Uniqueness of Landau levels and their analogs with higher Chern numbers

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(arXiv:2304.00866—Joint work with Tomoki Ozawa)

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Geometric and analytic aspects of the QHE
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Outline

Bloch bands and quantum geometry

Kähler bands

Ideal Kähler bands/Ideal flatbands

Flat Kähler bands

Bloch bands and quantum geometry

Abstracting the concept of a Bloch wavefunction

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$$|u_{\mathbf{k}+\mathbf{G}}\rangle = e_{\mathbf{G}}(\mathbf{k}) U_{\mathbf{G}} |u_{\mathbf{k}}\rangle, \text{ for all } \mathbf{G} \in \mathbb{Z}^2, \tag{1}$$

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where $(e_{\mathbf{G}})_{\mathbf{G}\in\mathbb{Z}^2}$ is a system of multipliers for a line bundle $L \to \mathrm{BZ}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $(U_{\mathbf{G}})_{\mathbf{G}\in\mathbb{Z}^2}$ a unitary representation of the reciprocal lattice.

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with $d \log h = \bar{A} + A$; i.e. L is an Hermitian line bundle with connection. Note that $A(\mathbf{k})$ is the opposite of the Berry connection.

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The transformation $|u_{\mathbf{k}}\rangle \mapsto |u_{\mathbf{k}}\rangle g(\mathbf{k})$ *induces the transformations* $e_{\mathbf{G}}(\mathbf{k}) \mapsto e_{\mathbf{G}}(\mathbf{k}) \left(g(\mathbf{k}+\mathbf{G})g^{-1}(\mathbf{k})\right)$ *which produces isomorphic line bundles.*

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Remark

The unitary operator U intertwines the unitary reps of $\mathbb{Z}^2$: $U'_{\mathbf{G}} U = U U_{\mathbf{G}}, \forall \mathbf{G} \in \mathbb{Z}^2$.

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The direct sum is over a collection of points $\mathbf{r} = (x, y) \in$ u.c.. This collection can be the whole unit cell, in which case the direct sum has to be replaced by the more general direct integral $\mathcal{H} \cong \int^{\oplus} d^2\mathbf{r}\, \mathcal{H}_{\mathbf{r}}$.

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$$\begin{aligned} u_{\mathbf{k}+\mathbf{G}}(\mathbf{r}) &= e_{\mathbf{G}}(\mathbf{k})\chi(\mathbf{G})u_{\mathbf{k}}(\mathbf{r}) \\ &= e_{\mathbf{G}}(\mathbf{k})e^{-2\pi i\mathbf{G}\cdot\mathbf{r}}u_{\mathbf{k}}(\mathbf{r}). \end{aligned} \tag{3}$$

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  and so determine sections of a line bundle $L_{\mathbf{r}} \to \mathrm{BZ}^2$ whose multipliers are $e_{\mathbf{G}}(\mathbf{k})\chi(\mathbf{G})$.

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  and so determine sections of a line bundle $L_{\mathbf{r}} \to \mathrm{BZ}^2$ whose multipliers are $e_{\mathbf{G}}(\mathbf{k})\chi(\mathbf{G})$.
- For each $\mathbf{r}$ occurring in the decomposition $\mathcal{H} = \bigoplus_{\mathbf{r}} \mathcal{H}_{\mathbf{r}}$ we have one such line bundle.

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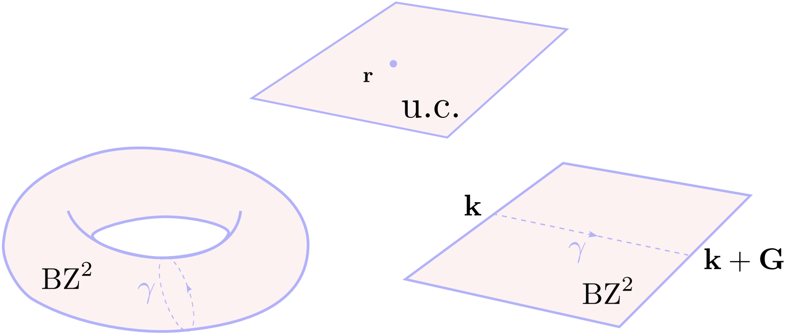
The $u_{\mathbf{k}}(\mathbf{r})$'s know of the spatial structure of the unit cell through the holonomy on loops $\mathbf{k}(t) \mapsto \mathbf{k} + t\mathbf{G}$, $t \in [0,1]$: $\chi^{-1}(\mathbf{G})$.

Spatial structure of Bloch wavefunctions (cont.)



$$e^{2\pi i \mathbf{G}\cdot\mathbf{r}} = \mathrm{hol}_\gamma : (\mathcal{L}_\mathbf{r})_\mathbf{k} \to (\mathcal{L}_\mathbf{r})_{\mathbf{k}+\mathbf{G}} = (\mathcal{L}_\mathbf{r})_\mathbf{k}$$
$$\gamma : t \mapsto \mathbf{k}(t) = \mathbf{k} + t\mathbf{G}$$

Figure: Geometric interpretation of the spatial structure of Bloch wavefunctions.

