CTEQ - Introduction to Monte Carlo

Lecture 2
Matrix Elements and Parton Showers
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Overview

- lecture 1 - introduction and Monte Carlo techniques
- **lecture 2 - matrix elements and parton showers**
- lecture 3 - multi-parton interactions, hadronization, and non-perturbative effects

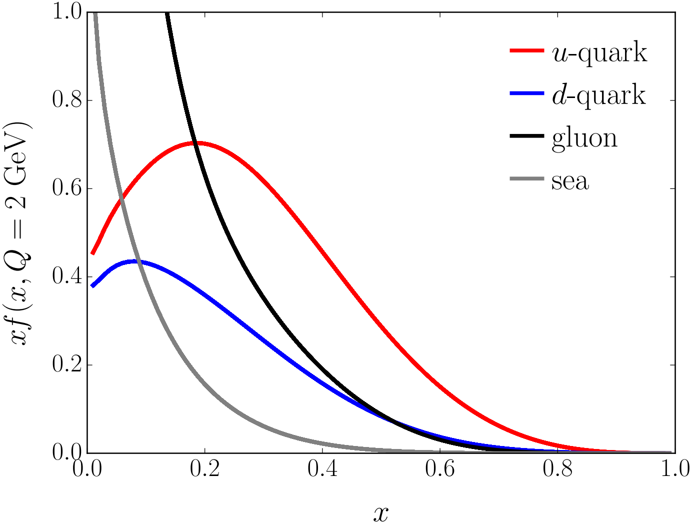
Factorization Theorem

$$\sigma = \int \mathrm{d}x_1 \int \mathrm{d}x_2\, f_1(x_1, Q^2) f_2(x_2, Q^2) \hat{\sigma}$$

- proven for some processes, assumed for many
- f_i - parton distribution function for parton i
- x_i - longitudinal momentum fraction for parton i
- Q^2 - factorization scale
- $\hat{\sigma}$ - partonic cross section

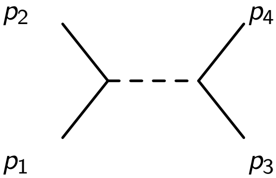
- to generate events we need to sample phase space according to differential cross section
- use the MC sampling techniques of lecture (1)
- first we need to define our kinematics, consider $2 \to 2$ process

Parton Distribution Functions

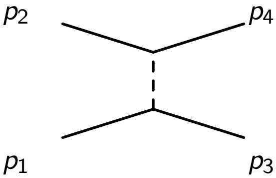


- match order of PDF with calculation (including parton showers)

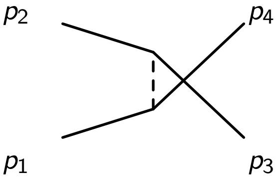
Kinematics



$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$



$$t = (p_1 - p_3)^2 = (p_4 - p_2)^2$$



$$u = (p_1 - p_4)^2 = (p_3 - p_2)^2$$

Kinematics

- define 4-momentum of the two beams

$$
\begin{aligned}
p_1 &= (0, 0, Ex_1, Ex_1) \\
p_2 &= (0, 0, -Ex_2, Ex_2) \\
\Rightarrow \hat{s} &= x_1 x_2 s
\end{aligned}
$$

- distributions are uniform in azimuthal angle ϕ with unpolarized beams, only care about $\hat{\theta}$

$$
\begin{aligned}
0 &= \hat{s} + \hat{t} + \hat{u} \\
\hat{t} &= -\frac{\hat{s}}{2}(1 - \cos\hat{\theta}) \\
\hat{u} &= -\frac{\hat{s}}{2}(1 + \cos\hat{\theta})
\end{aligned}
$$

Kinematics

- need to only consider x_1, x_2, and $\cos\hat{\theta}$
- typically transform x_1 and x_2, helps control general behaviour

$$\tau = x_1 x_2 = \frac{\hat{s}}{s}$$
$$y = \frac{1}{2}\ln\frac{x_1}{x_2}$$
$$z = \cos\hat{\theta}$$

- for the massless case, we then have

$$\sigma = \int \mathrm{d}\tau \int \mathrm{d}y \int \mathrm{d}z \, \frac{\hat{s}}{2\tau} x_1 f_1(x_1, Q^2) x_2 f_2(x_2, Q^2) \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\hat{t}}$$

Sampling Phase Space

- phase space, even for $2 \to 2$ can be complicated
- PDFs with peaking behaviour
- divergent cross sections regulated with cut-offs

$$
\begin{aligned}
qq' \to qq' :& \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\hat{t}} = \frac{\pi\alpha_s^2 4(\hat{s}^2 + \hat{u}^2)}{9\hat{s}^2\hat{t}^2} \\
qg \to qg :& \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\hat{t}} = \frac{\pi\alpha_s^2 \left(\hat{s}^2 + \hat{u}^2\right)\left(9\hat{s}\hat{u} - 4\hat{t}^2\right)}{9\hat{s}^3\hat{t}^2\hat{u}} \\
gg \to q\bar{q} :& \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\hat{t}} = \frac{\pi\alpha_s^2 \left(\hat{t}^2 + \hat{u}^2\right)\left(4\hat{s}^2 - 9\hat{t}\hat{u}\right)}{24\hat{s}^4\hat{t}\hat{u}}
\end{aligned}
$$

