be in a **pure state** $|\Psi\rangle$, in which case the expectation of an observable O is $\langle O \rangle = \langle \Psi | O | \Psi \rangle$.
- More generally a quantum system can be in a **statistical superposition** of states $|\Psi_i\rangle$, each with a probability p_i . This is conveniently described as a density matrix

$$\rho = \sum_j p_j |\Psi_j\rangle\langle\Psi_j| \qquad \text{in which case} \qquad \langle O \rangle = \sum_j p_j \langle \Psi_j | O | \Psi_j \rangle = \mathrm{Tr}(\rho O)$$

Fact : If the total system is a state ρ, the subsystem A is in the state $\rho_A = \mathrm{Tr}_{\mathscr{H}_B}(\rho)$

in the sense that for any observable O_A acting on $\mathscr{H}_A$:

$$\langle O_A \rangle \equiv \mathrm{Tr}_{\mathscr{H}_A \otimes \mathscr{H}_B}(\rho O_A) = \mathrm{Tr}_{\mathscr{H}_A}(\rho_A O_A)$$

As far as any measurement in A is concerned, the subsystem A is described by ρ_A.

Schmidt decomposition and reduced density matrix

Important remark : even if the total system is in a pure state $|\Psi\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$, subsystem A is generically a statistical superposition.

$$|\Psi\rangle = \sum_{j=1}^{r} \sqrt{p_j}\, |u_j\rangle \otimes |v_j\rangle \qquad \Rightarrow \qquad \rho_A = \sum_j p_j |u_j\rangle\langle u_j|$$

$\{p_j\}$ is the (non-zero) spectrum of $\rho_A = \mathrm{Tr}_{\mathscr{H}_B}\left(|\Psi\rangle\langle\Psi| \right)$

$$p_j > 0\,, \quad \sum_{j=1}^{r} p_j = 1\,.$$

The subsystem A is the statistical superposition of the states $|u_j\rangle$ with probability p_j.

How to quantify entanglement ?

The subsystem A is the statistical superposition of the states $|u_j\rangle$ with probability p_j.

In order to quantity the amount of entanglement between subsystems A and B, the most natural candidate is the Von Neumann entropy from classical information theory :

The (von Neumann) entanglement entropy of the state $|\Psi\rangle$ *w.r.t.* to the bipartition $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$ is

$$S(|\Psi\rangle) = -\sum_j p_j \log p_j = -\mathrm{Tr}\left(\rho_A \log \rho_A\right)$$

Some properties :

- $S \geq 0$, with equality iff there is no entanglement
- S is maximal (given a Schmidt rank) when the uncertainty is maximal : all p_j equal

Other measures include the Rényi entropy of order n

$$S_n(|\Psi\rangle) = \frac{1}{1-n} \log \sum_j p_j^n = \frac{1}{1-n} \log \mathrm{Tr}\left(\rho_A^n\right)$$

Area law

and subleading corrections

Generic features of EE : Area law

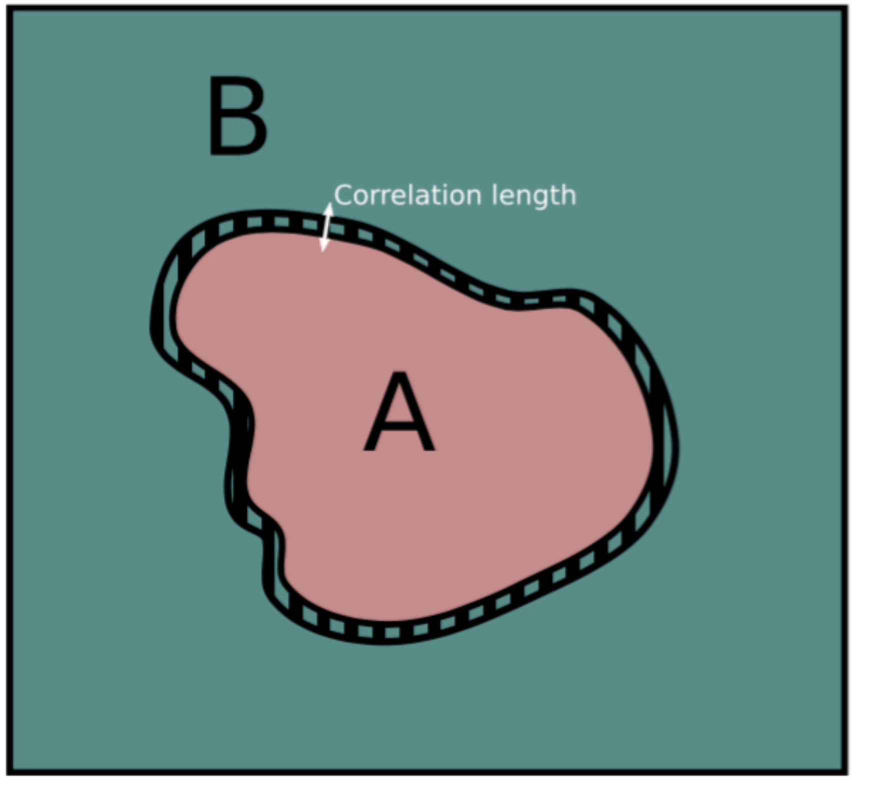
For ***gapped*** quantum systems

- spatial bipartition
- $|\Psi\rangle$ ground-state of a ***local*** Hamiltonian with a ***spectral gap***
- finite correlation length ξ

Expectation : the leading asymptotic behavior of EE is governed by a **area/boundary law**

$$S_n \sim C_n \frac{\text{Vol}(\,\partial A\,)}{\xi^{d-1}}, \qquad (\xi \to 0)$$

for regions A much larger than the correlation length ξ, and C_n some non-universal constant



Hastings 2007 : proof of area law in 1D (Von Neumann entropy). Implies the ability to approximate one-dimensional ground state by a matrix product state.

Topological EE

[Kitaev, Preskill 2006], [Levin Wen 2006]

In two spatial dimensions, it has been proposed that

$$S_n = C_n \frac{\text{Length}(\,\partial A\,)}{\xi} - \gamma + o(1), \qquad (\xi \to 0)$$

where γ is **universal** (*i.e.* insensitive to the short distance physics). γ is known as the ***topological entanglement entropy***, as it is expected to vanish for topologically trivial phases (phases not supporting anyons).

- For Laughlin $\nu = 1/m$ it is expected to be $\gamma = \log\sqrt{m}$.

- More generally $\gamma = \log D$ where $D \geq 1$ is the *total quantum dimension*, given by

$$D = \sqrt{\sum_a d_a^2}$$

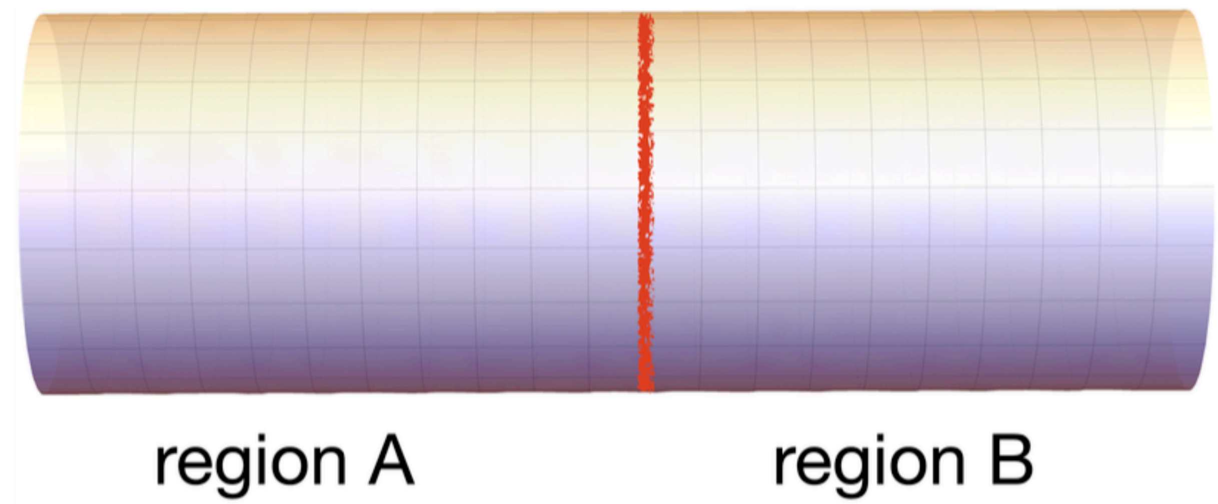
where d_a are the quantum dimensions of the anyons (labelled by a)

