MC Tutorial Background: Parton Showers, Matching, Merging

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Introduction: Why Monte Carlo Event Generators?

Theory

QFT: Lagrangian formulation of physics

- Standard Model: $\mathcal{L}_{\text{SM}}$
- Beyond the Standard Model: $\mathcal{L}_{\text{BSM}}$

Experiments

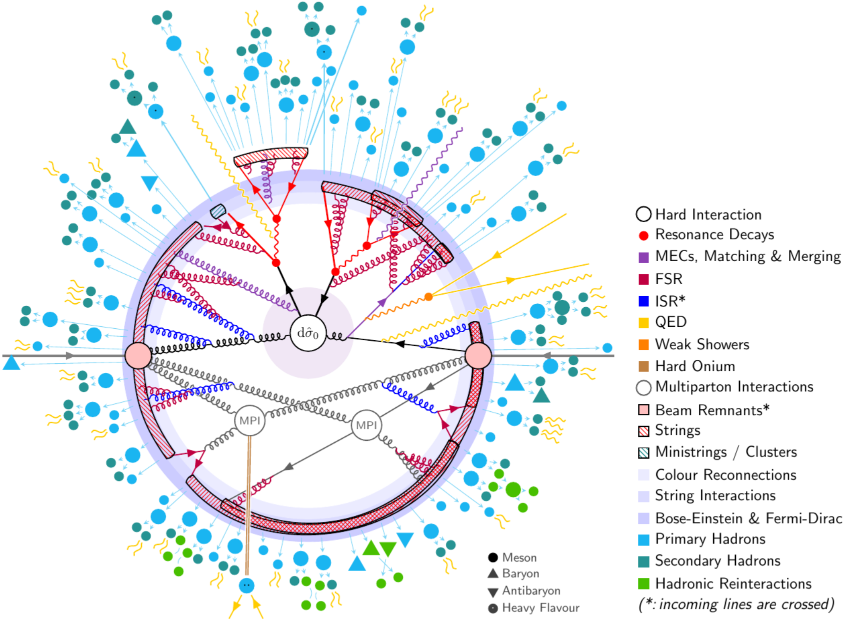
Collider experiments with complex detectors

- LHC with ATLAS, CMS, ...
- Reconstruction of individual events
- Very advanced counting experiments

Simulation

Linking theory & experiment

- MC generators: Stochastic simulation of events
- Allow to compare theory and experiment
- Predict event count by integrating differential cross section over specific phase space regions



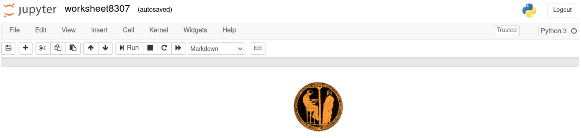
dσ̂0
MPI
MPI
Meson
Baryon
Antibaryon
Heavy Flavour
Hard Interaction
Resonance Decays
MECs, Matching & Merging
FSR
ISR*
QED
Weak Showers
Hard Onium
Multiparton Interactions
Beam Remnants*
Strings
Ministrings / Clusters
Colour Reconnections
String Interactions
Bose-Einstein & Fermi-Dirac
Primary Hadrons
Secondary Hadrons
Hadronic Reinteractions
(*: incoming lines are crossed)

Monte Carlo Tutorials

- Getting familiar with MC event generators Pythia, Herwig, Sherpa.
- Hands-on: Construction of a parton shower.

All resources and instructions at `https://gitlab.com/cteq-tutorials/2022/`

`git clone https://gitlab.com/cteq-tutorials/2022.git`



Pythia 8.3 Python Worksheet

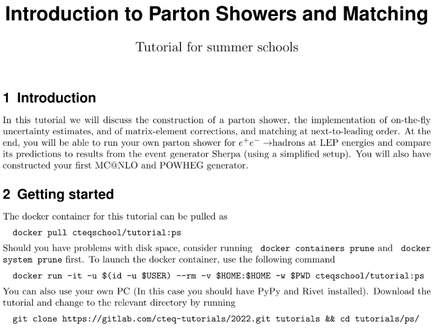
Written by:

- Torbjörn Sjöstrand (Department of Astronomy and Theoretical Physics, Lund University)
- Peter Skands (School of Physics, Monash University)
- Stefan Prestel
- Philip Ilten (School of Physics, University of Cincinnati)
- Leif Gellerson (Department of Astronomy and Theoretical Physics, Lund University)

Introduction

The Pythia 8.3 program is a standard tool for the generation of high-energy collisions (specifically, it focuses on centre-of-mass energies greater than about 10 GeV), comprising a coherent set of physics models for the evolution from a few-body high-energy ("hard") scattering process to a complex multihadronic final state. The particles are produced in vacuum. Simulation of the interaction of the produced particles with detector material is not included in Pythia but can, if needed, be done by interfacing to external detector-simulation codes.

The Pythia 8.3 code package contains a library of hard interactions and models for initial- and final-state parton showers, multiple parton-parton interactions, beam remnants, string fragmentation and particle decays. It also has a set of utilities and interfaces to external programs.



Introduction to Parton Showers and Matching

Tutorial for summer schools

1 Introduction

In this tutorial we will discuss the construction of a parton shower, the implementation of on-the-fly uncertainty estimates, and of matrix-element corrections, and matching at next-to-leading order. At the end, you will be able to run your own parton shower for $e^+e^- \to$hadrons at LEP energies and compare its predictions to results from the event generator Sherpa (using a simplified setup). You will also have constructed your first MC@NLO and POWHEG generator.

2 Getting started

The docker container for this tutorial can be pulled as

```
docker pull cteqschool/tutorial:ps
```

Should you have problems with disk space, consider running `docker containers prune` and `docker system prune` first. To launch the docker container, use the following command

```
docker run -it -u $(id -u $USER) --rm -v $HOME:$HOME -w $PWD cteqschool/tutorial:ps
```

You can also use your own PC (In this case you should have PyPy and Rivet installed). Download the tutorial and change to the relevant directory by running

```
git clone https://gitlab.com/cteq-tutorials/2022.git tutorials && cd tutorials/ps/
```

Shower Monte Carlo Event Generators

Three commonly used general purpose Event Generators

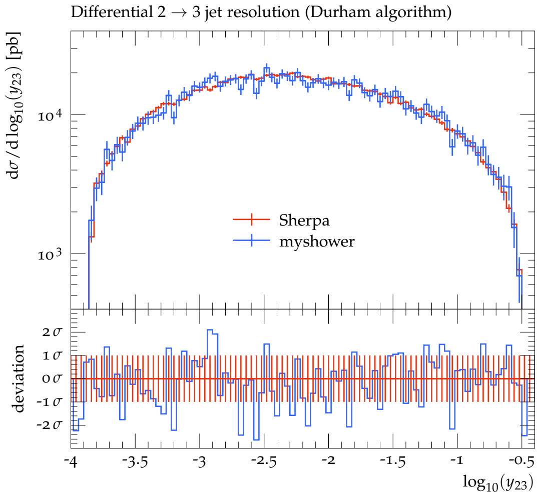
Pythia (begun 1978)
- Originated from hadronization studies: Lund string model
- Pythia 6 virtuality shower, Pyhtia 8 $p_\perp$ shower with ME corrections
- Also DIRE and VINCIA dipole/antenna showers
- Interleaved multi parton interactions

Herwig (begun 1984)
- Originated from coherence studies: Angular ordered shower
- Also $p_\perp$ orderd CS dipole shower
- Cluster hadronization

SHERPA (begun ∼2000)
- Originated from Matrix Element/Parton-Shower matching/merging (CKKW(L))
- CS dipole shower and DIRE parton shower
- Own cluster hadronization

Construct a parton shower

- Code your own parton shower in Python
- Basic ingredients:
  - The splitting functions
  - The kinematics
  - The veto algorithm
- Further features:
  - Scale uncertainties
  - Matrix element corrections
  - POWHEG matching
  - MC@NLO matching
- Follow-up: LL resummation



Goal of this lecture

Provide useful background for MC tutorials

Parton Showers

- Heuristic picture
- Technical ingredients: Phase space, Colors, Implementation
- Effects of the Parton Shower
- Color coherence
- Dipole radiation

Matching & Merging

- Matrix element corrections
- Multi-jet merging
- NLO Matching

Also: time to download the docker images for your favorite tutorial
https://gitlab.com/cteq-tutorials/2022/

Suggested reading

- R. K. Ellis, W. J. Stirling, B. R. Webber
  QCD and Collider Physics
  Cambridge University Press, 2003
- R. D. Field
  Applications of Perturbative QCD
  Addison-Wesley, 1995
- M. E. Peskin, D. V. Schroeder
  An Introduction to Quantum Field Theory
  Westview Press, 1995
- L. Dixon, F. Petriello (Editors)
  Journeys Through the Precision Frontier
  Proceedings of TASI 2014, World Scientific, 2015
- T. Sjöstrand, S. Mrenna, P. Z. Skands
  PYTHIA 6.4 Physics and Manual
  JHEP 05 (2006) 026

Additional references provided on the slides.