- importance sample independently in τ, y, and z
- carefully construct each $g(x)$ for maximum efficiency

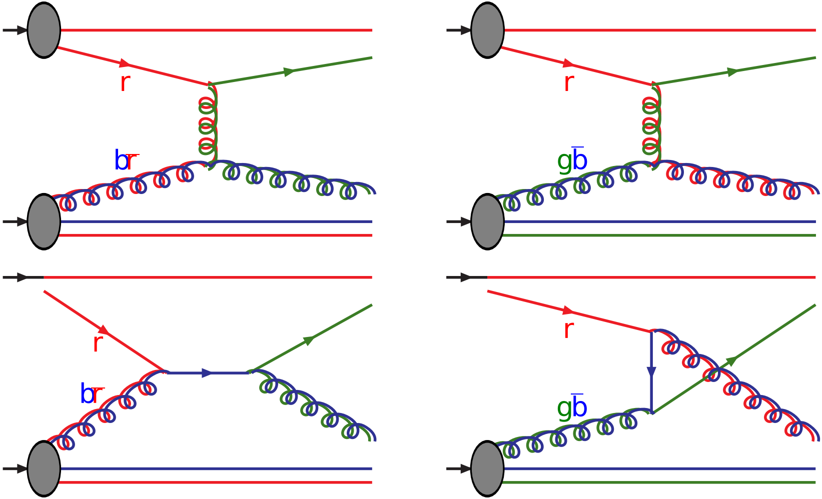
Sampling Phase Space

$$
\begin{aligned}
h_\tau(\tau) =& \frac{c_1}{\mathcal{I}_1}\frac{1}{\tau} + \frac{c_2}{\mathcal{I}_2}\frac{1}{\tau^2} + \frac{c_3}{\mathcal{I}_3}\frac{1}{\tau(\tau+\tau_R)} + \frac{c_4}{\mathcal{I}_4}\frac{1}{(s\tau - m_R^2)^2 + m_R^2\Gamma_R^2} \\
& + \frac{c_5}{\mathcal{I}_5}\frac{1}{\tau(\tau+\tau_{R'})} + \frac{c_6}{\mathcal{I}_6}\frac{1}{(s\tau - m_{R'}^2)^2 + m_{R'}^2\Gamma_{R'}^2} \\
h_y(y) =& \frac{c_1}{\mathcal{I}_1}(y - y_{\text{min}}) + \frac{c_2}{\mathcal{I}_2}(y_{\text{max}} - y) + \frac{c_3}{\mathcal{I}_3}\frac{1}{\cosh y} \\
h_z(z) =& \frac{c_1}{\mathcal{I}_1} + \frac{c_2}{\mathcal{I}_2}\frac{1}{a-z} + \frac{c_3}{\mathcal{I}_3}\frac{1}{a+z} + \frac{c_4}{\mathcal{I}_4}\frac{1}{(a-z)^2} + \frac{c_5}{\mathcal{I}_5}\frac{1}{(a+z)^2}
\end{aligned}
$$

- handle up to two resonances in h_τ, plus inteference
- relatively flat for h_y except third term for peak at 0 from PDFs
- h_z handles divergent cross-section behaviour
- $\mathcal{I}_\rangle$ are normalization terms, c_i are optimized

Color Flows

- processes have multiple diagrams, *e.g.* $qg \to qg$



Color Flows

$$
\begin{aligned}
|\mathcal{M}|^2 &= |\mathcal{M}_1 + \mathcal{M}_2|^2 \\
&= |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + \mathrm{Re}(\mathcal{M}_1\mathcal{M}_2*)
\end{aligned}
$$