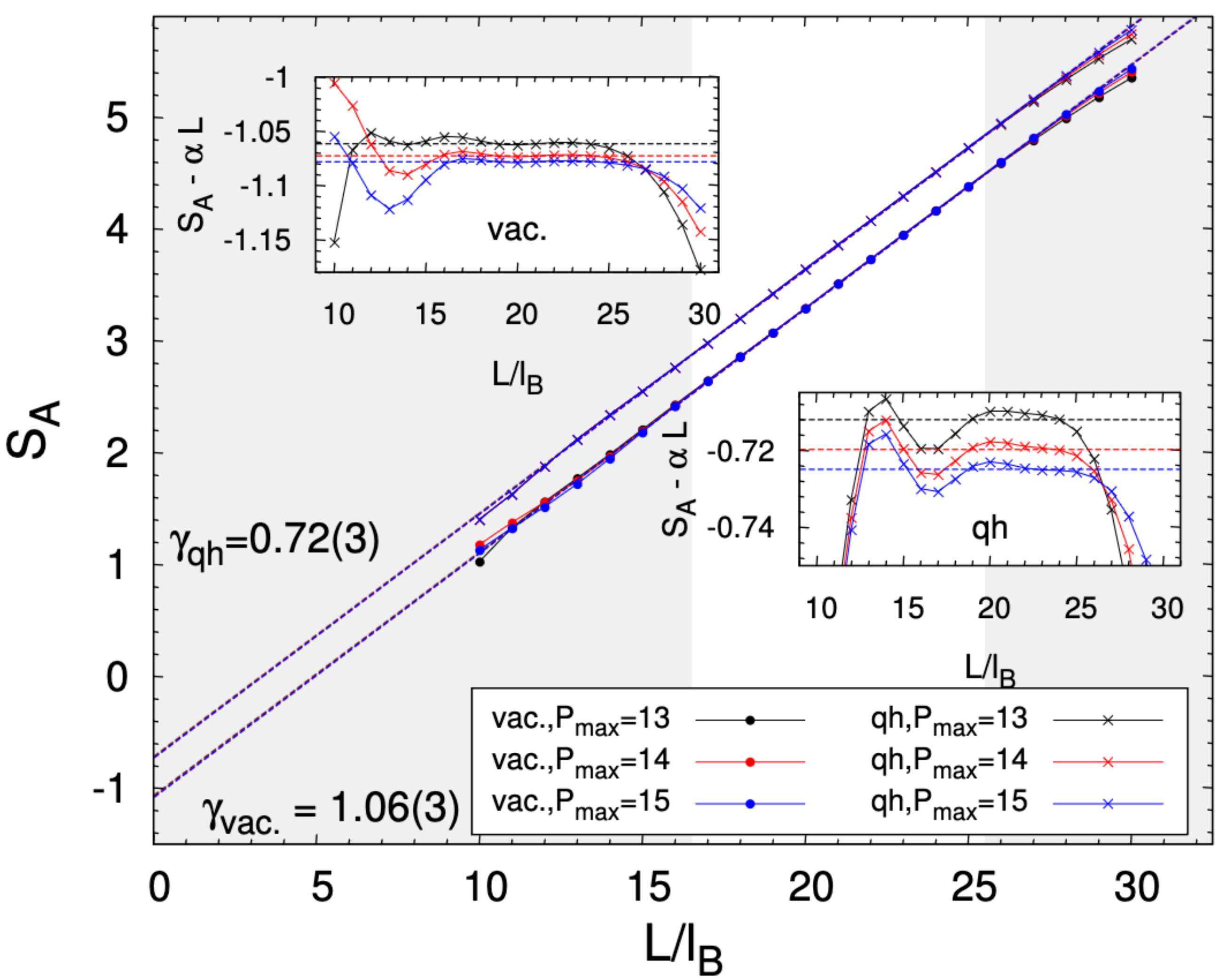
- For a CFT model state $D = 1/S_{00}$ and $d_a = S_{a0}/S_{00}$ in terms the modular S matrix of the underlying conformal field theory.

This is supported by a few exactly solvable models (*e.g.* quantum stabilizers such as the toric code), numerics on FQH states and also TQFT mumbo jumbo.

Topological EE from microscopic models

Exact Matrix Product State approach to CFT model states on the cylinder [Zaletel Mong 2012]





Scaling of the entanglement entropy for the Moore-Read state.

TQFT prediction :

- $\gamma_{\text{vac}} = \log(2\sqrt{2}) \simeq 1.0397$
- $\gamma_{\text{qh}} = \log(2) \simeq 0.6931$

$4\,\%$ error in the range $15\xi \leq L \leq 25\xi$

[From BE, Regnault, Bernevig 2015]

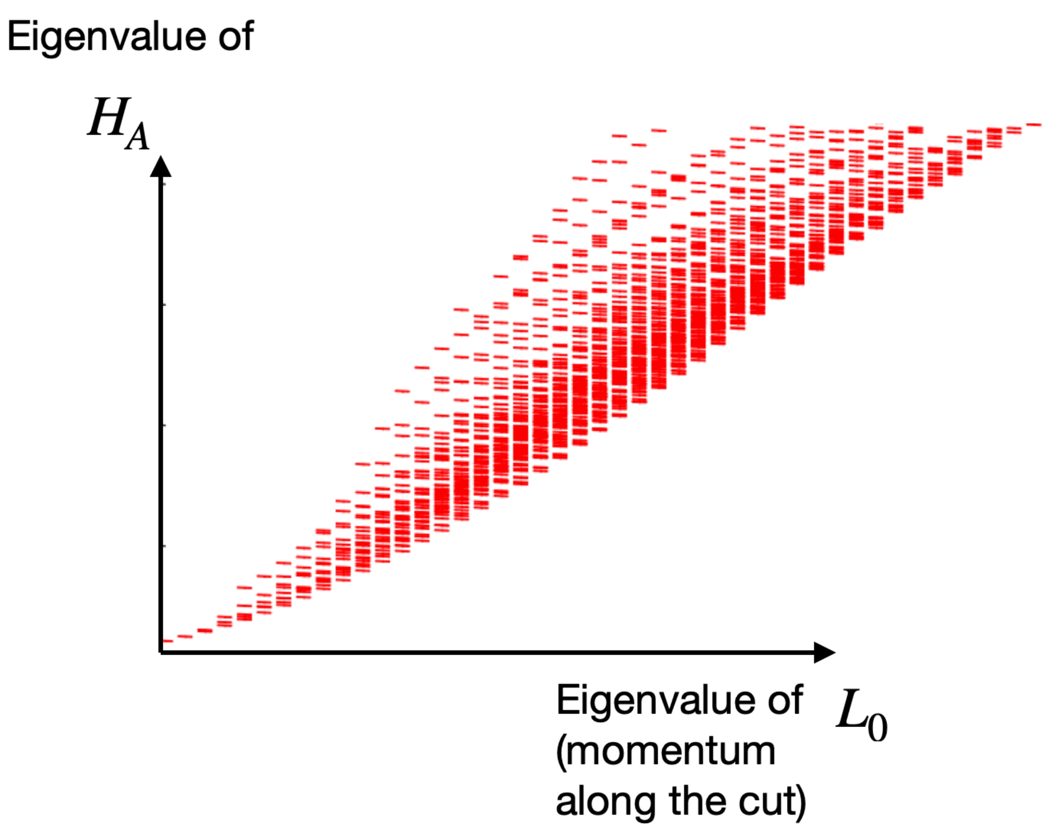
Entanglement spectrum and bulk-edge correspondence

The entanglement or modular Hamiltonian H_A is defined via $\rho_A = \frac{1}{Z} e^{-H_A}$ and it spectrum is known as the **entanglement spectrum**.

