Integer Quantum Hall Effect on complex manifolds: a probabilistic view

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Summary

Warm-up and motivations

1.1 - IQHE on the plane

The model

Consider a system of electrons in the complex plane subject to a constant perpendicular magnetic field B. If one restricts to the first N states in the lowest Landau level (LLL), the N-particle wavefunction is

$$\Psi(z_1, \dots, z_N) = \frac{1}{\sqrt{Z_N}} \prod_{i<j} (z_i - z_j) e^{-\frac{1}{4} B \sum_i |z_i|^2},$$

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Complex Gaussian random matrices

Consider a $N \times N$ matrix A whose entries are $A_{jk} = x_{jk} + i y_{jk}$, where (x_{jk}, y_{jk}) are i.i.d. real random variables with distribution $\mathcal{N}(0, \sqrt{B})$. Then (Ginibre, '65) the distribution density of the eigenvalues of A is given by

$$p(z_1, \dots, z_N) = \frac{1}{Z_N} \prod_{i<j} |z_i - z_j|^2 e^{-\frac{1}{2} B \sum_i |z_i|^2} = \|\Psi(z_1, \dots, z_N)\|^2.$$

Correlation functions

Using the well-known theory of Determinantal Point Processes (DPP) developped by Macchi ('75), we know that the n-point correlation functions $(1 \leqslant n \leqslant N)$ of such process satisfy

$$\rho_n(z_1, \ldots, z_n) = \det(K_N(z_i, z_j))_{1 \leqslant i,j \leqslant n}$$

with

$$K_N(z, w) = \frac{1}{\pi} \sum_{j=0}^{N-1} \frac{1}{j!} (z\overline{w})^j e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2}.$$

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- If we let $N \to \infty$ we obtain a DPP whose correlation kernel is

$$K_\infty(z, w) = \frac{1}{\pi} e^{z\overline{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2},$$

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- Still hot topic: various confining potentials, adiabatic transport...

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- Space: a Kähler manifold (M, ω) of dimension d.

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- Prequantization condition

$$\omega = \frac{i}{2\pi} R^L = c_1(L, h).$$

Lowest Landau level

Replace L by $L^k := L^{\otimes k}$ and h by h^k. The lowest Landau level is $H^0(M, L^k)$, which is a Hilbert space of finite dimension N_k for the L^2 inner product

$$\langle \psi_1, \psi_2 \rangle = \int_M h_x^k(\psi_1(x), \psi_2(x)) \frac{\omega^d(x)}{d!}.$$

Given a basis $\{\psi_\ell\}$, the N_k-particle wavefunction is the Slater determinant

$$\Psi(x_1, \ldots, x_{N_k}) = \frac{1}{\sqrt{N!}} \det\left(\psi_\ell(x_m)\right).$$

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Questions

- Does it yield a known point process?
- If so, is it a DPP?
- What happens in the regime $k \to \infty$ (= large-N limit)?

Determinantal point processes

2.1 - Definitions

Point processes

Let E be a locally compact Polish space, with its Borel σ-algebra $\mathcal{B}(E)$ and a reference Borel measure λ. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A **simple point process** (PP) is equivalently a random measure $\mu : \Omega \to \mathscr{M}_1(E)$ defined by

$$\mu = \sum_{i \in I} \delta_{x_i},$$

or a random configuration, *i.e.* a locally finite random subset $X : \Omega \to \mathcal{B}(E)$.