- combine based on large color limit, $N_c \to \infty$

$$
\frac{\text{interference}}{\text{total}} \propto \frac{1}{N_c^2 - 1}
$$

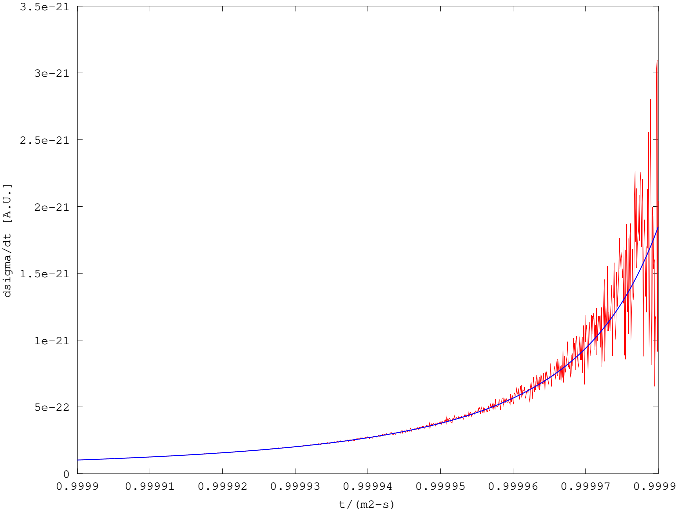
$$
\begin{aligned}
|\mathcal{M}'|^2 &= |\mathcal{M}_1'|^2 + |\mathcal{M}_2'|^2 \\
|\mathcal{M}_i'|^2 &= |\mathcal{M}|^2 \left(\frac{|\mathcal{M}_i|^2}{\sum_j |\mathcal{M}_j|^2} \right)_{N_c \to \infty}
\end{aligned}
$$

Numerical Precision

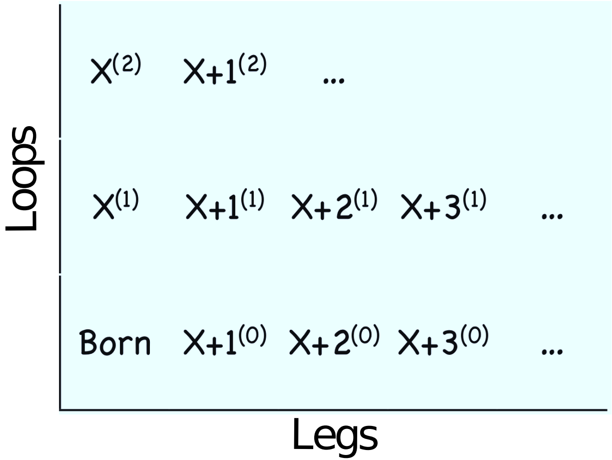
- need to be careful about form of $\mathrm{d}\hat{\sigma}\,/\mathrm{d}\hat{t}$

$$
\begin{aligned}
F(gg \to {}^3D_1 g) = {} & \frac{16\alpha_s^3\pi^2}{81M^3s^2(M^2-s)^5(M^2-t)^5(s+t)^5}\{102M^{20}s^3 + 302M^{20}s^2t + 302M^{20}st^2 \\
& + 102M^{20}t^3 - 286M^{18}s^4 - 1732M^{18}s^3t - 2844M^18s^2t^2 - 1732M^{18}st^3 \\
& - 286M^{18}t^4 + 275M^{16}s^5 + 3840M^{16}s^4t + 10289M^{16}s^3t^2 + 10289M^{16}s^2t^3 \\
& + 3840M^{16}st^4 + 275M^{16}t^5 - 227M^{14}s^6 - 5004M^{14}s^5t - 19569M^{14}s^4t^2 \\
& - 29536M^{14}s^3t^3 - 19569M^{14}s^2t^4 - 5004M^{14}st^5 - 227M^{14}t^6 + 410M^{12}s^7 \\
& + 5137M^{12}s^6t + 23585M^{12}s^5t^2 + 47908M^{12}s^4t^3 + 47908M^{12}s^3t^4 \\
& + 23585M^{12}s^2t^5 + 5137M^{12}st^6 + 410M^{12}t^7 - 470M^{10}s^8 - 4220M^{10}s^7t \\
& - 19534M^{10}s^6t^2 - 47528M^{10}s^5t^3 - 63536M^{10}s^4t^4 - 47528M^{10}s^3t^5 \\
& - 19534M^{10}s^2t^6 - 4220M^{10}st^7 - 470M^{10}t^8 + 245M^8s^9 + 2190M^8s^8t \\
& + 10358M^8s^7t^2 + 28602M^8s^6t^3 + 47093M^8s^5t^4 + 47093M^8s^4t^5 \\
& + 28602M^8s^3t^6 + 10358M^8s^2t^7 + 2190M^8st^8 + 245M^8t^9 - 49M^6s^{10} \\
& - 580M^6s^9t - 2822M^6s^8t^2 - 8984M^6s^7t^3 - 17653M^6s^6t^4 - 21968M^6s^5t^5 \\
& - 17653M^6s^4t^6 - 8984M^6s^3t^7 - 2822M^6s^2t^8 - 580M^6st^9 - 49M^6t^{10} \\
& + 67M^4s^{10}t + 210M^4s^9t^2 + 774M^4s^8t^3 + 2006M^4s^7t^4 + 3147M^4s^6t^5 \\
& + 3147M^4s^5t^6 + 2006M^4s^4t^7 + 774M^4s^3t^8 + 210M^4s^2t^9 + 67M^4st^{10} \\
& + 25M^2s^{10}t^2 + 100M^2s^9t^3 + 220M^2s^8t^4 + 340M^2s^7t^5 + 390M^2s^6t^6 \\
& + 340M^2s^5t^7 + 220M^2s^4t^8 + 100M^2s^3t^9 + 25M^2s^2t^{10} + 5s^{10}t^3 \\
& + 25s^9t^4 + 60s^8t^5 + 90s^7t^6 + 90s^6t^7 + 60s^5t^8 + 25s^4t^9 + 5s^3t^{10}\},
\end{aligned}
$$