Quantum geometry in momentum space

- The assignment $\mathbf{k} \mapsto |u_{\mathbf{k}}\rangle \in \mathcal{H} - \{0\}$ determines a map $f : \mathbb{R}^2 \to \mathbb{P}\mathcal{H}$ which uniquely defines the Bloch wavefunction.

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$$F := f^*(2i\omega_{FS}) \text{ and } g := f^* g_{FS}. \quad (8)$$

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Proposition (Quantum geometry in the Brillouin zone)

The tensors $F \in \Omega^2(\mathbb{R}^2)$ *and* $g \in \mathrm{Symm}^2(\mathbb{R}^2)$ *go down to the Brillouin zone* $\mathsf{BZ}^2 = \mathbb{R}^2/\mathbb{Z}^2$*.*

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Proof.

Since $|u_{\mathbf{k}+\mathbf{G}}\rangle = U_{\mathbf{G}}|u_{\mathbf{k}}\rangle e_{\mathbf{G}}(\mathbf{k})$, then for $\phi_{\mathbf{G}} : \mathbf{k} \mapsto \mathbf{k} + \mathbf{G}$, $\mathbf{G} \in \mathbb{Z}^2$, we have

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where in the above formula $U_{\mathbf{G}} : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ denotes the induced map. Now $U_{\mathbf{G}} \in \mathrm{Aut}(\mathbb{P}\mathcal{H}, \omega_{FS})$ as a Kähler manifold so $\phi_{\mathbf{G}}^* F = F$ and $\phi_{\mathbf{G}}^* g = g$ for all $\mathbf{G} \in \mathbb{Z}^2$.

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where in the above formula $U_{\mathbf{G}} : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ denotes the induced map. Now $U_{\mathbf{G}} \in \mathrm{Aut}(\mathbb{P}\mathcal{H}, \omega_{FS})$ as a Kähler manifold so $\phi_{\mathbf{G}}^* F = F$ and $\phi_{\mathbf{G}}^* g = g$ for all $\mathbf{G} \in \mathbb{Z}^2$. The second part follows from the equivalence of representatives being done through elements of $\mathrm{Aut}(\mathbb{P}\mathcal{H}, \omega_{FS})$. □

Kähler bands

Kähler bands

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A Kähler band is determined by a Bloch wavefunction with the property that $f : \mathbb{R}^2 \to \mathbb{P}\mathcal{H}$ *is a holomorphic immersion with respect to a complex structure* J *that is invariant under reciprocal lattice translations* ($\phi_{\mathbf{G}}^* J = J$ *for all* $\mathbf{G} \in \mathbb{Z}^2$).

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Remark

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Remark

For a Kähler band $(f^*\omega_{FS}, f^*g_{FS}, J)$ *determines a Kähler structure in the Brillouin zone* BZ^2, *hence the adjective Kähler.*

Ideal Kähler bands/Ideal flatbands

Ideal Kähler bands

Definition (Ideal Kähler bands)

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Ideal Kähler bands, also known as ideal flat bands, are Kähler bands with respect to a translation invariant complex structure J_τ*, determined by a modular parameter* $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} : Im(\tau) > 0\}$ *(complex coordinate* $z_\tau = k_x + \tau k_y$*).*

Structure theory of ideal Kähler bands

Holomorphicity condition

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$$\frac{\partial}{\partial \bar{z}_\tau}|u_{\mathbf{k}}\rangle - \frac{\langle u_{\mathbf{k}}|\frac{\partial}{\partial \bar{z}_\tau}|u_{\mathbf{k}}\rangle}{\langle u_{\mathbf{k}}|u_{\mathbf{k}}\rangle}|u_{\mathbf{k}}\rangle = 0, \tag{10}$$

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We can then assume *holomorphic multipliers* $e_\gamma(z_\tau)$ with $\gamma \in \Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{Z}^2$, so $L \to \mathbb{C}/\Lambda_\tau$ and, in fact, all the $L_{\mathbf{r}} \to \mathbb{C}/\Lambda_\tau$ are *Hermitian holomorphic line bundles*.

Structure theory of ideal Kähler bands (cont.): Theta functions

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- Then

$$f_{\alpha,\mathbf{r}}(\mathbf{k}) = \sum_{\beta=0}^{\mathcal{C}-1} a_{\alpha,\beta}(\mathbf{r})\vartheta\left[\begin{array}{c} \frac{\beta}{\mathcal{C}}-\frac{y}{\mathcal{C}} \\ x \end{array}\right](\mathcal{C}z_\tau,\mathcal{C}\tau), \; \alpha = 1,2,\ldots \quad (14)$$

Structure theory of ideal Kähler bands (cont.)