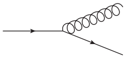
Lectures based on previous work by Stefan Höche and others. Thanks!

What is a parton shower?

The heuristic view

Radiative corrections as a branching process

- Make two well motivated assumptions
  - Parton branching can occur in two ways



observed

unobserved

  - Evolution conserves probability
- The consequence is Poisson statistics
  - Let the decay probability be λ
  - Assume indistinguishable particles $\rightarrow$ naive probability for n emissions

$$P_{\mathrm{naive}}(n, \lambda) = \frac{\lambda^n}{n!}$$

  - Probability conservation (i.e. unitarity) implies a no-emission probability

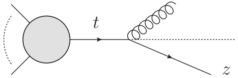
$$P(n, \lambda) = \frac{\lambda^n}{n!} \exp\{-\lambda\} \qquad \longrightarrow \qquad \sum_{n=0}^{\infty} P(n, \lambda) = 1$$

- In the context of parton showers $\Delta = \exp\{-\lambda\}$ is called a Sudakov factor

Radiative corrections as a branching process

- Decay probability for parton state in collinear limit

$$\lambda \to \frac{1}{\sigma_n} \int_t^{Q^2} \mathrm{d}\bar{t} \, \frac{\mathrm{d}\sigma_{n+1}}{\mathrm{d}\bar{t}} \approx \sum_{\text{jets}} \int_t^{Q^2} \frac{\mathrm{d}\bar{t}}{\bar{t}} \int \mathrm{d}z \frac{\alpha_s}{2\pi} P(z)$$



Parameter t identified with evolution "time"

- Splitting function $P(z)$ spin & color dependent

$$P_{qq}(z) = C_F \left[\frac{2z}{1-z} + (1-z) \right] \qquad P_{gq}(z) = T_R \left[z^2 + (1-z)^2 \right]$$

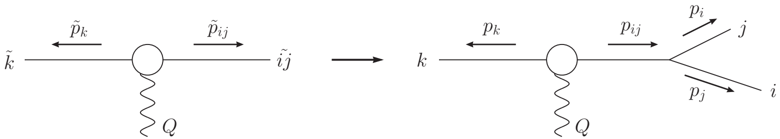
$$P_{gg}(z) = C_A \left[\frac{2z}{1-z} + z(1-z) \right] + (z \leftrightarrow 1-z)$$

- When adding partons
  - On-shell conditions must be maintained
  - Overall four-momentum must be conserved
  - Color must be conserved

How to deal with the phase space

Example momentum mapping

Final state momentum mapping



- Generate off-shell momentum by rescaling

$$p_{ij}^\mu = \tilde{p}_{ij}^\mu + \frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\,\tilde{p}_k^\mu\,, \qquad p_k^\mu = \left(1 - \frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\right)\tilde{p}_k^\mu$$

- Then branch into two on-shell momenta

$$p_i^\mu = \tilde{z}\,\tilde{p}_{ij}^\mu + (1-\tilde{z})\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^\mu + k_\perp^\mu, \qquad p_j^\mu = (1-\tilde{z})\,\tilde{p}_{ij}^\mu + \tilde{z}\,\frac{p_{ij}^2}{2\tilde{p}_{ij}\tilde{p}_k}\tilde{p}_k^\mu - k_\perp^\mu$$

- On-shell conditions require that

$$\vec{k}_T^2 = p_{ij}^2\,\tilde{z}(1-\tilde{z}) \qquad \leftrightarrow \qquad \tilde{z}_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\vec{k}_T^2/p_{ij}^2}\right)$$

$\rightarrow$ for any finite $\vec{k}_T$ we have $0 < \tilde{z} < 1$

How to color a shower

The improved large-N_c approximation

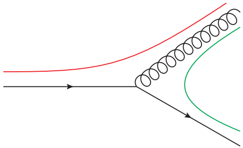
Color flow

- Write gluon propagator using completeness relations

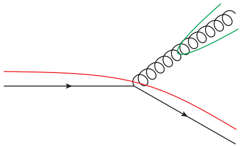
$$\underbrace{\delta^{ab}}_{\text{standard}} = 2\,\mathrm{Tr}(T^a T^b) = 2\,T^a_{ij} T^b_{ji} = T^a_{ij}\ \underbrace{2\,\delta_{ik}\delta_{jl}}_{\text{color flow}}\ T^b_{lk}$$

- Quark-gluon vertex

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right) =$$

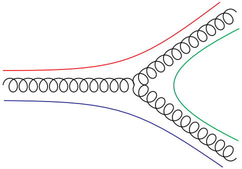


$$-\frac{1}{N_c}$$



- Gluon-gluon vertex

$$f^{abc}\,T^a_{ij} T^b_{kl} T^c_{mn} = \delta_{il}\delta_{kn}\delta_{mj} - \delta_{in}\delta_{ml}\delta_{kj} =$$



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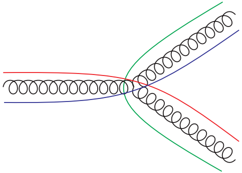
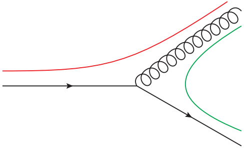


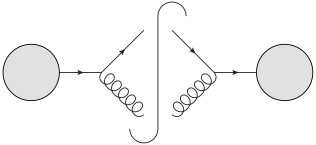
Color flow

- Typically, parton showers also make the leading-color approximation

$$T^a_{ij} T^a_{kl} \to \frac{1}{2}\delta_{il}\delta_{jk} \qquad \leftrightarrow$$



- If used naively, this would overestimate the color charge of the quark:
  Consider process $q \to qg$ attached to some larger diagram



$$\propto \qquad T^a_{ij} T^a_{jk} = C_F \delta_{ik}$$

  but now we have $\dfrac{1}{2}\,\delta_{il}\delta_{jm}\delta_{mj}\delta_{lk} = \dfrac{C_A}{2}\,\delta_{ik}$

- Color assignments in parton shower made at leading color
  but color charge of quarks actually kept at C_F

How to implement the algorithm

Monte-Carlo methods for parton showers

Monte-Carlo methods: Poisson distributions

- Assume decay process described by $g(t)$
- Decay can happen only if it has not happened already

  Must account for survival probability $\leftrightarrow$ Poisson distribution

$$\mathcal{G}(t) = g(t)\Delta(t,t_0) \qquad \text{where} \qquad \Delta(t,t_0) = \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, g(t')\right\}$$

- If $G(t)$ is known, then we also know the integral of $\mathcal{G}(t)$

$$\int_t^{t_0} \mathrm{d}t' \mathcal{G}(t') = \int_t^{t_0} \mathrm{d}t' \, \frac{\mathrm{d}\Delta(t',t_0)}{\mathrm{d}t'} = 1 - \Delta(t,t_0)$$

- Can generate events by requiring $1 - \Delta(t,t_0) = 1 - R$

$$t = G^{-1}\Big[G(t_0) + \log R\Big]$$

You will use this formula in the tutorial

Monte-Carlo methods: Poisson distributions

- Importance sampling for Poisson distributions
  - Generate event according to $\mathcal{G}(t)$
  - Accept with $w(t) = f(t)/g(t)$
  - If rejected, continue starting from t
- Probability for immediate acceptance

$$\frac{f(t)}{g(t)}\, g(t) \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, g(t')\right\}$$