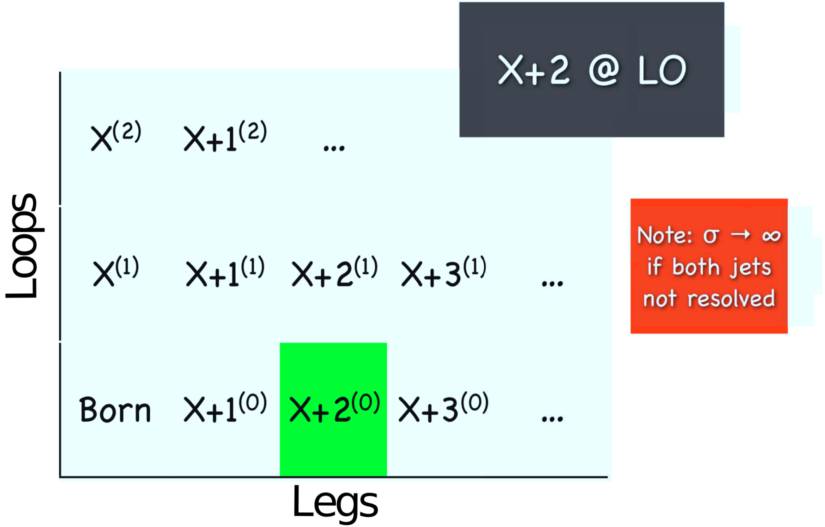
Numerical Precision



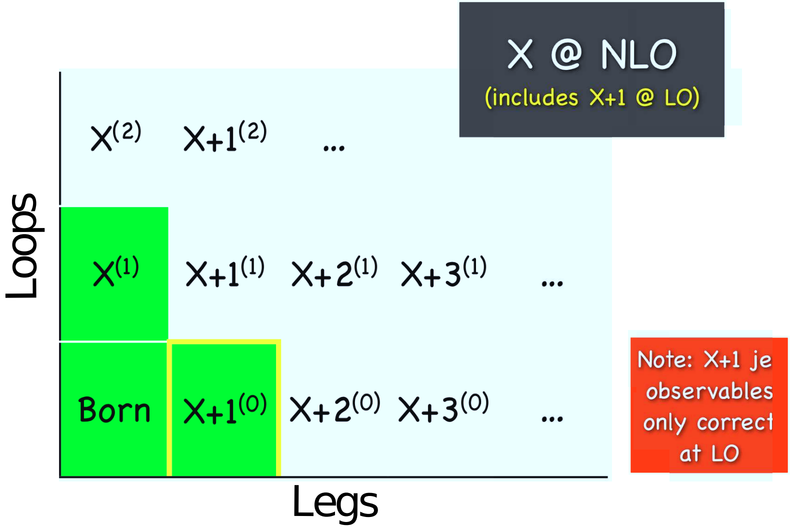
Loops and Legs



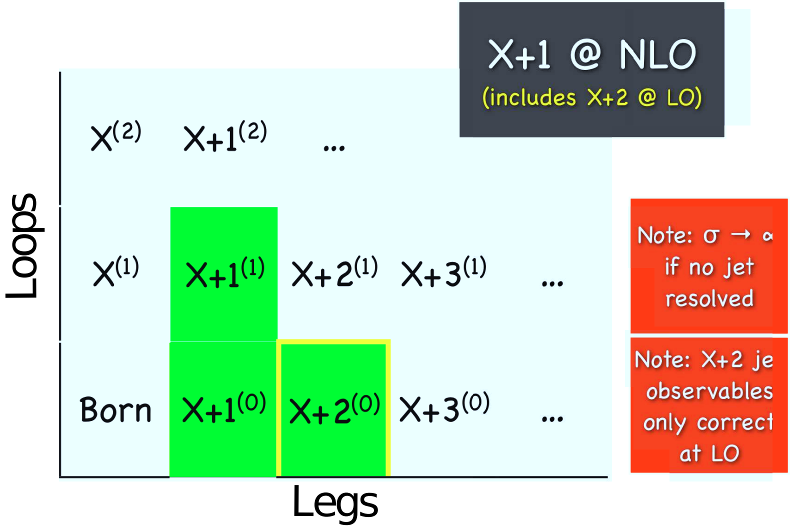
Loops and Legs



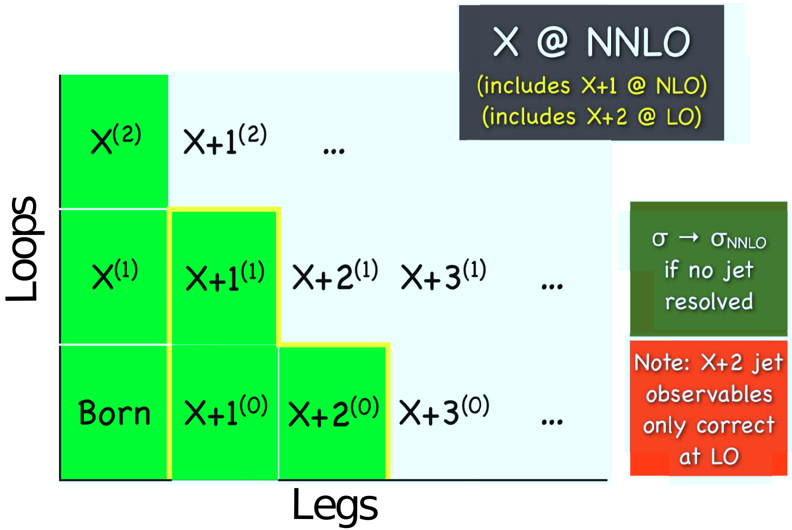
Loops and Legs



Loops and Legs

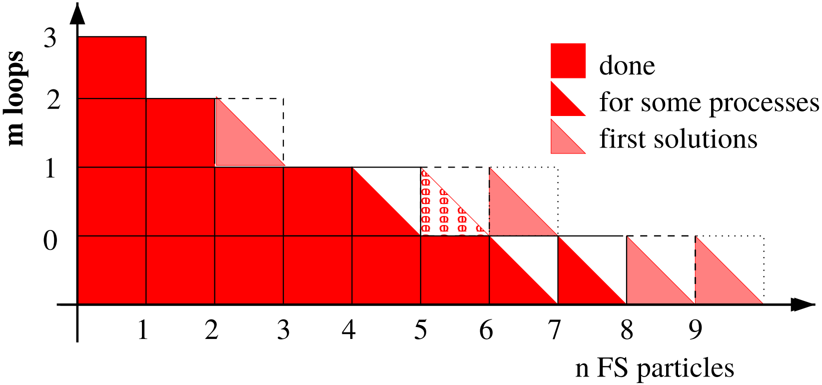


Loops and Legs



Loops and Legs

