Vector-boson production at hadron colliders — Lecture II

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Outline of the lectures

Lecture I:

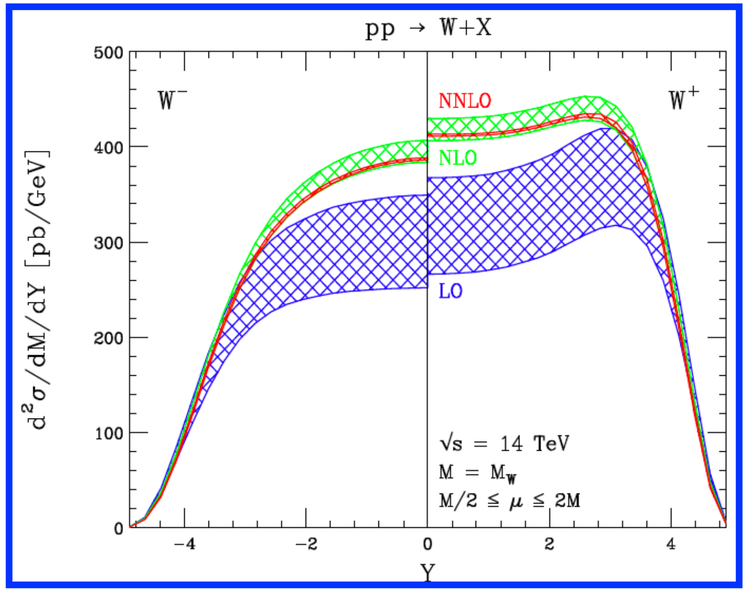
- Drell-Yan like gauge boson production: overview of the basics and historical importance
- Underlying theory framework for W/Z production
- Importance of higher order corrections: QCD, electroweak, mixed QCD-EW
- Modern applications: W-mass, PDFs, BSM searches
- Drell-Yan in the future

Lecture II:

- W-production at NLO in QCD: calculation and lessons learned
- Vector boson production in association with jets: phenomenological importance
- Transverse momentum resummation for W/Z production
- The importance of multi-boson production and the special role of self-interactions — unitarity

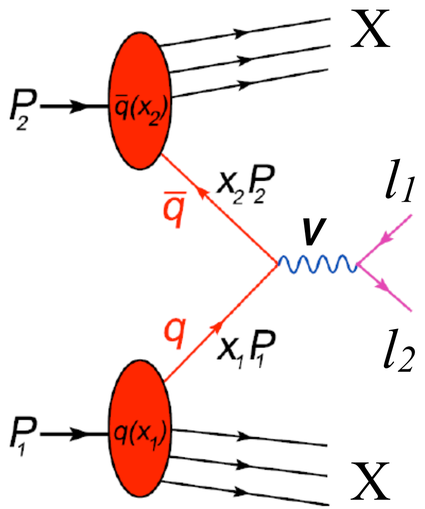
Calculation of W production at NLO

- The Drell-Yan process is an excellent example that shows the importance of calculations to higher orders in QCD perturbation theory.
- The NLO calculation is simple enough to do by hand
- Such calculations can be performed these days using publicly available codes. But sometime these codes need to be extended. It's important to understand what is happening under the hood
- We'll do W-production to NLO together, to understand the features of such calculations, both the technical aspects and how to understand if what you're getting makes physical sense



Leading-order

- We'll begin with the leading order cross section to establish our conventions



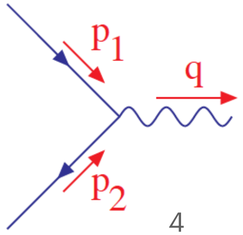
$$h(P_1) + h'(P_2) \to W^+ + X$$

- Recall the QCD factorization theorem:

$$\sigma_{hh' \to WX} = \int dx_1 dx_2 f_{q/h}(x_1, \mu_F^2) f_{\bar{q}/h'}(x_2, \mu_F^2) \underbrace{\hat{\sigma}_{q\bar{q} \to W}}_{\text{partonic cross section}}$$

- The leading-order partonic cross section is an integral over the phase space of the matrix element squared

- To get the matrix element we need to compute the following diagram:

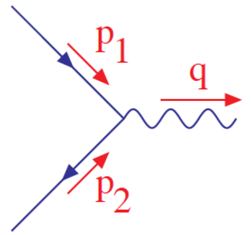


$$p_1 = x_1 P_1, \quad p_2 = x_2 P_2$$

Leading-order amplitude

- We should average the matrix element squared over initial-state colors and spins

$$|\bar{\mathcal{M}}_{LO}|^2 = \underbrace{\left(\frac{1}{3}\cdot\frac{1}{3}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right)}_{\text{color, spin average}} \sum_{spin}\sum_{color} \left|\quad\right|^2$$