- Probability for acceptance after one rejection

$$\frac{f(t)}{g(t)}\, g(t) \int_t^{t_0} \mathrm{d}t_1 \exp\left\{-\int_t^{t_1} \mathrm{d}t'\, g(t')\right\} \left(1 - \frac{f(t_1)}{g(t_1)}\right) g(t_1) \exp\left\{-\int_{t_1}^{t_0} \mathrm{d}t'\, g(t')\right\}$$

- For n intermediate rejections we obtain n nested integrals $\int_t^{t_0} \int_{t_1}^{t_0} \ldots \int_{t_{n-1}}^{t_0}$
- Disentangling yields $1/n!$ and summing over all possible rejections gives

$$f(t) \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, g(t')\right\} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\int_t^{t_0} \mathrm{d}t' \left[g(t') - f(t')\right]\right]^n = f(t) \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, f(t')\right\}$$

Monte-Carlo method for parton showers

- Start with set of n partons at scale t', which evolve collectively

  Sudakovs factorize, schematically

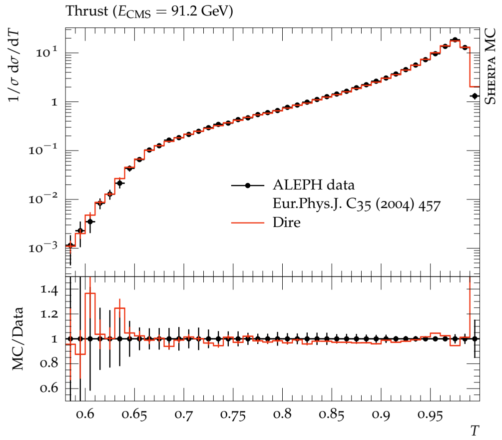
$$\Delta(t,t') = \prod_{i=1}^{n} \Delta_i(t,t')\,, \qquad \Delta_i(t,t') = \prod_{j=q,g} \Delta_{i\to j}(t,t')$$

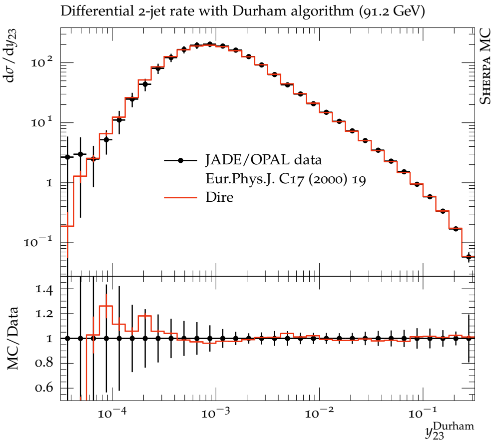
- Find new scale t where next branching occurs using veto algorithm
  - Generate t using overestimate $\alpha_s^{\max} P_{ab}^{\max}(z)$
  - Determine "winner" parton i and select new flavor j
  - Select splitting variable according to overestimate
  - Accept point with weight $\alpha_s(k_T^2) P_{ab}(z) / \alpha_s^{\max} P_{ab}^{\max}(z)$
- Construct splitting kinematics and update event record
- Continue until t falls below an IR cutoff

You will use this algorithm in the tutorial

Effects of the parton shower

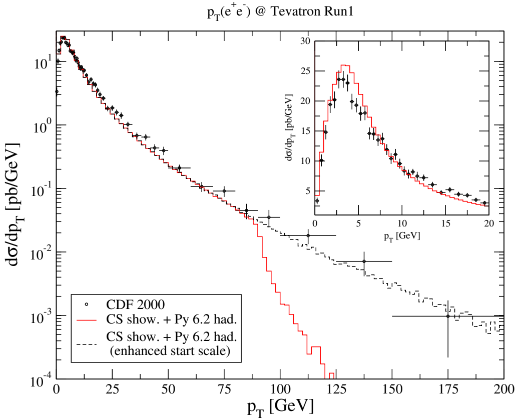
Effects of the parton shower





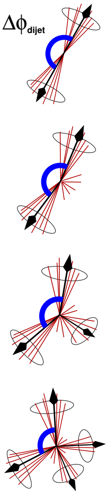
- Thrust and Durham 2 $\to$ 3-jet rate in $e^+e^- \to$hadrons
- Hadronization region to the right (left) in left (right) plot

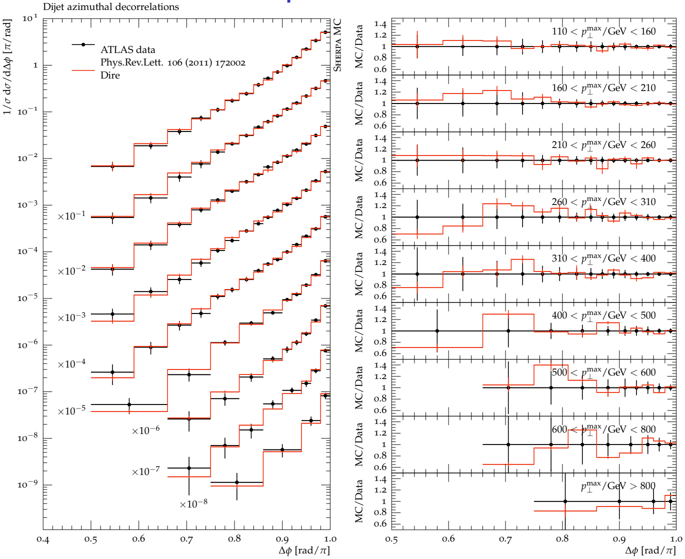
Effects of the parton shower



- Drell-Yan lepton pair production at Tevatron
- If hard cross section computed at leading order, then parton shower is only source of transverse momentum

Effects of the parton shower

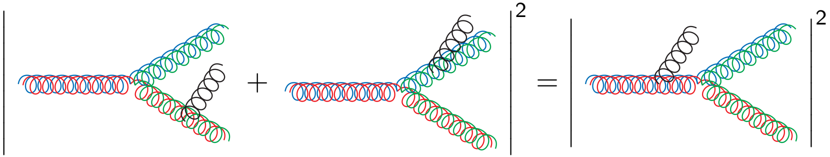




Color coherence

Soft Gluons: Physical Picture of Color Coherence

- Soft gluons (large wavelength) not able to resolve charges of emitting color dipole individually



- Emission with combined color charge of mother parton
  $\Rightarrow$ destructive interference outside cone with opening angle defined by dipole
- Can be solved by
  - Angular ordering (Herwig) [Marchesini, Webber (1988)]
  - Additional ordering constraint (approximately)
  - Dipole showers with transverse momentum ordering [Gustafsson,Pettersson (1988)]

Evidence for Color Coherence in 3-jet Events

Pseudorapidity of third jet [CDF (1994)]

- Very old Pythia: purely virtual ordered: too much radiation in central region
- Very old Pythia+: additional phase space constraint on initial-final dipole (angular veto) ok
- Herwig angular ordered ok

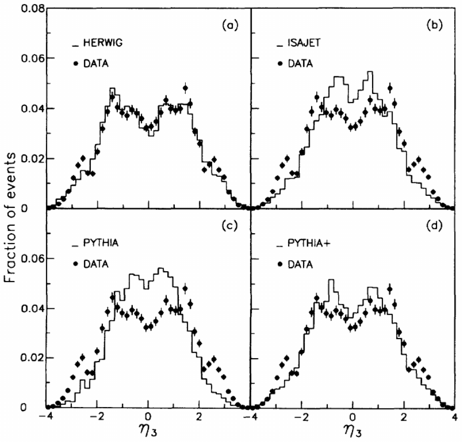


FIG. 13. Observed η_3 distribution compared to the predictions of (a) HERWIG; (b) ISAJET; (c) PYTHIA; (d) PYTHIA+.