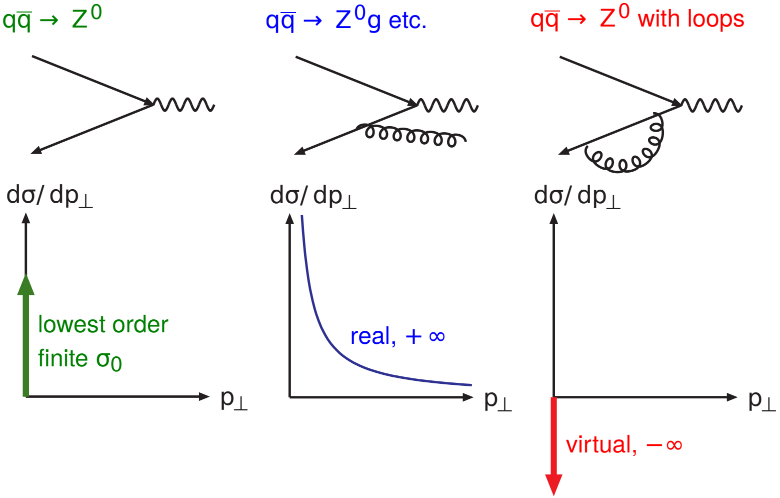
- significant work has gone into filling out this schematic
- state of the art, as of 2019



Loops and Legs

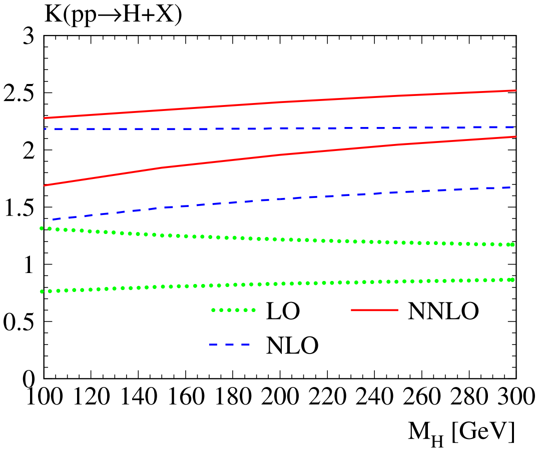
- number of tools provide automated legs with up to one loop
- SHERPA with COMIX and external loop generators
- HERWIG 7 with MATCHBOX
- MADGRAPH 5 with aMC@NLO