<u>**Li-Haldane (2008)**</u> the entanglement Hamiltonian **"mimics"** the chiral edge CFT, that is

$$H_A \simeq v \frac{2\pi}{L} \left(L_0 - \frac{c}{24} \right)$$

where v is a non-universal constant proportional to ξ, and $L =$ length(∂A).



Quote : *the (entanglement)spectrum is "gapless", and has the same count of states (characters) at each momentum (Virasoro level), although there is some splitting of the "energies".*

Entanglement spectrum of 1/3 Laughlin on the sphere with 8 particles [N.Regnault] in the $\Delta N_A = 0$ sector.

Entanglement spectrum and bulk-edge correspondence

Dubail Read Rezayi (2012): the entanglement Hamiltonian is in the same universality class as the chiral edge CFT, that is

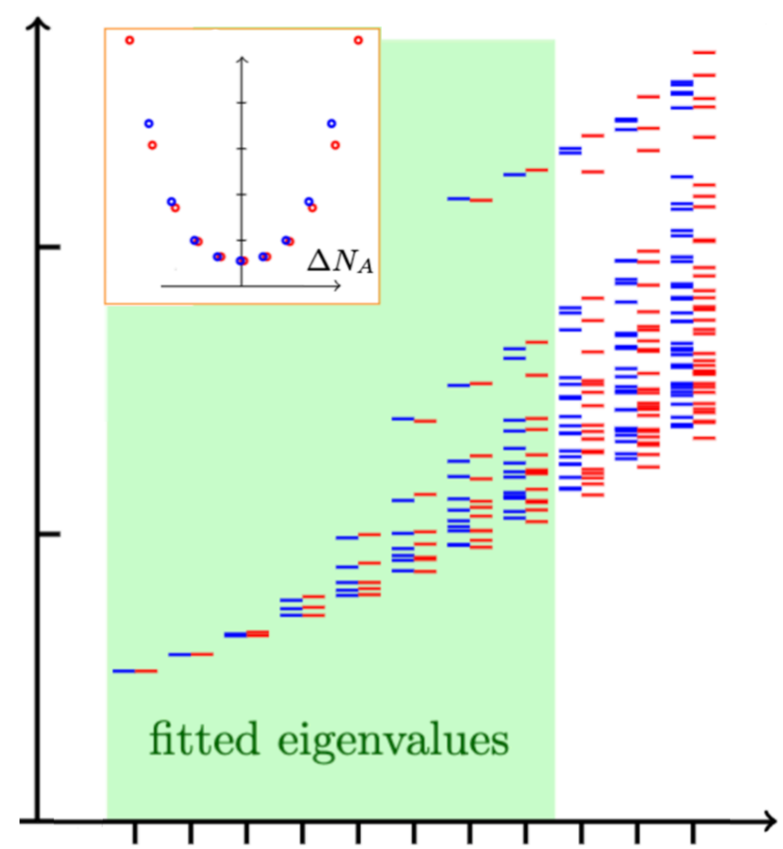
$$H_A = v\frac{2\pi}{L}\left(L_0 - \frac{c}{24}\right) \quad + \quad \textbf{local irrelevant perturbations}$$

where v is a non-universal constant proportional to ξ, and $L = \text{length}(\partial A)$.

Entanglement spectrum of 1/3 Laughlin on the sphere with 12 particles [Dubail *et al.*] in the $\Delta N_A = 0$ sector.

VS

spectrum of H_{CFT} + fine tuned perturbation (6 parameters)



Integer quantum Hall state

exact results

Integer quantum Hall effect

Setup :

- two-dimensional (oriented) surface M with metric g_{ij}
- magnetic field $F_{ij} = B\sqrt{g}\epsilon_{ij}$
- no interactions (but Pauli principle)

One body Hamiltonian = magnetic Laplacian

$$H = \frac{1}{2}\nabla^* \nabla = -\frac{1}{2}\frac{1}{\sqrt{g}}\nabla_i \sqrt{g} g^{ij} \nabla_j$$

acting on a Hermitian line bundle $L \to M$, with connection ∇ (whose curvature is the magnetic field F).

Facts :

- M is naturally a Riemann surface (the metric induces a complex structure J)
- L has a natural holomorphic structure $\bar\partial$ such that ∇ is the Chern connection

$$H = \bar\partial^* \bar\partial + \frac{B}{2}$$

Provided the magnetic field B is uniform (Kälher condition), then

Lowest Landau level = holomorphic sections

Integer quantum Hall state is gapped

LLL is spanned by holomorphic section $\{|\psi_n\rangle\}$ of L

IQH= fully occupied LLL

The corresponding quantum state is a slater determinant : $|\Psi\rangle = \bigwedge_n |\psi_n\rangle$

This is a gapped state. The projector onto the occupied states is known as the Bergman kernel $\Pi(z,\bar{w}) = \sum_n \langle z|\psi_n\rangle\langle\psi_n|w\rangle$

This kernel falls off faster than any power law. On the plane for instance

$$\left|\Pi(z,\bar{w})\right| = \frac{B}{2\pi} e^{-B\frac{|z-w|^2}{2}}$$

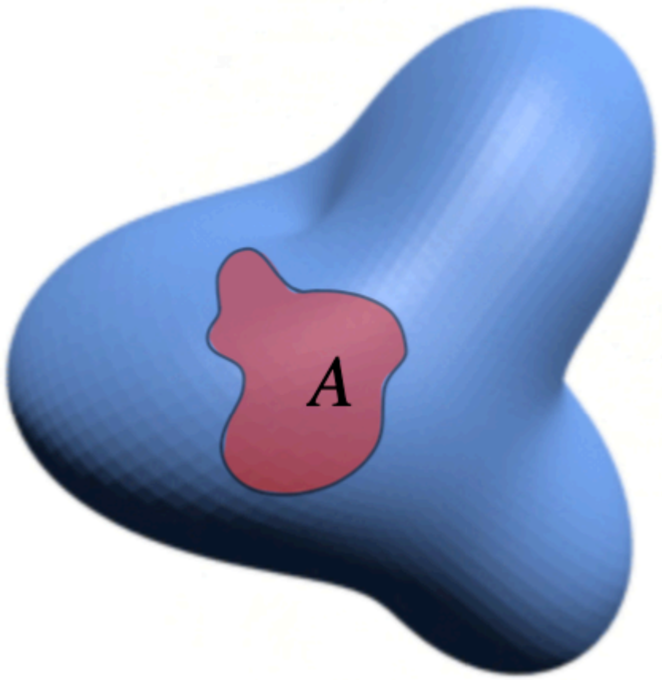
Correlation length (after restoring $\hbar$ and the electric charge q) $\xi = \sqrt{\frac{\hbar}{qB}}$

Area law and geometric corrections

<u>Theorem :</u> **Area Law holds** **[L.Charles, BE, 2018]**

$$S_A \sim C \frac{\text{Length}(\,\partial A\,)}{\xi}, \qquad (\xi \to 0)$$

for some explicit constant C, independently of the shape of A, as long as ∂A is smooth.



<u>Subleading corrections (exact result, not a theorem :):</u>

$$S_A = \int_{\partial A} \frac{d\sigma}{\xi} \left(C + \xi^2 \left[C_1 \kappa^2 + C_2 R \right] \right) + O(\xi^3)$$