Dipole radiation pattern

Geometric properties of semi-classical result

Structure of semi-classical matrix element

- Universal, semi-classical integrand (Eikonal) $\dfrac{2p_ap_b}{(p_ap_c)(p_bp_c)}$
  - Leads to double logarithm $1/2\log^2(2p_ap_b/\mu^2)$
  - Originates in gauge boson radiation off conserved charge
- Matrix element can be written in terms of energies and angles [Marchesini,Webber] NPB310(1988)461

$$\frac{2p_ap_b}{(p_ap_c)(p_cp_b)} = \frac{W_{ab,c}}{E_c^2} \qquad \text{with} \qquad W_{ab,c} = \frac{1-\cos\theta_{ab}}{(1-\cos\theta_{ac})(1-\cos\theta_{bc})}$$

- Divergent as $\theta_{ac} \to 0$ and as $\theta_{bc} \to 0$

  $\to$ Expose individual singularities using $W_{ab,c} = \tilde{W}^a_{ab,c} + \tilde{W}^b_{ba,c}$

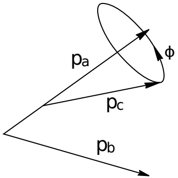
$$\tilde{W}^a_{ab,c} = \frac{1}{2}\left[\frac{1-\cos\theta_{ab}}{(1-\cos\theta_{ac})(1-\cos\theta_{bc})} + \frac{1}{1-\cos\theta_{ac}} - \frac{1}{1-\cos\theta_{bc}}\right]$$

  - Divergent as $\theta_{ac} \to 0$, but regular as $\theta_{bc} \to 0$
  - Convenient properties upon integration over azimuthal angle

Structure of semi-classical matrix element

- Work in a frame where direction of $\vec{p}_a$ aligned with z-axis

$$\cos\theta_{bc} = \cos\theta_b \cos\theta_c + \sin\theta_b \sin\theta_c \cos\phi_c$$



- Integration over ϕ_c yields

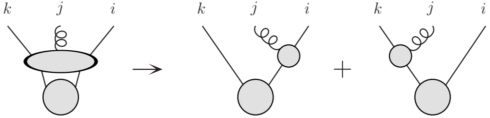
$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi_c \tilde{W}^a_{ab,c} = \frac{1}{1-\cos\theta_c} \times \begin{cases} 1 & \text{if} \quad \theta_c < \theta_b \\ 0 & \text{else} \end{cases}$$

  - On average, no radiation outside cone defined by parent dipole
  - Differential radiation pattern more intricate:
    Positive & negative contributions outside cone sum to zero

Structure of semi-classical matrix element

- Alternative approach: partial fraction matrix element & match to collinear sectors
  [Ellis,Ross,Terrano] NPB178(1981)421, [Catani,Seymour] hep-ph/9605323

$$\frac{p_i p_k}{(p_i p_j)(p_j p_k)} \to \frac{1}{p_i p_j}\frac{p_i p_k}{(p_i + p_k)p_j} + \frac{1}{p_k p_j}\frac{p_i p_k}{(p_i + p_k)p_j}$$

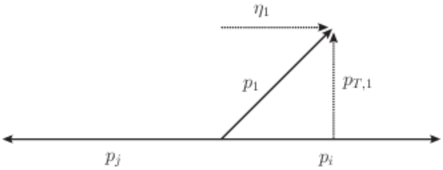


  - Convenient, Lorentz invariant formulation
  - Easy to integrate and use in NLO IR subtraction
  - Captures matrix element both in angular ordered and unordered region

The Lund Plane

The Lund plane

- Compute everything in center-of-mass frame of quarks



- Write momenta in Sudakov decomposition $\quad p_1 = p_1^+ + p_1^- + p_{T,1}$
  - On-shell condition: $p_1^2 = 2(p_1^+ p_1^- - p_{T,1}^2)$
  - "$-$"-projection: $p_1^- = 2p_i p_1 / \sqrt{2p_i p_j}$
  - "$+$"-projection: $p_1^+ = 2p_j p_1 / \sqrt{2p_i p_j}$
- Simple expressions for transverse momentum and rapidity
  - $p_{T,1}^2 = \dfrac{2(p_i p_1)(p_j p_1)}{p_i p_j}$
  - $\eta_1 = \dfrac{1}{2} \ln \dfrac{p_i p_1}{p_j p_1}$
- Semi-classical abelian matrix element squared $\propto 1/p_T^2$

The Lund plane

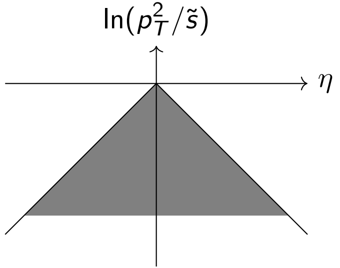
- Rewrite rapidity using transverse momentum

$$\eta_1 = \frac{1}{2}\ln\frac{p_i p_1}{p_j p_1} = \frac{1}{2}\ln\frac{s_{i1}^2}{p_{T,1}^2 s_{ij}} = \frac{1}{2}\ln\frac{p_{T,1}^2 s_{ij}}{s_{j1}^2}$$

- In momentum conserving parton branching $(\tilde{p}_i, \tilde{p}_j) \to (p_i, p_j, p_1)$

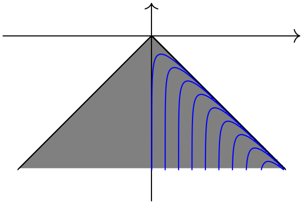
$$-\frac{1}{2}\ln\frac{\tilde{s}_{ij}}{p_{T,1}^2} \leq \eta_1 \leq \frac{1}{2}\ln\frac{\tilde{s}_{ij}}{p_{T,1}^2}$$

- Differential phase-space element $\propto \mathrm{d}p_T^2\,\mathrm{d}\eta$
- The Lund plane
  - $\eta, \ln(p_T^2/\tilde{s})$ plane
  - Phase space bounded by diagonals
  - Single-emission semi-classical radiation probability a constant



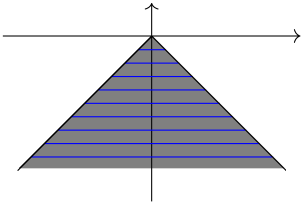
Ordering Variables in the Lund plane

Angular Ordering:
Lund plane filled from center to edges



- Dipole ends evolve separately: Parton shower
- Not ordered in $p_\perp^2$
- Color factors correct if observable insensitive to azimuthal correlations

Dipole Showers:
Lund plane filled from top to bottom



- Unified dipole and parton evolution
- Not ordered in η
- Color factors in improved leading color approximation

Matching & Merging

Improving parton showers with fixed order matrix elements

Recap: Parton Showers

Start from hard $2 \to 2$ scattering, dress with extra partons to get exclusive $2 \to n$ cross section

$$\mathrm{d}\sigma_n^{\mathrm{ex}} = F_0^+ F_0^- |M_0|^2 \mathrm{d}\phi_0 \times \left[\prod_{i=1}^{n} \frac{\alpha_\mathrm{s}(\rho_i)}{2\pi} \frac{F_i}{F_{i-1}} P_i \frac{\mathrm{d}\rho_i}{\rho_i} \mathrm{d}z \Pi_{i-1}(\rho_{i-1}, \rho_i) \right] \Pi_n(\rho_n, \rho_{\mathrm{min}})$$

- $|M_0|^2 \mathrm{d}\phi_0$: Born-level ME and phase space
- $F_i = x_i f_i(x_i, \rho_i)$: PDF's from both sides of i-parton state, $\pm$ for $\pm p_z$ beams
- $P_i \mathrm{d}z \mathrm{d}\rho_i / \rho_i$: Differential emission rate, correct for soft/collinear splittings
- ρ, z: Splitting variables, ρ jet resolution scale, z energy/momentum fraction
- $\Pi(\rho_{i-1}, \rho_i)$: No-emission probabilities
- ρ_{min}: Minimal resolution scale / shower cut-off scale

Recap: No-emission Probabilities

$$\Pi_i(\rho_i, \rho_{i+1}) = \exp\left(- \int_{\rho_{i+1}}^{\rho_i} \frac{\mathrm{d}\rho}{\rho} \frac{\alpha_\mathrm{s}(\rho)}{2\pi} \int_{z_\mathrm{min}}^{z_\mathrm{max}} \mathrm{d}z \frac{F_{i+1}}{F_i} P_i(z) \right)$$

- Probability of not having any emissions harder than ρ_{i+1} when starting shower from ρ_i
- Introduces all order corrections in α_s
- F_{i+1}/F_i only included for ISR
- Exclusive description of final state needs no-emission probabilities