NLO Contributions



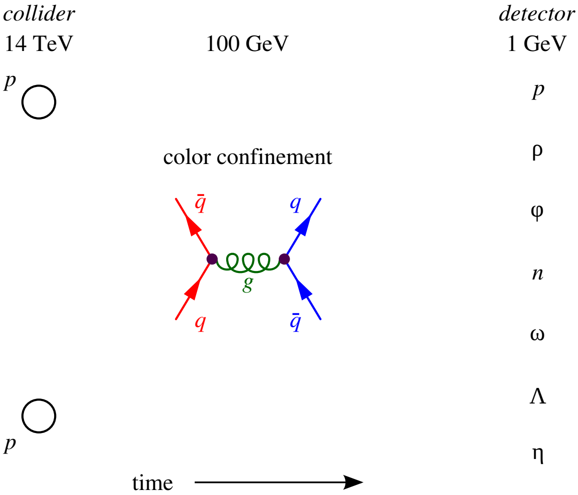
- high p_T tails are critical for new high mass searches and precision measurements

NLO Contributions

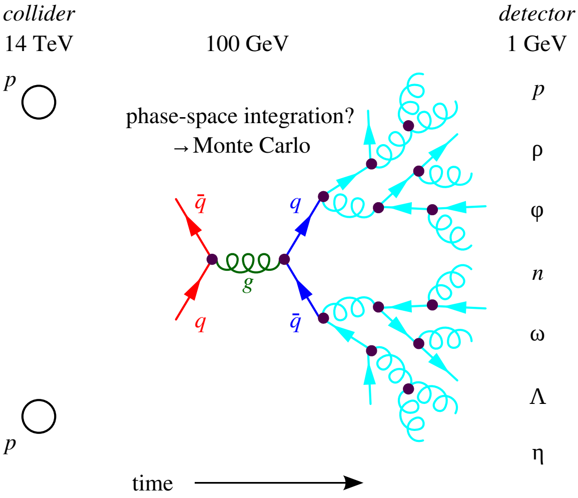


- NLO results typically begin to converge
- work well for inclusive cross sections
- need to be careful for differential cross sections

A Perturbing Problem



A Perturbing Problem

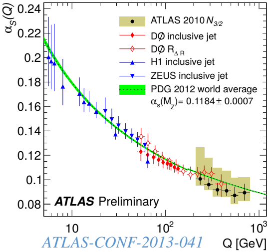


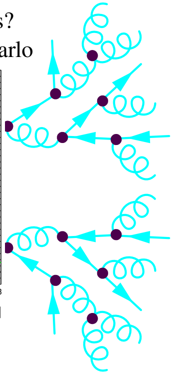
A Perturbing Problem

collider		*detector*
14 TeV	100 GeV	1 GeV

p

converges?
→Monte Carlo



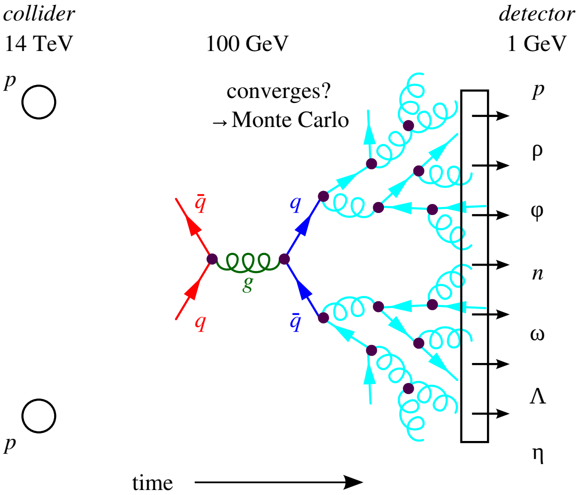


p

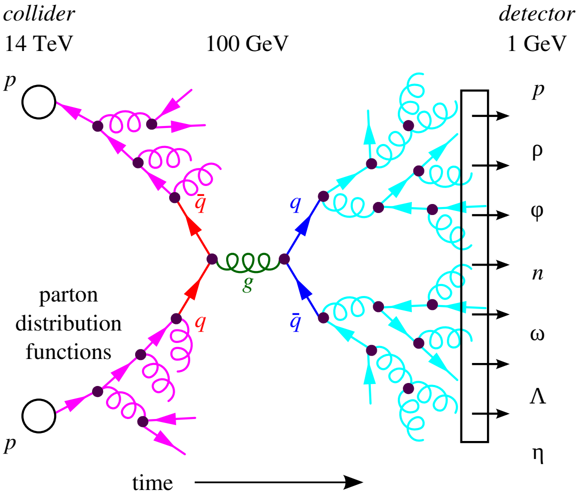
p
ρ
φ
n
ω
Λ
η

time →

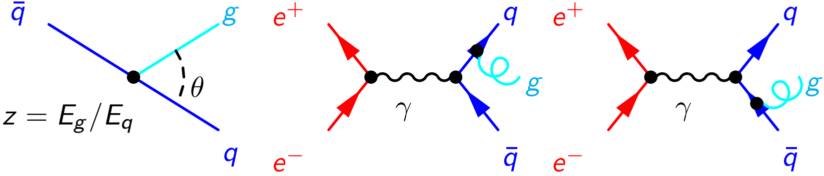
A Perturbing Problem



A Perturbing Problem



QCD Structure



$$\mathrm{d}\sigma \approx \sigma \left(\frac{2\,\mathrm{d}\cos\theta}{\sin^2\theta}\right)\left(\frac{\alpha_s}{2\pi}\right)\left(\frac{N_c^2-1}{2N_c}\right)\left(\frac{1+(1-z)^2}{z}\right)\mathrm{d}z$$