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Its n**-th factorial moment measure** is

$$\mu^{(n)} = \mathbb{E}\Bigg[\sum_{\substack{1 \leqslant i_1, \ldots, i_n \leqslant \mu(E) \\ i_1 \neq \cdots \neq i_n}} \delta_{X_{i_1}, \ldots, X_{i_n}} \Bigg],$$

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Proposition (Johansson '05)

If $(X_1, \dots, X_N)$ *has joint symmetric density* $p : E^N \to \mathbb{R}_+$ *with respect to* $\lambda^{\otimes N}$*, then for all* $n \leqslant N$*,*

$$\rho_n(x_1, \dots, x_n) = \frac{N!}{(N-n)!} \int_{E^{N-n}} p(x_1, \dots, x_N) \lambda^{\otimes N-n}(dx_{n+1} \cdots dx_N).$$

Determinantal point processes

A point process μ is called **determinantal with kernel** $K : E^2 \to \mathbb{C}$ if for all $n \geqslant 1$, its n-point correlation function satisfies

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Examples

- Eigenvalues of classical random matrix ensembles (random matrix theory)
- Zeros of some Gaussian analytic functions (random analytic functions)
- Uniform spanning trees (random graphs)
- Schur measures on Young diagrams (representation theory & statistical mechanics)

2.2 - Orthogonal ensembles

Orthogonal ensembles

An **orthogonal ensemble** is a particular case of DPP, given by a family $(X_1, \ldots, X_N)$ of random variables on E (often $\mathbb{R}$ or $\mathbb{C}$) with joint density

$$p(x_1, \ldots, x_N) = \frac{1}{N!} |\det(\phi_i(x_j))|^2 d\lambda^{\otimes N}(x_1, \ldots, x_N),$$

with (ϕ_i) an orthonormal family in $L^2(E, \lambda)$.

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The point process $\mu = \sum_i \delta_{X_i}$ is almost-surely simple, and it is determinantal with kernel

$$K_N(x, y) = \sum_{i=1}^{N} \phi_i^{(N)}(x) \overline{\phi_i^{(N)}}(y),$$

which is the reproducing kernel of $\mathcal{H} = \mathrm{Span}(\phi_1^{(N)}, \ldots, \phi_N^{(N)}) \subset L^2(E)$.

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Examples

The quantum Hele–Shaw flow (Wiegmann '02, Agam–Bettelheim–Wiegmann–Zabrodin '02, Hedenmalm–Makarov '04), the random normal matrix models (Ameur–Hedenmalm–Makarov '11, '15)

Christoffel–Darboux kernel

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Theorem (Christoffel 1858, Darboux 1878)

If $(\phi_i^{(N)})$ is a sequence of unitary orthonormal polynomials on $\mathbb{R}$,

$$K_N(x,y) = \frac{\phi_N^{(N)}(x)\phi_{N-1}^{(N)}(y) - \phi_{N-1}^{(N)}(x)\phi_N^{(N)}(y)}{x-y}.$$

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- The asymptotic study of the kernel K_N turns into a study of the orthonormal polynomials $\phi_N^{(N)}$ and $\phi_{N-1}^{(N)}$.
- Similar formula for orthonormal polynomials on the unit circle, but no general formula on $\mathbb{C}$ or $\mathbb{C}^n$. Instead: heavy complex analysis.

DPP on complex manifolds

3.1 - Bergman kernel

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The Hilbert space of holomorphic sections

If μ is a finite positive measure on M, $H^0(M, L^k) = \{s \in \mathscr{C}^\infty(M, L^k), \overline{\partial} s = 0\}$ is a Hilbert space of dimension $N_k < \infty$ for the inner product

$$\langle s_1, s_2 \rangle_{(k\phi,\mu)} = \int_M \langle s_1(x), s_2(x) \rangle_{k\phi} d\mu(x).$$

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The **Bergman kernel** is the Schwartz kernel B_k of the orthogonal projection $P_k : L^2(M, L^k) \to H^0(M, L^k)$. If $\{\psi_\ell\}$ is an ONB of $H^0(M, L^k)$,

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B_k is a reproducing kernel on the Hilbert space $H^0(M, L^k)$ and generalizes the Christoffel–Darboux kernel: if $M = \mathbb{C}P^1$ and $L = \mathscr{O}(-1)$, $H^0(M, L^k)$ is the space of homogeneous polynomials of degree k on $\mathbb{C}$.