- Use the Feynman rule from earlier in the lectures to find

$$\mathcal{M}_{LO} = -\frac{g_w}{2\sqrt{2}}\bar{v}(p_2)\gamma^\mu\epsilon^*_\mu(q)(1-\gamma_5)u(p_1)$$

- Trace technology from a standard QFT text book gives

$$|\bar{\mathcal{M}}_{LO}|^2 = \frac{g_w^2}{12}\left[p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu}p_1\cdot p_2\right]\sum_{spins}\epsilon_\mu(q)\epsilon^*_\nu(q)$$

Leading-order amplitude

- A massive gauge boson has three polarizations. At rest, we can write these polarization vectors as:

$$\epsilon_x = (0,1,0,0)$$
$$\epsilon_y = (0,0,1,0)$$
$$\epsilon_z = (0,0,0,1)$$

  - A Lorentz-covariant way of summing over these three states in an arbitrary frame is:

$$\sum_{spins} \epsilon_\mu(q)\epsilon^*_\nu(q) = -g_{\mu\nu} + \frac{q^\mu q^\nu}{M_W^2}$$

  - We will anticipate the need to later have this result in *d-dimensions* as well as 4-dimensions:

$$|\bar{\mathcal{M}}_{LO}|^2 = \frac{g_w^2}{12} p_1 \cdot p_2 \left[g_{\mu\nu} g^{\mu\nu} - 2\right] = \begin{cases} \frac{g_w^2}{12}\hat{s} \text{ for } d = 4 \\ \frac{g_w^2}{12}(1-\epsilon)\hat{s} \text{ for } d = 4 - 2\epsilon \end{cases}$$

  - We have introduced the partonic Mandelstam invariant

$$\hat{s} = (p_1 + p_2)^2 = x_1 x_2 S$$

Leading-order phase space and result

- We can integrate over the allowed phase space of the final-state W boson using the Lorentz-invariant phase space:

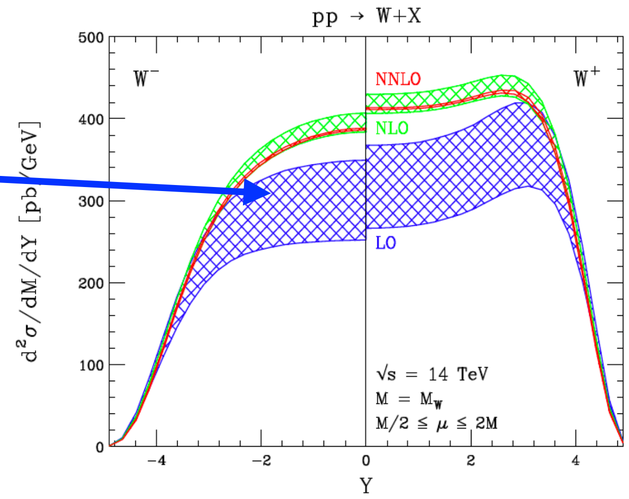
$$\text{LIPS} = \underbrace{\frac{1}{2\hat{s}}}_{\text{flux factor}} \int \frac{d^d q}{(2\pi)^{d-1}} \underbrace{\delta(q^2 - M_W^2)}_{\text{on-shell condition}} (2\pi)^d \delta^d(p_1 + p_2 - q)$$

- Final result:

$$\hat{\sigma}_{q\bar{q}\to W} = \frac{\pi g_w^2}{12\hat{s}} \delta(1-z)$$

$$z = \frac{M_W^2}{\hat{s}}$$

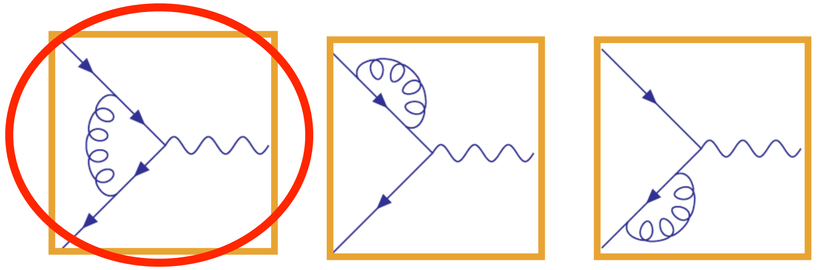
z measures how much of the initial state partonic energy goes into producing the W-boson. At LO z=1