Unitarity of Parton Shower: Fixed Order Expansion

Expand to $\mathcal{O}(\alpha_s^2)$

Use $\frac{1}{2\pi\rho}\frac{F_{i+1}}{F_i}P_i(z) = \bar{P}_i$ for ISR, $\frac{1}{2\pi\rho}P_i(z) = \bar{P}_i$ for FSR to simplify notation

$$\frac{\mathrm{d}\sigma_0^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \left[1 - \alpha_\mathrm{s} \int_{\rho_\mathrm{min}}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1 + \frac{\alpha_s^2}{2}\left(\int_{\rho_\mathrm{min}}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1\right)^2\right]$$

$$\frac{\mathrm{d}\sigma_1^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \alpha_\mathrm{s} \mathrm{d}\rho_1 \mathrm{d}z_1 \bar{P}_1 \left[1 - \alpha_\mathrm{s} \int_{\rho_1}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1 - \alpha_\mathrm{s} \int_{\rho_\mathrm{min}}^{\rho_1} \mathrm{d}\rho \mathrm{d}z \bar{P}_2\right]$$

$$\frac{\mathrm{d}\sigma_2}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \alpha_\mathrm{s}^2 \mathrm{d}\rho_1 \mathrm{d}z_1 \bar{P}_1 \mathrm{d}\rho_2 \mathrm{d}z_2 \bar{P}_2 \Theta(\rho_1 - \rho_2)$$

$\Rightarrow$ Unitarity in every order of α_s, total cross-section

$$\frac{\mathrm{d}\sigma_0^{\mathrm{inc}}}{\mathrm{d}\phi_0} = \frac{\mathrm{d}\sigma_0^{\mathrm{ex}}}{\mathrm{d}\phi_0} + \int \frac{\mathrm{d}\sigma_1^{\mathrm{ex}}}{\mathrm{d}\phi_0} + \int\int \frac{\mathrm{d}\sigma_2}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2$$

But 1-jet cross section not correct for hard/wide-angle emissions

Matrix Elements vs. Parton Showers

Matrix Elements

Fixed order good for hard jets

- + Contains all terms in given order of α_s
- + Valid also for high relative $p_\perp^2$
- - Only feasible for a few emissions

Parton Showers

Approx. excl. multi-parton cross section

- + Always finite
- + Can produce any number of emissions
- - Is only valid in soft/collinear regions

Combine strengths of Matrix Elements and Parton Showers

Experiments measure both high and low $p_\perp^2$ phenomena

- Describe hard emissions by fixed order predictions
- Add further emissions and include no-emission probabilities from PS

Matching & Merging Overview

Combine Matrix Element calculations and Parton Showers. Improve in different ways:

Matrix Element Corrections Oldest scheme, correct first emission of parton shower according to full process-dependent real emission calculation

Multi-jet Merging Improve radiation pattern of parton shower by adding higher-multiplicity matrix elements

NLO Matching Improve the perturbative precision by one higher order (NLO in α_s) cross section matched to parton showers

NLO Multi-jet Matching/Merging Combine multiple higher-multiplicity and higher-order cross sections in parton shower

Matrix Element Corrections

Matrix Element Corrections / Tree-level Matching

Want improved parton shower with full matrix elements for hard emissions
$\Rightarrow$ First step: Use full real-emission matrix element for hardest emission, process-dependent!

$$\alpha_{\mathrm{s}} \bar{P}_i \to \alpha_s \bar{P}_i^{\mathrm{ME}} \equiv \frac{|M_i|^2 \mathrm{d}\phi_i}{|M_{i-1}|^2 \mathrm{d}\phi_{i-1} \mathrm{d}\rho \mathrm{d}z}$$

- Old, but very good! [Bengtsson, Sjöstrand (1987)]
- + Natural and efficient within PS: Use modified acceptance probability
- - Difficult to generalize beyond one emission
- Vincia & Dire parton showers exponentiate n-parton matrix elements

  [Giele, Kosower, Skands (2008)] [Fischer, Prestel (2017)]

Leading Order Multi-Jet Merging

Multi-jet Merging: The Naive (and Wrong) Way

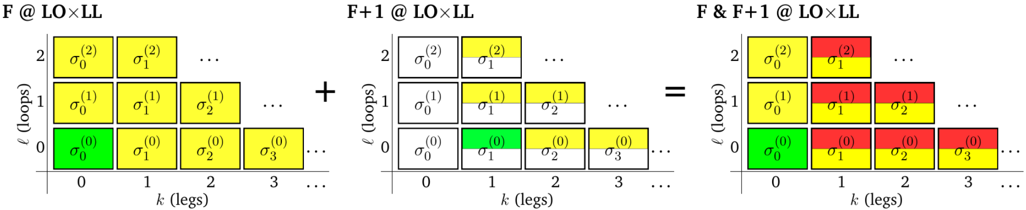
Want to improve PS emissions for more than hardest emission. Naive approach:

- Generate $[X]_{\mathrm{ME}}$ + parton shower
- Generate $[X + 1\text{jet}]_{\mathrm{ME}}$ + parton shower
- Generate $[X + 2\text{jet}]_{\mathrm{ME}}$ + parton shower
- ...

And combine everything into one sample. Does not work, double counting!

- $[X]_{\mathrm{ME}}$ + parton shower is inclusive
- $[X + 1\text{jet}]_{\mathrm{ME}}$ + parton shower is inclusive
- ...

See also Skands: Introduction to QCD



Multi-jet Merging: Exclusive Description without Double-counting

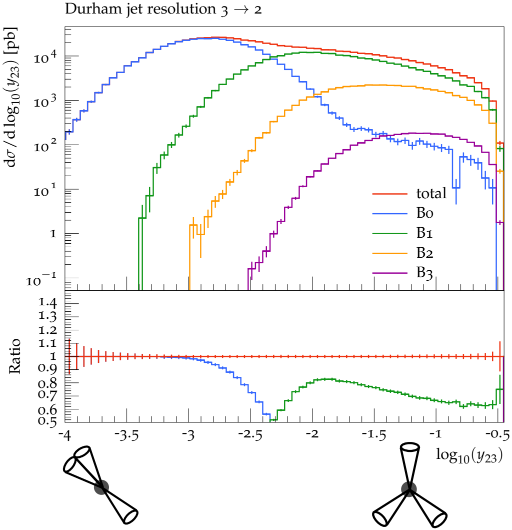
Solve double-counting issue by dividing phase space in “hard and soft region”:

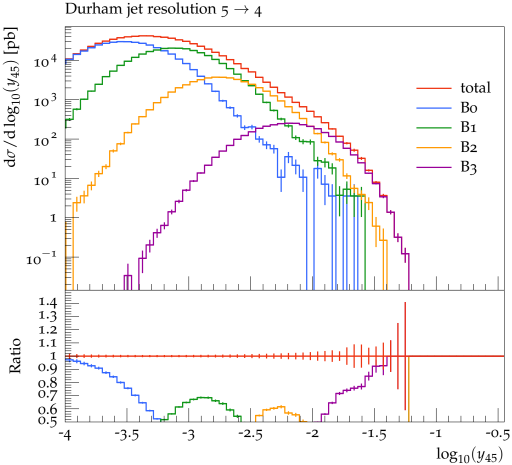
- Generating inclusive few jet samples according to exact tree-level $F_n^+ F_n^- |M_n|^2 \equiv B_n$ in “hard region”
- Using some merging scale ρ_{ms} to cut off divergences
- Making exclusive by reweighting with no-emission probabilities (and α_{s} and PDF ratios), i.e. how would PS have produced this configuration
- Using normal shower in “soft region” below ρ_{ms}

Remaining issues:

- Merging scale dependence
- Merging scale might not be defined in terms of shower evolution variable
- Might break unitarity of shower

Multi-jet Merging: $e^+e^- \to q\bar{q}$ + jets example

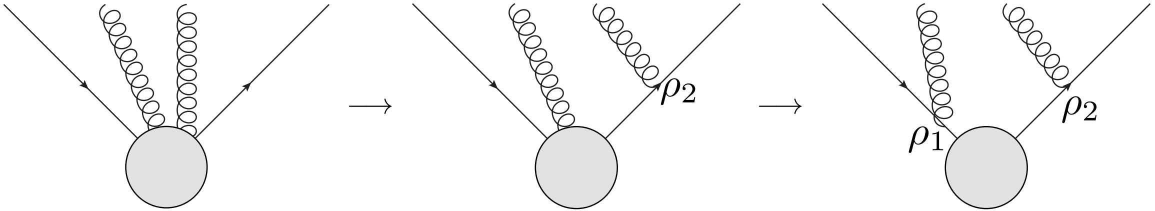




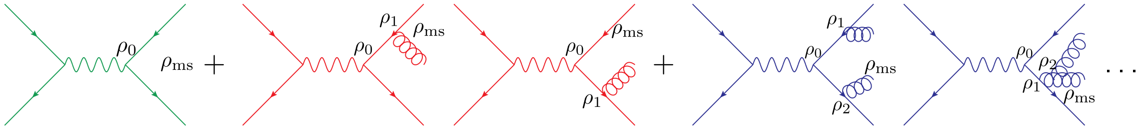
How to Reweight: Parton Shower Histories