Asymptotic expansion of the Bergman kernel

We now assume that μ is the measure associated to a Riemannian volume form on M, and that iR^L is a positive $(1,1)$-form (or equivalently the matrix $\left(\frac{\partial^2\phi(x)}{\partial z_i \partial \overline{z}_j}\right)$ is positive definite for all x).

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- Diagonal expansion

Theorem (Tian '90, Catlin '97, Zelditch '98)

There exist smooth functions $(b_j)_{j\geqslant 1}$ such that for any $r \geqslant 1$,

$$B_k(x,x) = \sum_{j=0}^{r} b_j(x) k^{d-j} + O(k^{d-r-1}).$$

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- Near-diagonal expansion:

Theorem (Bleher–Schiffman–Zelditch '00, Shiffman–Zelditch '02, Ma–Marinescu '06, Berman–Berndtsson–Sjöstrand '08)

Let $(z_1, \ldots, z_N)$ be a system of local coordinates in a neighborhood of $x_0 \in M$. There exists a limit kernel B_∞ such that

$$B_k\left(\frac{z}{\sqrt{k}}, \frac{z'}{\sqrt{k}}\right) = B_\infty(z,z')k^d + O(k^{d-\frac{1}{2}}).$$

Determinant of the Bergman kernel

Formally,

$$\det(B_k(x_i, x_j))_{1 \leqslant i,j \leqslant n} = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \sum_{i_1, \dots, i_n = 1}^{N_k} \bigotimes_{j=1}^{n} s_{i_j}(x_j) \otimes \overline{s_{i_j}}(x_{\sigma(j)}).$$

- Ill-defined because the summands live in different vector spaces!

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Duality and tensor products

For any finite-dimensional Hilbert space E, the following contraction is a canonical isomorphism:

$$\left\{ \begin{array}{ccc} L_x \otimes E \otimes \overline{L_x} & \longrightarrow & E \\ u_x \otimes u \otimes \overline{v_x} & \longmapsto & \langle u_x, v_x\rangle_\phi u \end{array} \right. ,$$

where v_x satisfies $(\overline{v_x}, u_x) = \langle u_x, v_x\rangle_\phi$ by the Riesz representation theorem. We can then set

$$\det(B_k(x_i, x_j))_{1\leqslant i,j\leqslant n} = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) \sum_{i_1,\dots,i_n=1}^{N_k} \prod_{j=1}^{n} \langle s_{i_j}(x_j), s_{i_{\sigma^{-1}(j)}}(x_j)\rangle_{k\phi}.$$

3.2 - The point process

The random process

We consider a family $(X_1, \dots, X_{N_k})$ of M-valued random variables with joint distribution

$$d\mathcal{P}_\phi(x_1, \dots, x_{N_k}) = \frac{1}{N_k!} \| \det(\psi_i(x_j)) \|^2_{k\phi} d\mu^{\otimes N_k}(x_1, \dots, x_{N_k}),$$

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Example: S^2

If we take $M = S^2$ the Riemann sphere with its round metric, such a process corresponds in stereographic coordinates to

$$\frac{1}{(k+1)!} |\det(z_\ell^{m-1})|^2 \prod_{\ell=1}^{k+1} \frac{i dz_\ell \wedge d\overline{z_\ell}}{2(1+|z_i|^2)^{k+2}}$$

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- Relation to random matrices: distribution of the eigenvalues of AB^{-1}, where A, B independent Ginibre matrices (Krishnapur '09)
- Bergman kernel:

$$B_k(z_1, z_2) = (k+1) \frac{(1 + z_1 \overline{z}_2)^k}{(1+|z_1|^2)^{\frac{k}{2}} (1+|z_2|^2)^{\frac{k}{2}}}.$$

3.3 - Universality

Theorem (TL '22)