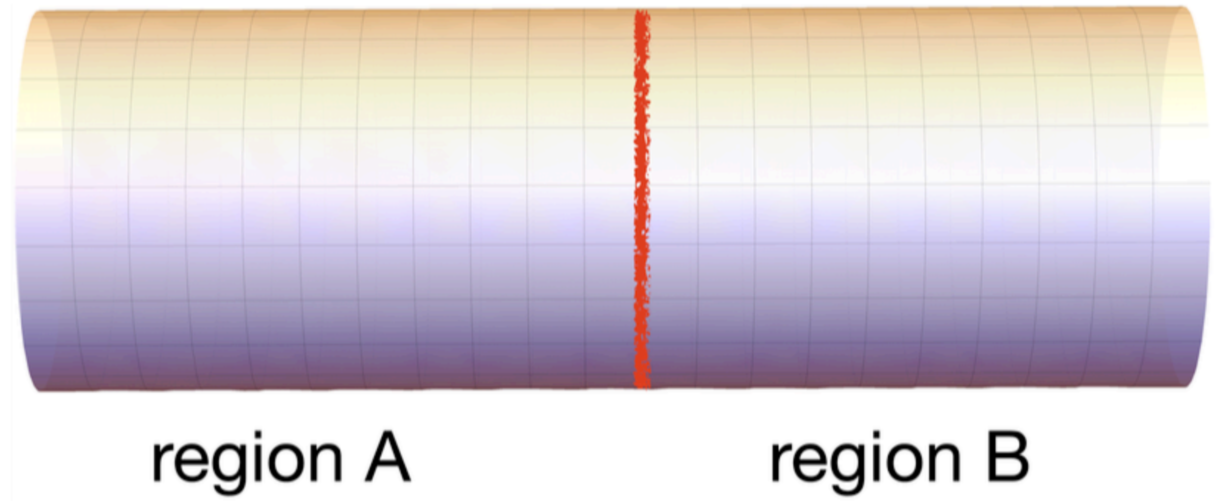
where

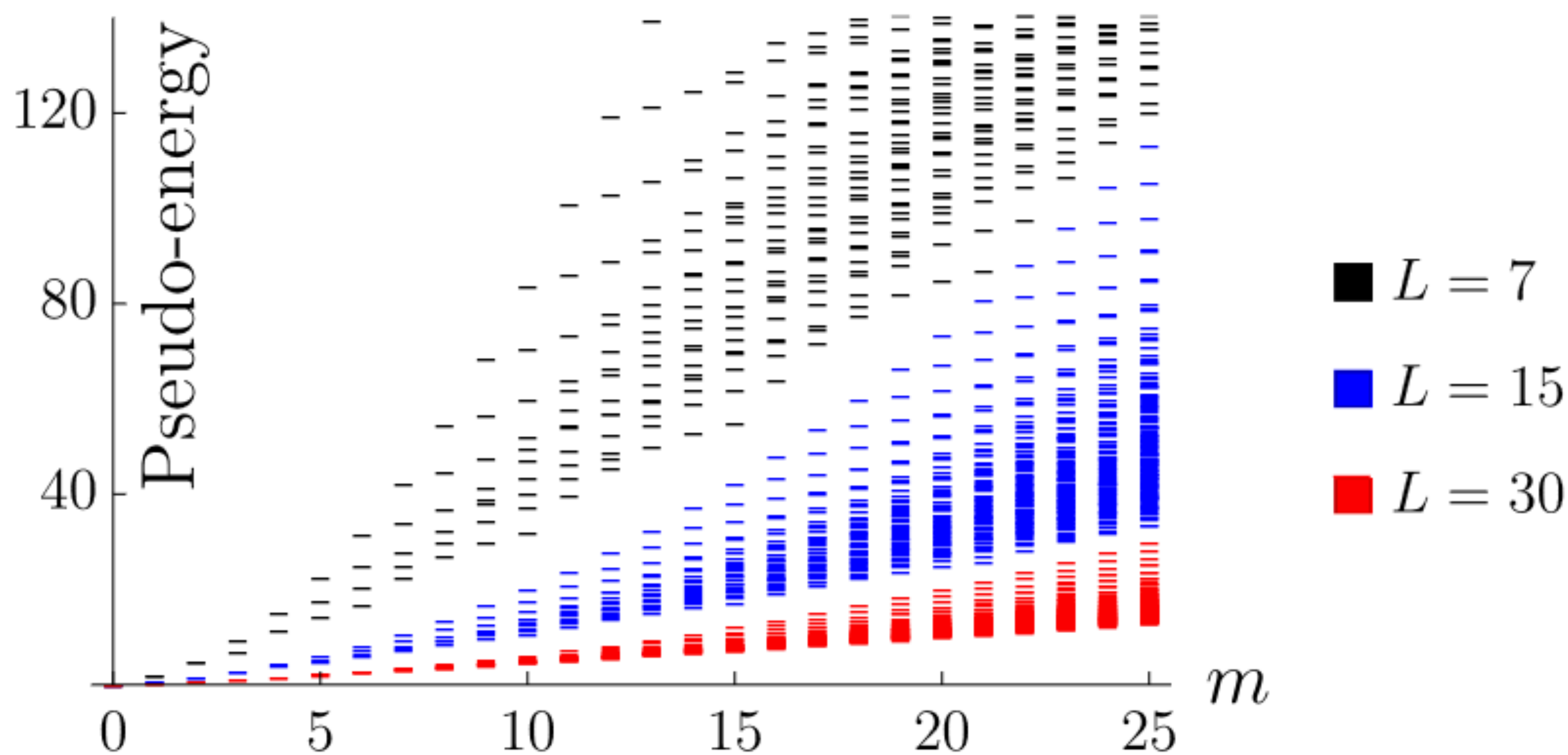
- κ is the geodesic curvature of ∂A
- R is the scalar curvature of the underlying surface

Entanglement spectrum on the cylinder

On the flat infinite cylinder of perimeter L :

$$S_n \sim C_n \frac{\text{Length}(\,\partial A\,)}{\xi} + O(\xi^{-\infty})$$





From [**B.Oblak, N. Regnault and BE (2021)**]

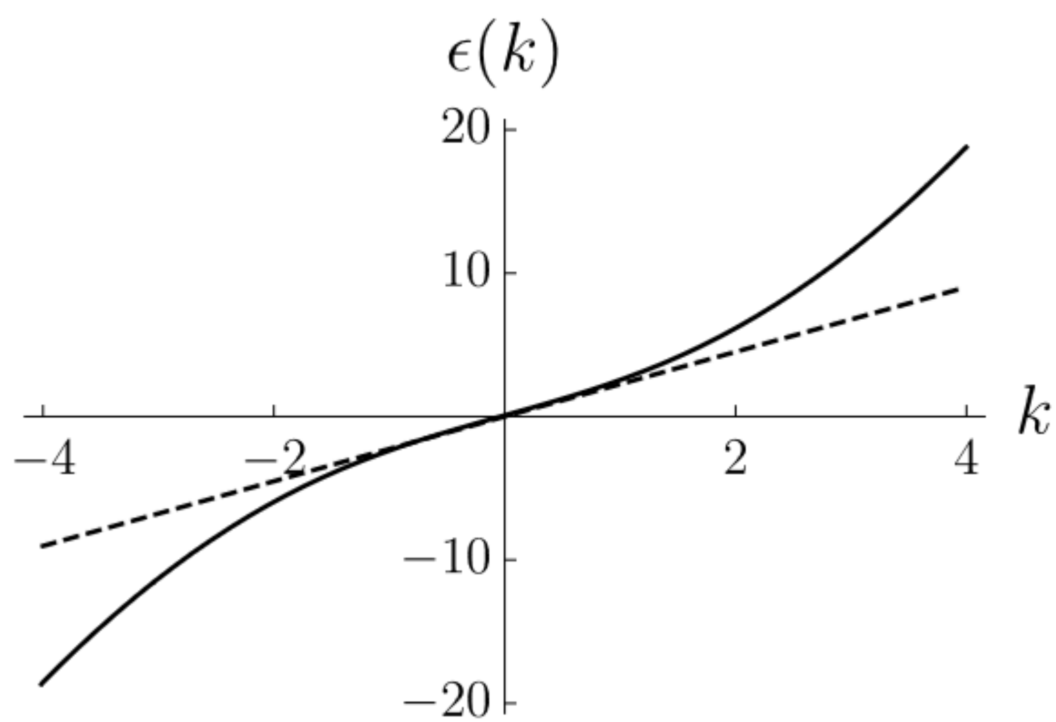
Entanglement Hamiltonian

Fact : for free fermions the entanglement Hamiltonian is quadratic [Peschel]

$$H_A = \sum_k \epsilon(k) c_k^\dagger c_k$$

IQH on the flat infinite cylinder of perimeter L :

$$\epsilon(k) = \log\left[\frac{2}{\text{erfc}\,(\xi\, k)} - 1\right], \qquad k \in \frac{2\pi}{L}\mathbb{Z}$$