Structure of ideal Kähler bands

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Structure of ideal Kähler bands

For an ideal Kähler band with flat complex structure determined by $\tau \in \mathbb{H}$ and Chern number $\mathcal{C}$, we have $|u_{\mathbf{k}}\rangle = \int_{\text{u.c.}}^{\oplus} d^2\mathbf{r}\; u_{\mathbf{k}}(\mathbf{r})$, with

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where $a_{\alpha,\beta}(\mathbf{r})$, $\alpha = 1, 2, \ldots, \dim \mathcal{H}_{\mathbf{r}}$, $\beta = 0, \ldots, \mathcal{C} - 1$, depend on the particular ideal Kähler band (also on the particular choice of basis $\{|\alpha, \mathbf{r}\rangle\}$ chosen for the $\mathcal{H}_{\mathbf{r}}$'s).

Flat Kähler bands

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Definition (Flat Kähler bands)

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Flat Kähler bands are Kähler bands for which the quantum geometry is translation-invariant.

Flat Kähler bands (cont.)

Consequences of the definition

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- For every $\mathbf{q} \in \mathbb{R}^2$, we have a diffeomorphism of BZ^2 given by $\phi_{\mathbf{q}} : \mathbf{k} \mapsto \mathbf{k} + \mathbf{q} \mod \mathbb{Z}^2$.

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- Since an anti-unitary operator would change the sign of F it follows that $U_{\mathbf{q}}$ is unitary.

Flat Kähler bands (cont.)

Heisenberg groups

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Flat Kähler bands (cont.)

Heisenberg groups (cont.)

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$$\begin{aligned} \omega(\mathbf{k}_1, \mathbf{k}_2) &= D\mathbf{k}_1^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{k}_2 \\ &= D\left(k_{1,x}k_{2,y} - k_{1,y}k_{2,x}\right), \text{for some } D \in \mathbb{R}. \end{aligned} \tag{20}$$

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- If $D \neq 0$, then G is called a *Heisenberg group*.

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for $g : \mathbb{R}^2 \to \mathrm{U}(1)$, which gives the isomorphism of central extensions.

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Note that for $D = 0$ this essentially says that $G = \mathbb{R}^2 \times \mathrm{U}(1)$ as a group.

Flat Kähler bands (cont.)

Theorem (Stone-von Neumann)

The Heisenberg group G has a unique, up to isomorphism, unitary irreducible representation $\mathcal{H} = L^2(\mathbb{R})$, for which $\mathrm{U}(1)$ acts as $(0,\lambda)\cdot\psi = \lambda\cdot\psi$, $(0,\lambda)\in\mathrm{U}(1)\subset G$, with

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which is the standard Hilbert space of a particle moving in one dimension with coordinate q for which the momentum is $p = \frac{1}{iD} \frac{\partial}{\partial q}$, with $[q, p] = \frac{i}{D}$.

Constructing the flat Kähler bands

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- But recall that we have a holomorphicity constraint.

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The explicit form of the wavefunction is obtained by decomposing $L^2(\mathbb{R}) = \int_{\text{u.c.}}^{\oplus} d^2\mathbf{r}\, \mathcal{H}_{\mathbf{r}}$, the decomposition being determined by the *Bloch-Zak* transform

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Here $f_{\mathbf{r}}$ determines a square-integrable section of a Hilbert bundle over the unit cell torus $\mathcal{H} \to \mathbb{R}^2/\mathbb{Z}^2$, whose fiber over $\mathbf{r}$ is $\mathcal{H}_{\mathbf{r}}$, which is a space spanned by $\mathcal{C}$ independent distributions,

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$$\delta_{\alpha}^{\mathbf{r}}(q) = \sum_{p\in\mathbb{Z}} \delta(\mathcal{C}q + x - \alpha - \mathcal{C}p) e^{-2\pi i \frac{1}{\mathcal{C}} y(-x+\alpha+\mathcal{C}p)}, \tag{32}$$

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$\alpha = 0, \ldots, \mathcal{C} - 1$, and isomorphic (through a coherent-state transform) to $H^0(\mathrm{BZ}^2, L_{\mathbf{r}})$.

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$$u_{\mathbf{k}}(\mathbf{r}) = \left(\vartheta \begin{bmatrix} -\frac{y}{\mathcal{C}} \\ x \end{bmatrix} (\mathcal{C}z, \mathcal{C}\tau), \ldots, \vartheta \begin{bmatrix} \frac{\mathcal{C}-1}{\mathcal{C}} - \frac{y}{\mathcal{C}} \\ x \end{bmatrix} (\mathcal{C}z, \mathcal{C}\tau)\right), \tag{33}$$

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which for $\mathcal{C} = 1$ coincides with the LLL Bloch wavefunction and for $\mathcal{C} > 1$ determines the so-called color-entangled Landau level type wavefunctions.

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Summary

- Upon fixing a flat complex structure in the Brillouin zone, parametrized by $\tau \in \mathbb{H}$, and fixing a Chern number $\mathcal{C}$, flat Kähler bands are unique.

Thank you!