- factorize into general form given any splitting kernel $\mathcal{P}_i$

$$\mathrm{d}\sigma \approx \sigma \sum_i \frac{\mathrm{d}\theta^2}{\theta^2}\mathcal{P}_i\left(z,\alpha_s\right)\mathrm{d}z$$

- diverges when collinear $(\theta \to 0, \pi)$ or infrared $(z \to 0)$

Timelike Parton Shower

$$\Delta(Q^2, q^2) = \exp\left[- \int_{q^2}^{Q^2} \mathrm{d}q'^2 \, \frac{1}{q'^2} \int_{Q_0^2/q'^2}^{1-Q_0^2/q'^2} \mathrm{d}z \, \mathcal{P}_j(z, \alpha_s) \right]$$

1. pick a uniform R
2. solve $\Delta(Q^2, q^2) = R$ for q^2
3. if $q > Q_0$ generate emission and repeat from 1
4. if $q \leq Q_0$ terminate shower

- this is just the veto algorithm!

Veto Algorithm Refresher

$$N(t) = \exp\left(-\int_0^t \mathrm{d}t'\, f(t')\right) = R$$
$$\Rightarrow t = F^{-1}(F(0) - \ln R)$$

1. start with $i = 0$ and $t = 0$
2. increment i
3. $t_i = G^{-1}(G(t_{i-1}) - \ln R)$
4. $y = Rg(t_i)$
5. if $y > f(t_i)$ return to 2 otherwise accept point

- $t = q$
- $\Delta = N(t)$
- $f(t) = \frac{1}{q^2} \int_{Q_0^2/q^2}^{1-Q_0^2/q^2} \mathrm{d}z\, \mathcal{P}_j(z, \alpha_s)$

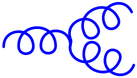
Spacelike Parton Shower

$$\Delta(Q^2, q^2, x) = \exp\Big[-\int_{q^2}^{Q^2} \mathrm{d}q'^2 \, \frac{1}{q'^2} \int_{Q_0^2/q'^2}^{1-Q_0^2/q'^2} \mathrm{d}z \\ \mathcal{P}_j(z, \alpha_s) \frac{x}{zx} \frac{f(x/z, q'^2, k)}{f(x, q'^2, j)} \Big]$$

- initial x is given by the hard scatter
- evolve from low x to high x
- evolve from high q to low q

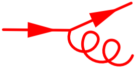
Splitting Kernels

- same splitting kernels as for DGLAP evolution
- only proportionality given here



$g \to gg$

$$\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)$$



$q \to qg$

$$\frac{1-z}{z}+\frac{z}{2}-2\mu$$

$g \to qq$

$$z^2+(1-z)^2+\mu^2$$

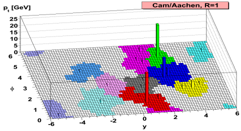
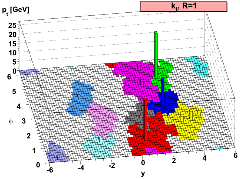
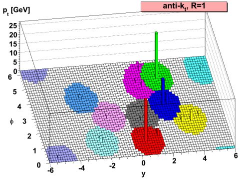
Reverse Engineering with Jets

- try to unfold initial hard partons from final state particles
  1. collinear safe $\rightarrow$ collinear emission changes nothing
  2. infrared safe $\rightarrow$ soft emission changes nothing
  3. insensitive to non-perturbative effects
  4. applicable to both parton and hadron level
- inclusive sequential clustering is algorithm of choice at LHC

$$d_{ij} = \min(p_{\mathrm{T}i}^{2p}, p_{\mathrm{T}j}^{2p}) \frac{\Delta R_{ij}^2}{R^2}, \quad d_{iB} = p_{\mathrm{T}i}^{2p}$$

1. select minimum d
2. if d_{ij}, combine particle i and j
3. if d_{iB}, consider particle as jet and remove from clustering
4. terminate if no particles otherwise return to 1

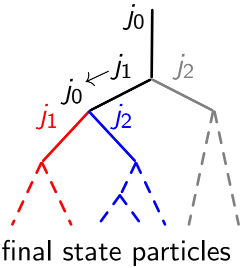
Flavors of Sequential Clustering

Cambridge/Aachen	k_t	anti-k_t
		
$p = 0$	$p = 1$	$p = -1$

- Cambridge/Aachen considers only geometry
- k_t and anti-k_t also consider momentum
- anti-k_t provides circular jets in R at high-p_{T}