H_A is the Hamiltonian of a **chiral Dirac fermion** in 1+1 dimensions

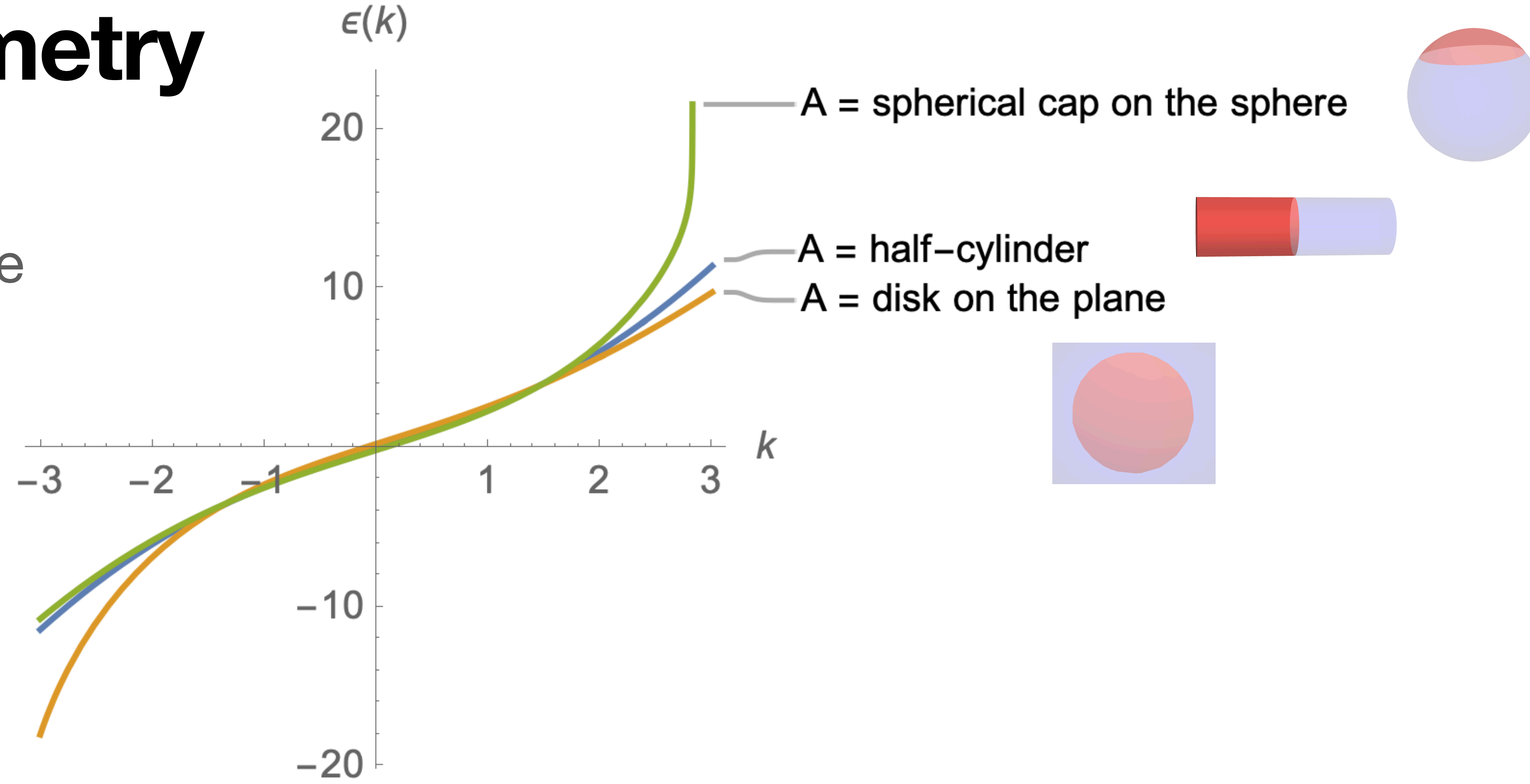
$$H_A = v \sum_k k c_k^\dagger c_k + \cdots$$

up to irrelevant perturbations.

Linear at small momenta

$$\epsilon(k) = \frac{4\xi}{\sqrt{\pi}} k + O(k^3)$$

Sensitivity to geometry



Linear, low energy part of the spectrum is robust.

Leading area law term is robust, subleading corrections sensitive to curvature(s)

$$S_n = \int_{\partial A} \frac{d\sigma}{\xi} \left(C_n + \xi^2 \left[C_n^{(1)} \kappa^2 + C_n^{(2)} R \right] \right) + O(\xi^3)$$

where

- κ is the geodesic curvature of ∂A
- R is the scalar curvature of the underlying surface

No correction of order $O(1)$ $\qquad S_n = C_n L/\xi + O(\xi^{-1})$

The $O(1)$ term γ =0 (the TEE) is also robust !

Symmetry-resolved entanglement entropy

Back to a bipartition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

We have a conserved quantity, namely the number of particles

$$N = \int \rho(\mathbf{r})\, d^2r = N_A + N_B$$

Even though the total number N is fixed, N_A fluctuates. Its distribution is known as the **Full Counting Statistics.** For convenience we work with $Q_A = N_A - \langle N_A \rangle$.

Facts :

ρ_A commutes with Q_A, and therefore it is block diagonal *w.r.t.* to the number of particles q in region A. In fact

$$\rho_A = \bigoplus_q p_q\, \rho_A(q)$$

where $\rho_A(q)$ is a bona fide density matrix (that is it is positive and has unit trace).

When measuring Q_A, the outcome q is obtained with probability p_q. After such a measurement, the reduced density matrix describing subsystem A collapses to $\rho_A(q)$.

Symmetry-resolved entropy

Idea : refine the EE by q sectors, according to

$$\rho_A = \bigoplus_q p_q \, \rho_A(q)$$

in order to measure how entanglement is distributed among the different sectors. This means introducing

$$S_n(q) = \frac{1}{1-n} \log \mathrm{Tr} \left(\rho_A(q)^n \right)$$

In particular for the total Von Neumann entropy splits into

$$S = -\sum_q p_q \log p_q + \sum_q p_q S(q)$$

- The first part is the amount of uncertainty coming from the fluctuations of Q_A
- The second term is the average entanglement entropy in each Q_A sector

Conjecture : equipartition of entanglement entropy **[Xavier, Alcaraz, Sierra (2018)]**

$S(q)$ does not depend on q

at least for typically values of q

Equipartition for Integer quantum Hall state

exact results

Full Counting Statistics for the IQHE

Theorem : **[L.Charles, BE (2018)]**

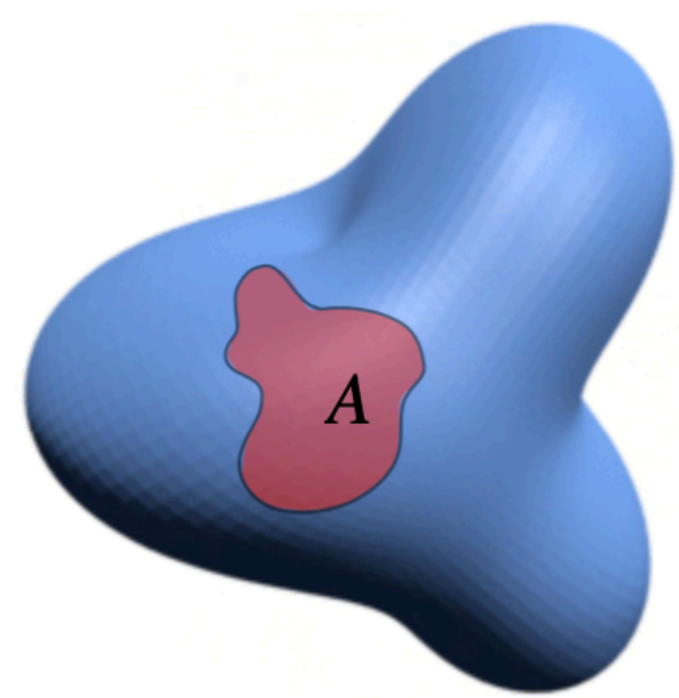
The random variable N_A is a sum of independent Bernoulli variables

$$N_A = \sum_j B_j$$

with success rate $\lambda_j \in (0,1)$ given by the spectrum of the Bergman kernel *restricted to region* A

All (even) cumulants obey the Area law. In particular the variance

$$\sigma^2 = \langle N_A^2 \rangle - \langle N_A \rangle^2 \sim (2\pi)^{-3/2} L/\xi, \qquad (\xi \to 0)$$