Want to apply no-emission probabilities and scale dependent ratios $\Rightarrow$ need ρ_i. Two ways:

- Find unique history by applying sequential $2 \to 1$ jet algorithm
- Find all possible parton shower histories by $3 \to 2$ clustering, choose one according to product of splitting probabilities
  - Choose one history according to product of splitting probabilities
  - Combine partons according to parton shower kinematics



Multi-jet Merging: Illustration in FSR



Combine MEs with different multiplicities, avoid overlap by reweighting

$$\langle \mathcal{O} \rangle = \int d\phi_0 \left\{ \mathcal{O}_0 B_0 w_0 + \int d\phi_1 \mathcal{O}_1 B_1 w_1 + \int d\phi_1 \int d\phi_2 \mathcal{O}_2 B_2 w_2 \right\}$$

with the weights

$$w_0 = \Pi_0(\rho_0, \rho_{\mathrm{ms}}), \; w_1 = \Pi_0(\rho_0, \rho_1) \frac{\alpha_s(\rho_1)}{\alpha_s(\mu_R)} \Pi_1(\rho_1, \rho_{\mathrm{ms}}),$$

$$w_2 = \Pi_0(\rho_0, \rho_1) \frac{\alpha_s(\rho_1)}{\alpha_s(\mu_R)} \Pi_1(\rho_1, \rho_2) \frac{\alpha_s(\rho_2)}{\alpha_s(\mu_R)}$$

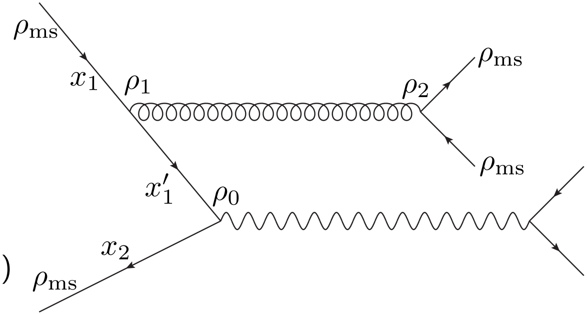
Multi-jet Merging: Illustration in ISR

Inclusive Matrix Element:

$$\frac{\mathrm{d}\sigma_2^{\mathrm{in}}}{\mathrm{d}\phi_{0+2}} = F_1(x_1, \rho_0) F_2(x_2, \rho_0) |M_2|^2$$

Exclusive Parton Shower:

$$\frac{\mathrm{d}\sigma_2^{\mathrm{ex}}}{\mathrm{d}\phi_0 \mathrm{d}\phi_{1,2}} = F_1'(x_1', \rho_0) F_2(x_2, \rho_0) |M_0|^2 \Pi_0(\rho_0, \rho_1) \frac{\alpha_{\mathrm{s}}(\rho_1)}{2\pi} \frac{F_1(x_1, \rho_1)}{F_1'(x_1', \rho_1)} \frac{P_1}{\rho_1} \Pi_1(\rho_1, \rho_2) \frac{\alpha_{\mathrm{s}}(\rho_2)}{2\pi} \frac{P_2}{\rho_2} \Pi_2(\rho_2, \rho_{\mathrm{ms}})$$



Find weight to make inclusive matrix element exclusive:

$$\frac{\mathrm{d}\sigma_2^{\mathrm{ex}}}{\mathrm{d}\phi_0 \mathrm{d}\phi_{1,2}} = w \frac{\mathrm{d}\sigma_2^{\mathrm{in}}}{\mathrm{d}\phi_{0+2}}$$

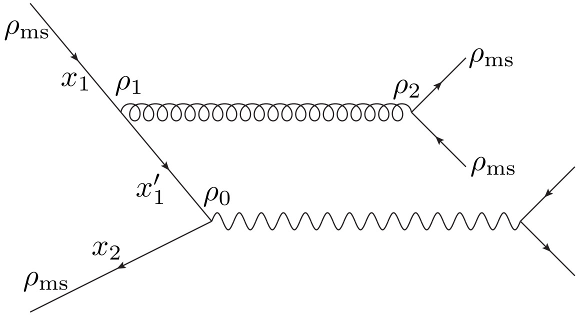
Multi-jet Merging: Merging Weight in ISR

$$w = w_{\alpha_s} w_{\mathrm{pdf}} w_{\mathrm{no-em}}$$

$$w_{\alpha_s} = \frac{\alpha_s(\rho_1)}{\alpha_s(\rho_0)} \frac{\alpha_s(\rho_2)}{\alpha_s(\rho_0)}$$

$$w_{\mathrm{pdf}} = \frac{f(x_1', \rho_0)}{f(x_1', \rho_1)} \frac{f(x_1, \rho_1)}{f(x_1, \rho_0)}$$

$$w_{\mathrm{no-em}} = \Pi_0(\rho_0, \rho_1)\Pi_1(\rho_1, \rho_2)\Pi_2(\rho_2, \rho_{\mathrm{ms}})$$



Multi-Jet Merging Algorithm

Summary of general multi-jet merging procedure:

1. Calculate inclusive cross sections for $X + n$ partons (with kinematic cut ρ_{ms} to avoid singularities)
2. Cluster according to jet algorithm or find parton shower history to find scales for no-emission probabilities and scale dependent ratios
3. Multiply with merging weight: α_{s}-ratios, no-emission probabilities (and PDF ratios for ISR)
4. If $n < N$, with N highest fixed order multiplicity, multiply no-emission probability towards merging scale ρ_{ms}
5. Allow further parton shower emissions below ρ_{ms}, for $n = N$ also above

CKKW Merging [Catani, Krauss, Kuhn, Webber (2001)]

- Cluster according to $k_\perp$ jet algorithm
- Apply analytic Sudakov factors (NLL accuracy) as no-emission probabilities
- Perform "truncated showering", since parton shower evolution variable not exactly identical to merging scale cut: Start shower from ρ_0, but forbid emissions above t_{ms}. Handle hard emissions (in ρ) below t_{ms} with care!
  - $+$ Appealing theoretical treatment
  - - Requires dedicated PS implementation
  - - Mismatch between analytical Sudakov and parton shower
  - Implemented in Sherpa (v 1.1) [Krauss (2002)]

CKKW-L Merging [Lönnblad (2001)]

- Cluster back to parton shower history according to splitting probabilities in PS
- Generate of no-emission probabilities using parton shower
- Perform showering step-by-step for each step in history, starting from respective clustering scale
- Veto event if emission at larger scale than next clustering scale or ρ_{ms} in last step
- Keep PS emissions below ρ_{ms} (and between ρ_n and ρ_{ms} at highest multiplicity)
    - $+$ Agreement between Sudakov and shower by construction $\Rightarrow$ Reduced merging scale dependence
    - $+$ Use simple veto in shower if ρ_{ms} in terms of PS evolution variable
    - - Requires dedicated PS implementation
    - Implemented in Sherpa ($\geq$1.2) [Höche, Krauss, Schumann, Siegert (2009)], Pythia8 [Lönnblad, Prestel (2012)] and Herwig7 [Bellm, Gieseke, Plätzer (2018)]

Unitarity in Multi-jet Merging

$$\frac{\mathrm{d}\sigma_0^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \left[1 - \alpha_s \int_{\rho_{\min}}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1 + \frac{\alpha_s^2}{2} \left(\int_{\rho_{\min}}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1\right)^2\right]$$

$$\frac{\mathrm{d}\sigma_1^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \alpha_\mathrm{s} \mathrm{d}\rho_1 \mathrm{d}z_1 \bar{P}_1^{\mathrm{ME}} \left[1 - \alpha_\mathrm{s} \int_{\rho_1}^{\rho_0} \mathrm{d}\rho \mathrm{d}z \bar{P}_1 - \alpha_\mathrm{s} \int_{\rho_{\min}}^{\rho_1} \mathrm{d}\rho \mathrm{d}z \bar{P}_2\right]$$

$$\frac{\mathrm{d}\sigma_2}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \alpha_\mathrm{s}^2 \mathrm{d}\rho_1 \mathrm{d}z_1 \bar{P}_1^{\mathrm{ME}} \mathrm{d}\rho_2 \mathrm{d}z_2 \bar{P}_2^{\mathrm{ME}} \Theta(\rho_1 - \rho_2)$$