SoftDrop and Jet Sub-structure

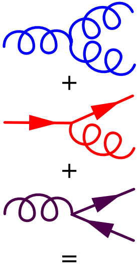
- what happens with boosted topology when $Q_{\rm hard} \gg Q_{\rm obs}$, *e.g.* $W, Z, H \to q\bar{q}$?
- anti-k_t produces a single jet $\to$ need jet sub-structure
- use jet sub-structure technique like SoftDrop



$$\frac{\min(p_{\rm T1}, p_{\rm T2})}{p_{\rm T1} + p_{\rm T2}} > z_{\rm cut} \left(\frac{\Delta R_{12}}{R} \right)^{\beta}$$

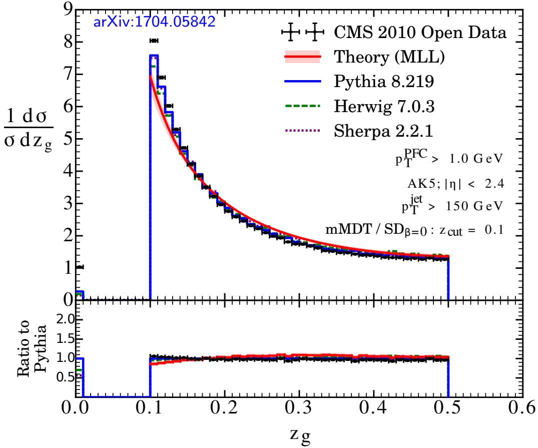
1. create fat anti-k_t jets
2. build Cambridge/Aachen tree for each fat jet
3. split j_0 into sub-jets j_1 and j_2
4. if j_1 and j_2 fulfil SoftDrop condition, terminate
5. otherwise, assign j_0 to larger $p_{\rm T}$ sub-jet and return to 3

Averaged Massless Splittings



$$\frac{1-z}{z}+\frac{z}{1-z}+\frac{1}{2}$$

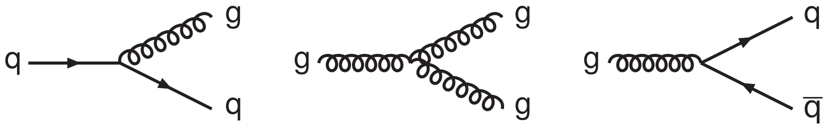
$$z_g \equiv \frac{p_{T1}}{p_{T1}+p_{T2}}$$



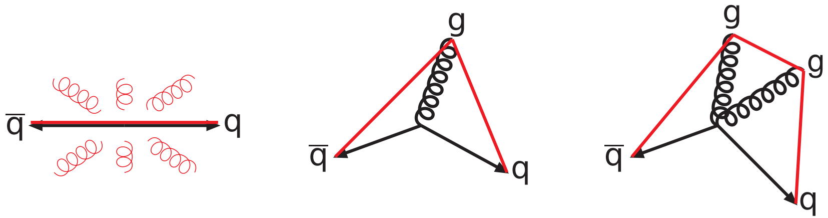
- SoftDrop provides direct access to the hardest $1 \to 2$ splitting

LEP Era Parton Showers

- standard $1 \to 2$ branching (Pythia, Herwig)



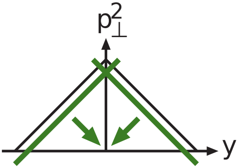
- dipole $2 \to 3$ emissions (Ariadne)



- most new parton showers are dipole motivated (Vincia, Dire)

Ordering in LEP Era Parton Showers

PYTHIA: $Q^2 = m^2$



large mass f rst

$\Rightarrow$ "hardness" ordered

coherence brute force

covers phase space

ME merging simple

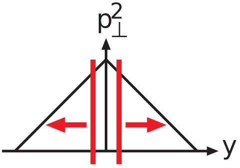
$g \to q\bar{q}$ simple

not Lorentz invariant

no stop/restart

ISR: $m^2 \to -m^2$

HERWIG: $Q^2 \sim E^2\theta^2$



large angle f rst

$\Rightarrow$ **hardness not ordered**

coherence inherent

gaps in coverage

ME merging messy

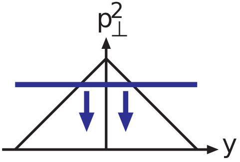
$g \to q\bar{q}$ simple

not Lorentz invariant

no stop/restart

ISR: $\theta \to \theta$

ARIADNE: $Q^2 = p_\perp^2$



large $p_\perp$ f rst

$\Rightarrow$ "hardness" ordered

coherence inherent

covers phase space

ME merging simple

$g \to q\bar{q}$ **messy**

Lorentz invariant

can stop/restart

ISR: more messy

Summary

- factorization theorem is crucial to allow for perturbative calculations
- sampling phase space for matrix elements can be suprisingly tricky
- be careful of numerical precision issues
- NL* terms can include both real and virtual contributions
- significant progress in automated n real with one virtual
- parton showers can fill in the gaps
- quite some choice (with corresponding pitfalls) in how to create a parton shower