Twisted moments $\hat{Z}_n(\alpha)$

Computing the SR Rényi entropy boils down to computing

$$Z_n(q) = \mathrm{Tr}\left(\rho_A^n \Pi_q\right) \qquad \text{where} \qquad \Pi_q \ \text{ is the projector onto the sector } Q_A = q$$

It turns out to be more convenient to work with the Fourier transform

$$\hat{Z}_n(\alpha) = \mathrm{Tr}\left(\rho_A^n \, e^{i\alpha Q_A}\right)$$

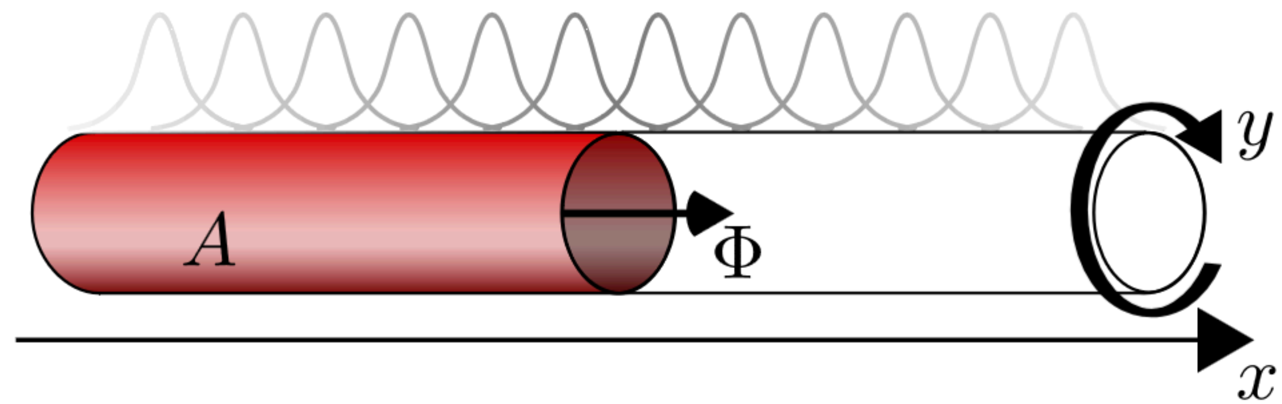
In particular $\hat{Z}_1(\alpha) = \mathrm{Tr}\left(\rho_A e^{i\alpha Q_A}\right) = \langle e^{i\alpha Q_A}\rangle$ is the generating function of the FCS

For the IQHE (Peschel trick)

$$\hat{Z}_n(\alpha) = \prod_j \left(\lambda_j^n e^{i\alpha(1-\lambda_j)} + (1-\lambda_j)^n e^{-i\alpha\lambda_j}\right)$$

where $\lambda_j \in (0,1)$ are the eigenvalues of the Bergman kernel restricted to A

Simple setup : the cylinder



Two parameters :

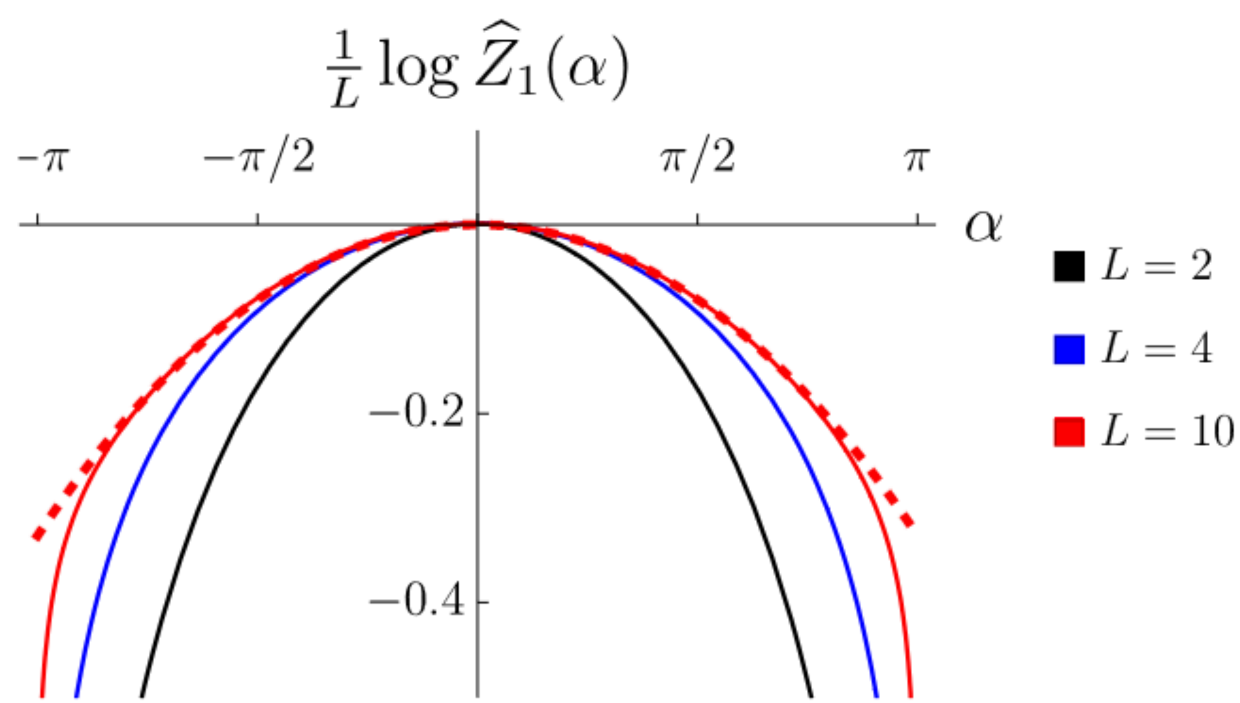
- the perimeter L/ξ
- the holonomy Φ

on the cylinder $\lambda_j = \frac{1}{2}\mathrm{erfc}\left(\frac{2\pi\xi}{L} j\right)$ with $j \in \mathbb{Z} + \frac{\Phi}{2\pi}$

$$\log \hat{Z}_n(\alpha) = \sum_j \log\left(\lambda_j^n e^{i\alpha(1-\lambda_j)} + (1-\lambda_j)^n e^{-i\alpha\lambda_j}\right) = \frac{L}{\xi} F_n(\alpha) + O\left(L^{-\infty}\right)$$

Where $F_n(\alpha)$ is explicit $F_n(\alpha) = \int_{-\infty}^{\infty} \frac{dk}{4\pi} \log\left(\lambda(k)^{2n} + \lambda(-k)^{2n} + 2\lambda(k)^n \lambda(-k)^n \cos\alpha\right)$ $\lambda(k) = \frac{1}{2}\mathrm{erfc}\left(\xi k\right)$

$$\hat{Z}_n(\alpha) \sim e^{-L(a_n + b_n\alpha^2 + c_n\alpha^4) + O(L\alpha^6)}$$