- Unitarity of parton shower broken in all multi-jet merging schemes mentioned above
- Inclusive cross-section only preserved if splitting probabilities in no-emission probability identical to full fixed order splitting probabilities

Unitary Merging: UMEPS [Lönnblad, Prestel (2012)]

Start from CKKW-L scheme, want to restore PS unitarity. Use:

$$\Pi_n(\rho_n, \rho_{\mathrm{ms}}) = 1 - \int_{\rho_{\mathrm{ms}}}^{\rho_n} \mathrm{d}\rho \mathrm{d}z \alpha_{\mathrm{s}} \bar{P}_{n+1}^{\mathrm{ME}}(\rho, z) \Pi_n(\rho_0, \rho)$$

i.e. probability of no emission is 1 - probability of at least one emission

$$\frac{\mathrm{d}\sigma_0^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \Pi_0(\rho_0, \rho_{\mathrm{ms}})$$

$$\frac{\mathrm{d}\sigma_1^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_1^+ F_1^- |M_1|^2 \mathrm{d}\rho_1 \mathrm{d}z_1 \Pi_0(\rho_0, \rho_1) \Pi_1(\rho_1, \rho_{\mathrm{ms}})$$

$$\frac{\mathrm{d}\sigma_2}{\mathrm{d}\phi_0} = F_2^+ F_2^- |M_2|^2 \mathrm{d}\rho_1 \mathrm{d}z_1 \Pi_0(\rho_0, \rho_1) \mathrm{d}\rho_2 \mathrm{d}z_2 \Pi_1(\rho_1, \rho_2)$$

Unitary Merging: UMEPS [Lönnblad, Prestel (2012)]

Start from CKKW-L scheme, want to restore PS unitarity. Use:

$$\Pi_n(\rho_n, \rho_{\mathrm{ms}}) = 1 - \int_{\rho_{\mathrm{ms}}}^{\rho_n} \mathrm{d}\rho \mathrm{d}z \alpha_{\mathrm{s}} \bar{P}_{n+1}^{\mathrm{ME}}(\rho, z) \Pi_n(\rho_0, \rho)$$

i.e. probability of no emission is 1 - probability of at least one emission

$$\frac{\mathrm{d}\sigma_0^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_0^+ F_0^- |M_0|^2 \, \cancel{\Pi_0(\rho_0, \rho_{\mathrm{ms}})} - \int F_1^+ F_1^- |M_1|^2 \mathrm{d}\rho_1 \mathrm{d}z_1 \Pi_0(\rho_0, \rho_1)$$

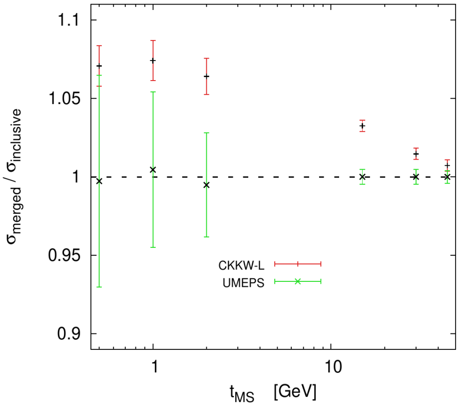
$$\frac{\mathrm{d}\sigma_1^{\mathrm{ex}}}{\mathrm{d}\phi_0} = F_1^+ F_1^- |M_1|^2 \mathrm{d}\rho_1 \mathrm{d}z_1 \Pi_0(\rho_0, \rho_1) \, \cancel{\Pi_1(\rho_1, \rho_{\mathrm{ms}})}$$

$$- \mathrm{d}\rho_1 \mathrm{d}z_1 \Pi_0(\rho_0, \rho_1) \int F_2^+ F_2^- |M_2|^2 \mathrm{d}\rho_2 \mathrm{d}z_2 \Pi_1(\rho_1, \rho_2)$$

$$\frac{\mathrm{d}\sigma_2}{\mathrm{d}\phi_0} = F_2^+ F_2^- |M_2|^2 \mathrm{d}\rho_1 \mathrm{d}z_1 \Pi_0(\rho_0, \rho_1) \mathrm{d}\rho_2 \mathrm{d}z_2 \Pi_1(\rho_1, \rho_2)$$

Unitary Merging: UMEPS [Lönnblad, Prestel (2012)]

- Still add CKKW-L reweighted samples
- Instead of last Sudakov, subtract +1 parton integrated sample
  $\Rightarrow$ Individual multiplicities still exclusive
- Can still add normal PS below merging scale
- + Procedure does not change inclusive cross section
- - UMEPS introduces negative weights $\Rightarrow$ less efficient



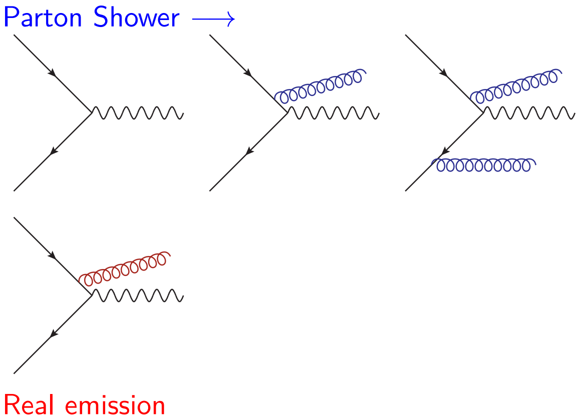
NLO Matching

MC@NLO & Powheg

Matching of NLO Matrix Elements & Parton Showers

We want precision predictions: Combine NLO fixed order calculations with Parton showers.

- Again problem of double counting of emissions by real emission matrix element and emissions generated by parton shower
- Also double counting of virtual terms through virtual corrections and Sudakov factors



Finite Numerical NLO Cross Section

NLO prediction for observable $\mathcal{O}$ given by

$$\langle \mathcal{O} \rangle = \int \mathrm{d}\phi_n (B_n + V_n) \mathcal{O}_n(\phi_n) + \int \mathrm{d}\phi_{n+1} B_{n+1} \mathcal{O}_{n+1}(\phi_{n+1})$$

but both V_n and B_{n+1} separately divergent, only sum is finite.
Use universal subtraction terms to get finite results: [Frixione, Kunszt, Siegner (1996)] [Catani, Seymour (1997)]

$$\langle \mathcal{O} \rangle = \int \mathrm{d}\phi_n (B_n + V_n + B_n \otimes I_1) \mathcal{O}_n(\phi_n) + \int \mathrm{d}\phi_{n+1} (B_{n+1} \mathcal{O}_{n+1}(\phi_{n+1}) - B_n \otimes D_1 \mathcal{O}_n(\phi_{n+1}))$$

Event interpretation not yet possible, $\mathcal{O}_n$ and $\mathcal{O}_{n+1}$ contributions must be finite separately

Shower Subtraction

Want to attach shower (include factor α_s in $\bar{P}$)

$$\mathcal{O}_n(\phi_n) \to \mathcal{F}_n(\mathcal{O}, \phi_n) = \Pi(\rho_n, \rho_\mathrm{min})\mathcal{O}_n(\phi_n) + \int \mathrm{d}\phi_{+1} \Pi(\rho_n, \rho_{n+1})\bar{P}_{n+1}\mathcal{F}_{n+1}(\mathcal{O}, \phi_{n+1})$$

$$\overset{\mathcal{O}(\alpha_\mathrm{s})}{\to} 1 - \int \mathrm{d}\phi_{+1} \bar{P}_{n+1}\mathcal{O}_n(\phi_{n+1}) + \int \mathrm{d}\phi_{+1} \bar{P}_{n+1}\mathcal{O}_{n+1}(\phi_{n+1})$$