Let M be a compact complex manifold of dimension d, $L \to M$ be a positive Hermitian line bundle over M, and μ be the volume measure associated to a Riemannian volume form on M. The associated point process is determinantal with kernel B_k, and its n-point correlation functions admit the following scaling limit in local coordinates around $x \in M$:

$$\frac{1}{k^{nd}} \rho_n \left(\frac{u_1}{\sqrt{k}}, \dots, \frac{u_n}{\sqrt{k}} \right) = \det(B_\infty(u_i, u_j)) + O(k^{-\frac{1}{2}}).$$

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Ideas of the proof

- Check that all combinatorial techniques used on orthogonal ensembles are compatible with our definition of $\det(B_k(x_i, x_j))$.

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Let M be a compact complex manifold of dimension d, $L \to M$ be a positive Hermitian line bundle over M, and μ be the volume measure associated to a Riemannian volume form on M. The associated point process is determinantal with kernel B_k, and its n-point correlation functions admit the following scaling limit in local coordinates around $x \in M$:

$$\frac{1}{k^{nd}} \rho_n \left(\frac{u_1}{\sqrt{k}}, \dots, \frac{u_n}{\sqrt{k}} \right) = \det(B_\infty(u_i, u_j)) + O(k^{-\frac{1}{2}}).$$

Ideas of the proof

- Check that all combinatorial techniques used on orthogonal ensembles are compatible with our definition of $\det(B_k(x_i, x_j))$.
- Use the asymptotic expansion of B_k and a control of the fluctuations when performing the Laplace expansion of $\det(B_k)$.

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Note

The results holds if we replace L^k by $L^k \otimes F$, where F trivial bundle endowed with a continuous metric.

3.3 - Universality

About universality

Universality is a conjecture, or a meta-theorem, stating that the scaling limit of the observables of many random objects has a universal form, often related to the Gaussian distribution.

- "[Universality] is widely found in the field of random matrix theory. The universality principle loosely states that the eigenvalues statistics of interest will behave asymptotically as if the matrix elements were Gaussian" [Edelman–Guionnet–Péché '16]

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In our setting

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In our setting

- Scaling limits of correlation functions are universal due to the properties of the Bergman kernel. In particular, relaxing the assumptions for the Bergman kernel expansion leads to a direct generalization (see Berman '18)
- The process with kernel B_∞, if M is a Kähler compact surface, is the infinite Ginibre ensemble, the asymptotic eigenvalue distribution of complex Gaussian matrices.

3.4 - Large deviations

New point process

Consider a random family $(X_1, \dots, X_{N_k})$ with joint distribution

$$d\mathcal{P}_\phi^V(x_1, \dots, x_{N_k}) = \frac{1}{Z_{N_k}(\phi, V)} \| \det(\psi_i(z_j))\|_{k\phi}^2 e^{-k\sum_i V(x_i)} d\mu^{\otimes N_k}(x_1, \dots, x_{N_k}),$$

where $\{\psi_i\}$ ONB of $H^0(M, L^k)$ for $\langle \cdot, \cdot \rangle_{(k\phi, \mu)}$.

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Consequences

- The partition function now depends on V and ϕ:

$$\begin{aligned} Z_{N_k}(\phi, V) =& N_k! \det(\langle \psi_i, \psi_j \rangle_{(\phi+V, \mu)} \\ =& N_k! \int_{M^{N_k}} e^{-\sum_i V(x_i)} d\mathcal{P}_\phi(x_1, \dots, x_{N_k}). \end{aligned}$$

- The process is not a DPP anymore.

3.4 - Large deviations

Convergence of Bergman measures

Define the **Bergman measure** on M by

$$\beta_k(x) = \frac{1}{N_k} B_k(x,x) d\mu(x).$$

Theorem (Berman '09)

Let (μ, ϕ) be a weighted measure on M satisfying the Bernstein–Markov condition. The Bergman measures converge weakly in the sense of currents to the equilibrium measure $\mu_{\mathrm{eq}}^{\phi} = \left(\omega + \frac{i}{2\pi}\partial\overline{\partial}\phi\right)^d$.