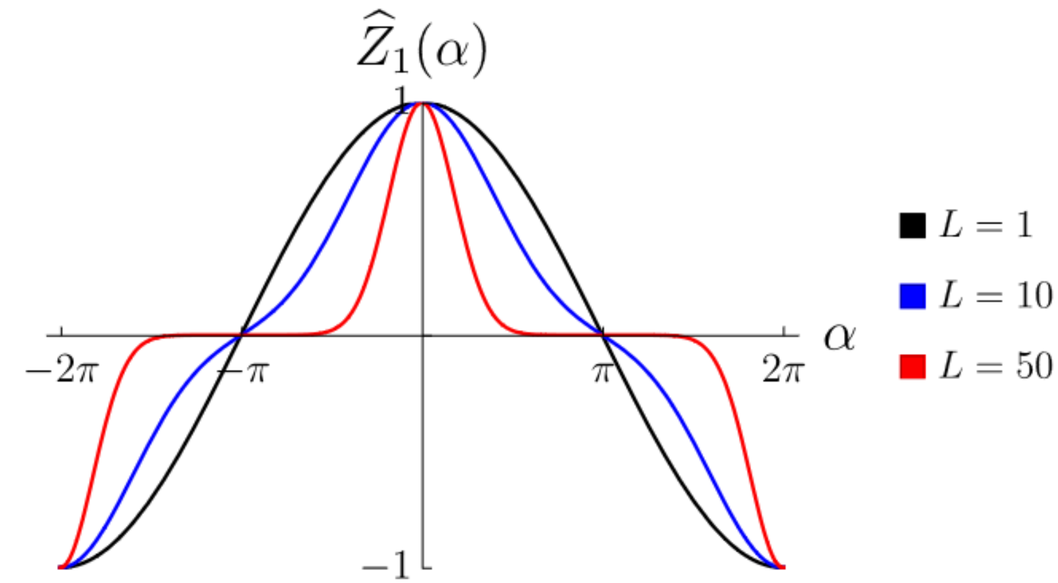


Full Counting Statistics for the IQHE on the cylinder

$\widehat{Z}_1(\alpha) \sim e^{-L(b_1\alpha^2 + c_n\alpha^4) + O(L\alpha^6)}$

and $p_q = \int \frac{d\alpha}{2\pi} e^{-i\alpha q} \widehat{Z}_1(\alpha)$

is readily obtained by saddle point



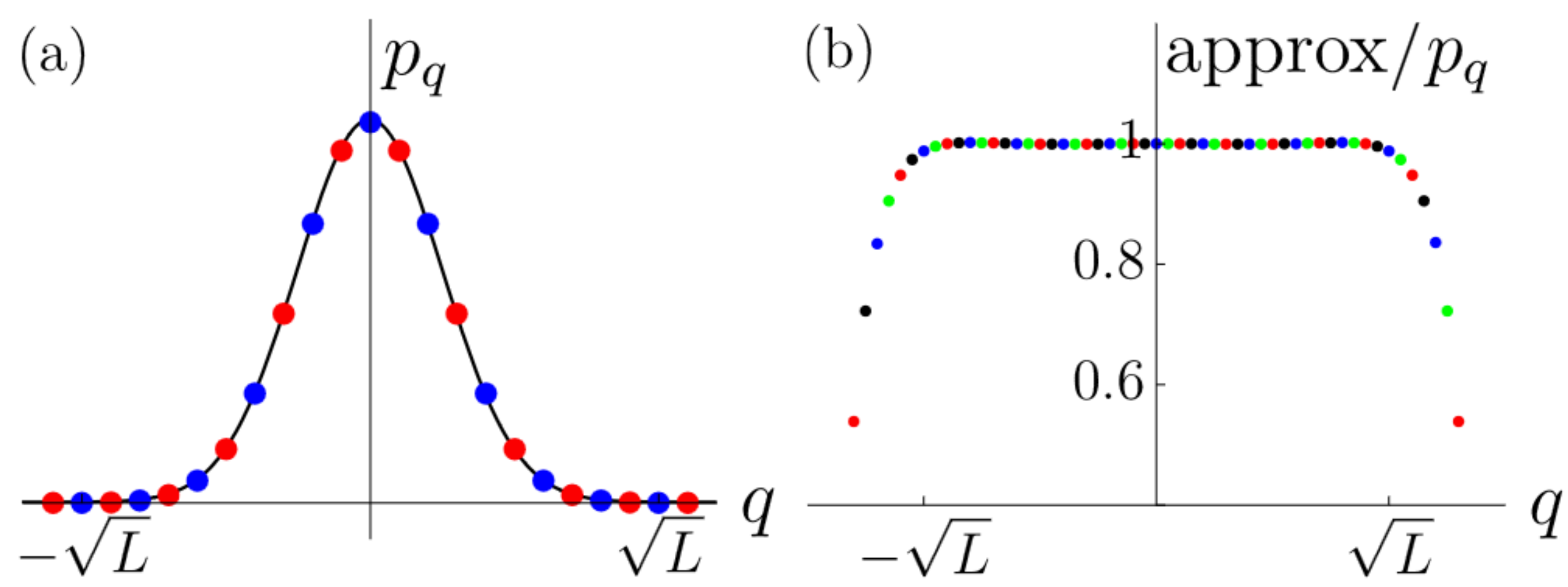
On the cylinder the distribution is nearly Gaussian

$$p_q \simeq \frac{1}{2\pi\sigma^2} e^{-\frac{q^2}{2\sigma^2}}$$

with variance $\sigma^2 = \frac{1}{(2\pi)^{3/2}} \frac{L}{\xi}$

at least for typical values, that is

$$q = O(\sqrt{L})$$

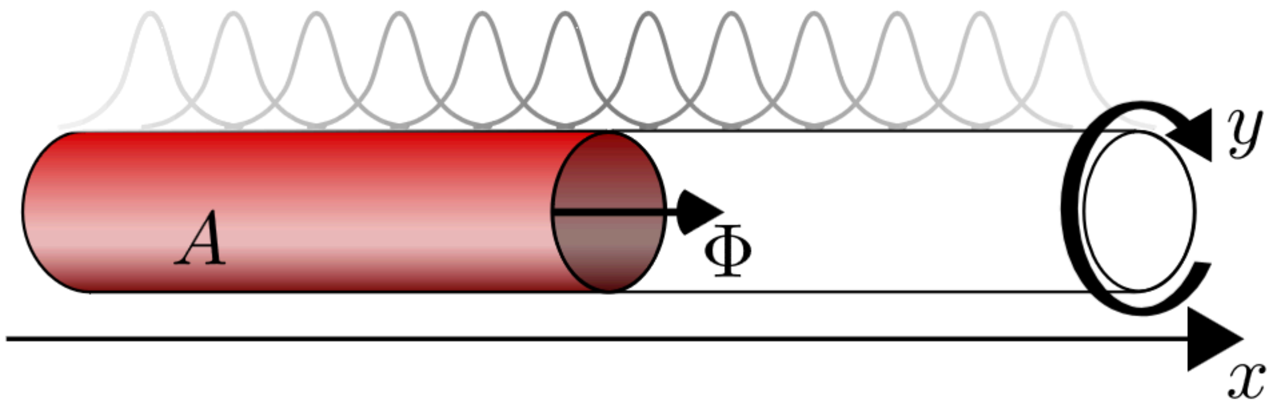


FCS on the cylinder for $L = 25\xi$ and various values of Φ

From [**B.Oblak, N. Regnault and BE (2021)**]

Full Counting Statistics for the IQHE on the cylinder

From [**B.Oblak, N. Regnault and BE (2021)**]



$$F_n(\alpha) = \int_{-\infty}^{\infty} \frac{dk}{4\pi} \log\left(\lambda(k)^{2n} + \lambda(-k)^{2n} + 2\lambda(k)^n \lambda(-k)^n \cos\alpha\right)$$

$$\hat{Z}_n(\alpha) \sim e^{-L(a_n + b_n\alpha^2 + c_n\alpha^4) + O(L\alpha^6)}$$

For typical values of q, namely $q = O(\sqrt{L})$, we have

$$S_n(q) \sim S_n - \frac{1}{2}\log L + A_n + B_n\frac{q^2}{L} + C_n\frac{q^4}{L^3} + O\left(\frac{1}{L^2}\right)$$

Equipartition holds for $q = o(\sqrt{L})$

Equipartition for Laughlin states

probing the Li-Haldane conjecture

Equipartition from Li-Haldane

From [**B.Oblak, N. Regnault and BE (2021)]**

Let's assume that Li-Haldane holds strictly (no irrelevant perturbation/finite size correction)

$$\rho_A = \frac{1}{Z_a} e^{-\frac{2\pi v\xi}{L}\left(L_0 - \frac{c}{24}\right)} \qquad \text{with} \qquad Z_a = \mathrm{Tr}_a\, e^{-\frac{2\pi v\xi}{L}\left(L_0 - \frac{c}{24}\right)}$$

For the Laughlin state the CFT is a chiral compact boson at radius $R = \sqrt{m}$

$$\rho_A = \frac{1}{Z_a} \mathrm{e}^{-\frac{\pi v p}{L} Q_A^2} \mathrm{e}^{-\frac{2\pi v}{L}\left(\sum_n a_{-n} a_n - 1/24\right)}$$

ρ_A is manifestly block-diagonal with respect to Q_A, and more remarkably

the normalized blocks $\rho_A(q)$ do not depend on q :

$$\rho_A(q) = \frac{1}{\eta(\tau)} \mathrm{e}^{-\frac{2\pi v}{L}\left(\sum_n a_{-n} a_n - 1/24\right)}$$

This would mean strict equipartition of the entanglement !