But $B_n\mathcal{F}_n$ contains $\mathcal{O}(\alpha_\mathrm{s})$ terms $\Rightarrow$ subtract shower terms to first order in α_s such that accuracy of NLO not spoiled by shower

MC@NLO [Frixione, Webber (2002)]

With shower subtraction, arrive at MC@NLO prescription

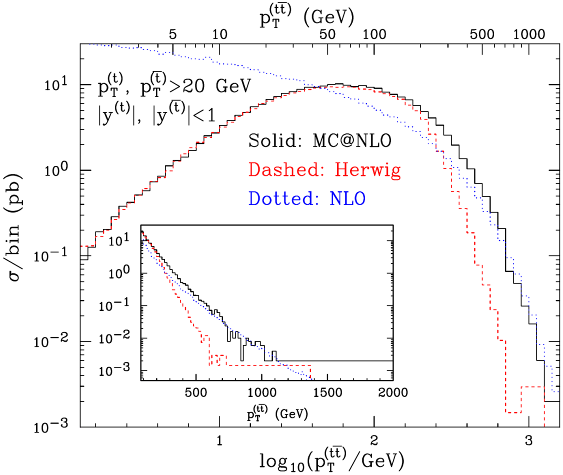
$$\langle \mathcal{O} \rangle_{\mathrm{MC@NLO}} = \int \mathrm{d}\phi_n (B_n + V_n + B_n \otimes I_1) \mathcal{F}_n(\mathcal{O}, \phi_n) \qquad \text{Born + subtracted virtual}$$

$$+ \int \mathrm{d}\phi_{n+1} (B_n \bar{P}_{n+1} - B_n \otimes D_1) \mathcal{F}_n(\mathcal{O}, \phi_{n+1})) \qquad \text{Shower virtual - subtraction}$$

$$+ \int \mathrm{d}\phi_{n+1} (B_{n+1} - B_n \bar{P}_{n+1}) \mathcal{F}_{n+1}(\mathcal{O}, \phi_{n+1}) \qquad \text{Real - shower real}$$

- Event generation possible since $\mathcal{O}_n$ and $\mathcal{O}_{n+1}$ separately finite
- Sudakov supression agrees with shower prediction
- Distribution correct only if parton shower is attached to cancel MC counterterms
- Can lead to many events with negative weights
- Needs to be implemented for each shower separately
- Automated in aMC@NLO [Frederix, Frixione, Hirschi, Maltoni, Pittau, Torrielli (2012)]

MC@NLO



- MC@NLO gives smooth transition between real emission pattern at high scales and parton shower at low scales
- Inclusive cross section correct at NLO

Plot form [Nason, Webber (2012)]

POWHEG [Nason (2004)] [Frixione, Nason, Oleari (2007)]

Positive Weight Hardest Emission Generator

$$\langle\mathcal{O}\rangle_{\mathrm{POWHEG}} = \int \mathrm{d}\phi_n (B_n + V_n + B_n \otimes I_1)\mathcal{F}_n^{\mathrm{HI}}(\mathcal{O}, \phi_n) \qquad \text{Born + subtracted virtual}$$
$$+ \int \mathrm{d}\phi_{n+1}(B_{n+1} - B_n \otimes D_1)\mathcal{F}_n^{\mathrm{HI}}(\mathcal{O}, \phi_{n+1})) \qquad \text{Shower virtual - subtraction}$$

Based on MC@NLO, modify shower to get “shower real” = “real” for hardest emission (similar to matrix element corrections)

- Less negative weights ⇒ Improved efficiency
- Hardest emission modified ⇒ Differences compared to MC@NLO, but both NLO correct
- Implementation process by process, but independent of attached shower

Summary

Matching and Merging Summary

Goal: Combine matrix elements and parton showers. The Problem:

- Parton showers generate singular terms of higher-order matrix elements
- Same terms present in X + jet(s) matrix elements
- Combination must not double count

ME Corrections

- Oldest scheme, correct PS emissions to match full real emission ME
- Hard to iterate beyond one emission
- Developments: higher multiplicity, NLO in VINCIA

Multi-jet Merging

- Combine multiple LO ME samples by reweighting
- Separate phase space regions to deal with divergence
- Different schemes available

NLO Matching

- MC subtraction allows for NLO ME + PS
- MC@NLO and POWHEG
- Can be combined with multi-jet merging

Summary

Discussed background for MC tutorials

- Basic ingredients to parton shower
- Improvements through matching & merging

Now it's your turn: follow instructions at https://gitlab.com/cteq-tutorials/2022/:

Get familiar with Pythia, Herwig, Sherpa

Code your own parton shower

Please also download docker container and training data for next week's ML tutorials by Josh
https://gitlab.com/cteq-tutorials/2022/-/tree/main/ml

Backup

Collinear Factorization of QCD Cross Sections

Hadronic cross section for scattering $ab \to n$ given by

$$\sigma = \sum_{a,b} \int_0^1 \frac{\mathrm{d}x_a}{x_a} \frac{\mathrm{d}x_b}{x_b} \int x_a f_a^{h_1}(x_a, \mu_\mathrm{F}) x_b f_b^{h_2}(x_b, \mu_\mathrm{F}) \mathrm{d}\hat{\sigma}_{ab \to n}(\mu_\mathrm{F}, \mu_\mathrm{R})$$

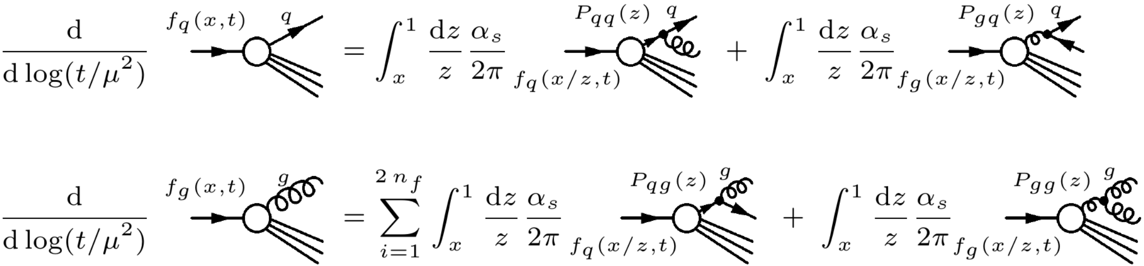
- $\hat{\sigma}$ Partonic cross section
- $f_a^h(x_a, \mu_\mathrm{F})$ parton distribution functions (PDFs)
- x_a light cone momentum fraction $\to$ $x_a f_a$ momentum flux of parton a at x_a
- μ_F factorization scale

Need to take PDFs into account in initial state radiation (ISR), since they change flux

See [Collins, Soper, Sterman (1989)] for factorization theorems in QCD

DGLAP Equations

[Dokshitzer (1977)] [Gribov, Lipatov (1972)] [Altarelli, Parisi (1977)]



- Coupled differential equations describing the parton flux of a hadron at different resolution scales

Initial State Radiation and PDFs

- Modify emission and no-emission probabilities to include PDFs: $x_{\mathrm{new}} = x/z$:

$$\mathrm{d}\mathcal{P}_{\mathrm{emission}}(\rho) = \frac{\mathrm{d}f_j}{f_j} = \frac{\mathrm{d}\rho}{\rho}\frac{\alpha_{\mathrm{s}}}{2\pi}\int_{z_{\min}}^{z_{\max}} \mathrm{d}z P_{ij}(z)\frac{\frac{x}{z} f_i(\frac{x}{z},\rho)}{x f_j(x,\rho)}$$

$$\mathcal{P}_{\mathrm{no-em}}(\rho_1,\rho_2) = \exp\left(-\int_{\rho_2}^{\rho_1}\frac{\mathrm{d}\rho}{\rho}\frac{\alpha_{\mathrm{s}}}{2\pi}\int_{z_{\min}}^{z_{\max}} \mathrm{d}z P_{ij}(z)\frac{\frac{x}{z} f_i(\frac{x}{z},\rho)}{x f_j(x,\rho)}\right) := \Pi(\rho_1,\rho_2)$$

- Initial state shower (more or less) reproduces DGLAP
- DGLAP evolution from small to large scale
- ISR usually uses “backwards evolution”: from large to small scales
  $\Rightarrow$ makes sure we can start from partonic process of interest at high scale [Sjöstrand (1985)]