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- Expectation of the empirical measures of the unweighted process ($V = 0$):

$$\mathbb{E}\left[\hat{\mu}_k\right] = \beta_k(x) \to \mu_{\mathrm{eq}}^{\phi}.$$

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- Using the scaling limit of the correlation functions and standard limit theorems in probability, we can replace the convergence of the expectation by a convergence in probability.
- What about the new weighted process?

Mabuchi functional

The *equilibrium energy* is the energy $\mathcal{E}_{\mathrm{eq}} : \mathscr{C}^0(M) \to \mathbb{R}$ associated with the equilibrium measure:

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{E}_{\mathrm{eq}}(\phi + tU) = \int_M U(x) d\mu_{\mathrm{eq}}^{\phi}(x).$$

The *Mabuchi functional* (Mabuchi '86, Donaldson '05) arises from a variational approach of this energy:

$$\mathcal{L}_{\mathrm{eq}}(\phi, U) = \int_0^1 \int_M U(x) d\mu_{\mathrm{eq}}^{\phi + tU}(x) dt = \mathcal{E}_{\mathrm{eq}}(\phi + U) - \mathcal{E}_{\mathrm{eq}}(\phi).$$

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Rate function

$$\mathcal{I}_{\text{eq}}^{\phi, V}(\nu) = \sup_{U \in \mathscr{C}^0(M)} \left(\int_M U(x) d\nu(x) - \mathcal{L}_{\text{eq}}(\phi + V, -U) \right)$$

Theorem (TL '22)

The empirical measures $\mu_k = \frac{1}{N_k}\sum_i \delta_{X_i}$ *of the weighted process satisfy a large deviation principle with good rate function* $\mathcal{I}_{\mathrm{eq}}^{\phi,V}$ *and speed* $kN_k \sim N_k^{1+\frac{1}{d}}$*: for any Borel set* $\Gamma \subset \mathscr{M}_1(M)$*,*

$$- \inf_{\nu \in \mathring{\Gamma}} \mathcal{I}_{\mathrm{eq}}^{\phi,V}(\nu) \leqslant \underline{\lim}_k \frac{1}{kN_k} \log \mathbb{P}[\widehat{\mu}_k \in \Gamma] \leqslant \overline{\lim}_k \frac{1}{kN_k} \log \mathbb{P}[\widehat{\mu}_k \in \Gamma] \leqslant - \inf_{\nu \in \overline{\Gamma}} \mathcal{I}_{\mathrm{eq}}^{\phi,V}(\nu)$$

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Similar results

- Bloom–Levenberg '13 (in $\mathbb{C}^d$)
- Berman '18 (for the unweighted process)

3.4 - Large deviations

Elements of proof

The *cumulant-generating function* associated with a Borel probability measure μ on $\mathcal{E}$ is the function

$$\Lambda_\mu : \lambda \in \mathcal{E}^* \mapsto \log \mathbb{E}[e^{\langle \lambda, X_1 \rangle}] = \log \int_{\mathcal{E}} e^{\langle \lambda, x \rangle} d\mu(x).$$

Theorem (Gärtner–Ellis)

Let (μ_ϵ) be an exponentially tight family of probability measures on a Banach space $\mathcal{E}$. Suppose $\Lambda(\cdot) = \lim_{\epsilon \to 0} \epsilon \Lambda_{\mu_\epsilon}(\cdot/\epsilon)$ is finite-valued, Gateaux differentiable and lower semicontinuous in $\mathcal{E}^*$ with respect to the weak-* topology. Then (μ_ϵ) satisfies a LDP in $\mathcal{E}$ with speed $\frac{1}{\epsilon}$ and with good rate function Λ^*.

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- The cumulant-generating function of the empirical measures is

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- Combine with estimates by Berman–Boucksom '10 $\to$ Mabuchi functional.

Thank You!