Furthermore the charge distribution would be **exactly Gaussian** !

Probing Li-Haldane

From [**B.Oblak, N. Regnault and BE (2021)**]

Let's now restore irrelevant perturbations to the entanglement Hamiltonian

$$\rho_A = \frac{1}{Z_a} e^{-H_A} \qquad H_A = \frac{2\pi v}{L}\left(L_0 - \frac{c}{24}\right) + \sum_j g_j \underbrace{\int_0^L \phi_j(y)\,\mathrm{d}y}_{\equiv (\frac{\pi}{L})^{\Delta_j - 1} V_j}$$

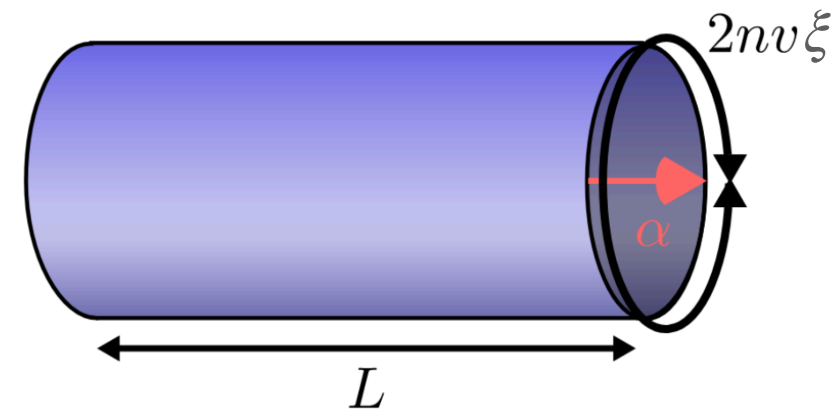
where

- all perturbations are $U(1)$ neutral (here polynomials in derivatives of the scalar field)
- the scaling dimensions Δ_j of the perturbations are $\Delta_j > 2$
- V_j stands for the zero mode of the field ϕ_j
- the coupling constants g_j are not known and depend on the geometry

we need to evaluate the charged moments :

$$\widehat{Z}_n(\alpha) = \frac{1}{Z_a^n} \mathrm{Tr}_a\left(\mathrm{e}^{i\alpha Q_A} \mathrm{e}^{-\frac{2\pi v n}{L}(L_0 - 1/24) + \cdots}\right)$$

Mapping to partition functions in 2+0 critical systems



Tr $\left(e^{i\alpha Q_A} e^{-\frac{2\pi n\xi v}{L}(L_0 - 1/24) + \cdots}\right)$ can be interpreted as

the partition function of a critical 1D system

- on an *open* chain of length L
- at inverse temperature $\beta_n = 2nv\xi$
- with *twisted boundary conditions* (in the compact imaginary time)

Interchanging space and imaginary time, one obtains a periodic system of size $2nv\xi$, with **twisted** periodic boundary conditions. Thus

$$\text{Tr}\left(e^{i\alpha Q_A} e^{-\frac{2\pi\xi v}{L}(L_0 - 1/24) + \cdots}\right) = \langle B(\alpha) | e^{-LH} | B(\alpha) \rangle \sim \left| \langle B(\alpha) | \Psi_0 \rangle \right|^2 e^{-LE_0(\alpha)}$$

one recovers the same phenomenology as for the IQHE $\widehat{Z}_n(\alpha) \sim g^{n-1}\, e^{-L(a_n + b_n\alpha^2 + c_n\alpha^4) + O(L\alpha^6)}$

up to an additional prefactor g = Ludwig-Affleck boundary entropy $\Rightarrow \quad \gamma = \log g$

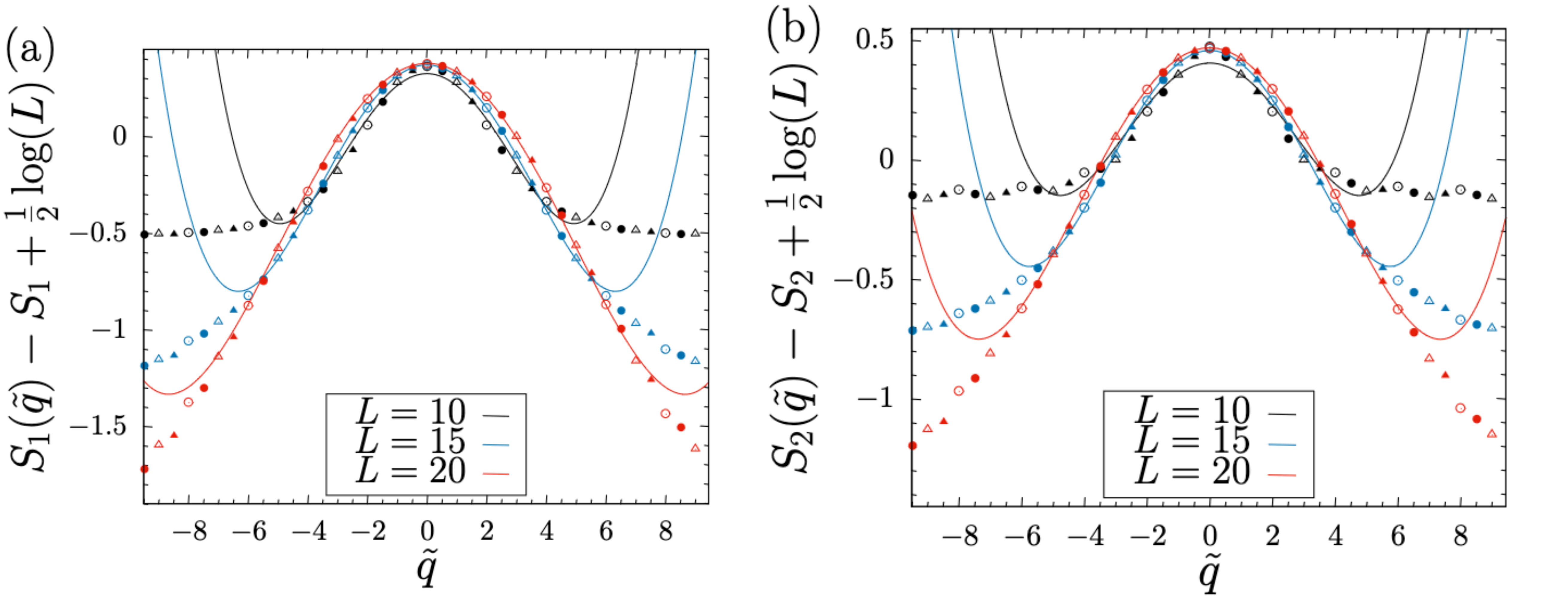
Same conclusion as for the IQHE :

- **equipartition holds for** $q = o(\sqrt{L})$
- **FCS is gaussian for** $q = O(\sqrt{L})$

Numerical check (using MPS on the cylinder)

For typical values of q, namely $q = O(\sqrt{L})$, we have

$$S_n(q) \sim S_n - \frac{1}{2}\log L + A_n + B_n\frac{q^2}{L} + C_n\frac{q^4}{L^3} + O\left(\frac{1}{L^2}\right)$$



Numerical values for Laughlin $\nu = 1/2$ ($\tilde{q}$ stands for $q/2$).

The solid lines are quartic fits with three parameters A_n, B_n, C_n. As anticipated the quartic approximation holds for $q = O(\sqrt{L})$.

To conclude

For both the IQHE and the Laughin state

- typical charge fluctuations are gaussian
- Equipartition of entanglement entropy holds for $q = o(\sqrt{L})$
- finite size corrections are important

This offers compelling evidence in support of the Li-Haldane conjecture in the strong sense of Dubail Read and Rezayi, that is

$$\rho_A = \frac{1}{Z_a} e^{-H_A} \qquad H_A = \frac{2\pi v}{L}\left(L_0 - \frac{c}{24}\right) + \sum_j g_j \underbrace{\int_0^L \phi_j(y)\, \mathrm{d}y}_{\equiv (\frac{\pi}{L})^{\Delta_j - 1} V_j}$$

Thank you !