LO blue-band scale variation entirely from PDFs through μ_F

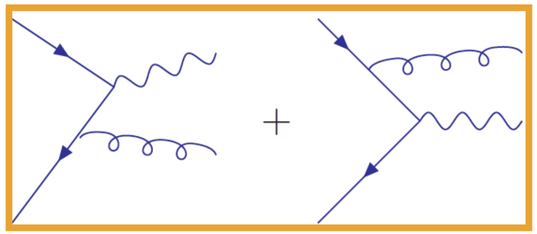
NLO diagrams

- Several types of diagrams appear first at NLO
  - *Virtual corrections:*



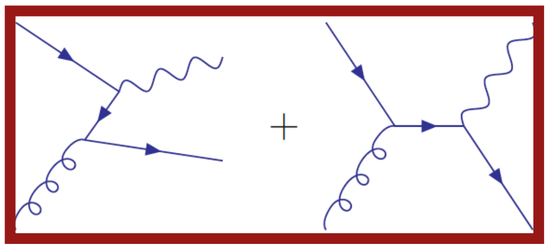
We'll focus on the first diagram; it turns out to be the only one that contributes in our method of calculation.

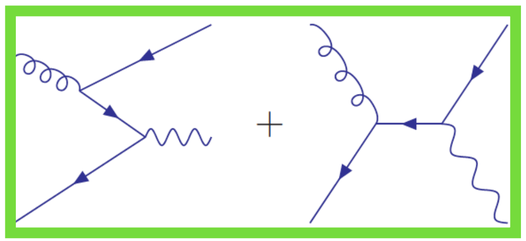
  - *Real radiation corrections:*



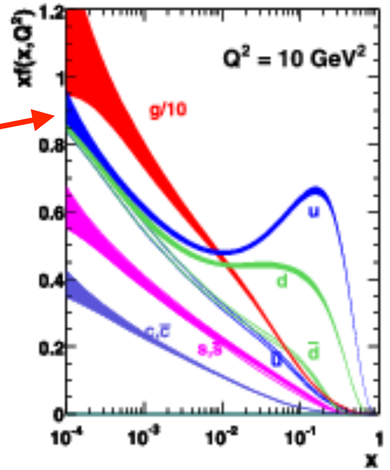
Physically clear that we can't demand no gluons produced along with the W; too many jets at the LHC! We'll also see a theoretical reason for including this in the calculation.

  - New partonic channels first appear at NLO; qg and gq̄ !



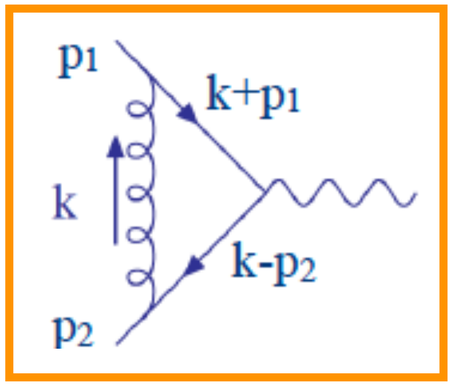


Can take a gluon from the proton as easily as a quark; this is an important contribution!



Virtual correction

- We'll evaluate the virtual correction first. The Feynman rules introduced in these lectures straighforwardly lead to the amplitude:



$$\mathcal{M}_V = i\frac{g_w g_s^2}{2\sqrt{2}}\mathrm{Tr}[T^a T^a]\int \frac{d^d k}{(2\pi)^d}\frac{\bar{v}(p_2)\gamma^\mu(\not{k}-\not{p}_2)\not{\epsilon}_q(1-\gamma_5)(\not{k}+\not{p}_1)\gamma_\mu u(p_1)}{k^2(k+p_1)^2(k-p_2)^2}$$

There are a standard set of techniques, detailed in QFT texts, for handling such expressions

- Combine denominators using *Feynman parameters*

$$\frac{1}{abc} = 2\int_0^1 dx\,dy\,dz\,\delta(1-x-y-z)\frac{1}{(xa+yb+zc)^3}$$

- Shift the loop momentum k so that only k^2 appears in the denominator. Then, all terms in the numerator that are linear in k vanish by spherical symmetry

Divergences

$$\mathcal{M}_V = -2ig_s^2 \mathrm{Tr}[T^a T^a] \mathcal{M}_{LO} \int_0^1 dxdydz\delta(1-x-y-z) \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+xy\hat{s})^3} \times \left\{ \frac{2(1-\epsilon)^2}{2-\epsilon} k^2 - 2\hat{s}\left[(1-x)(1-y)-\epsilon xy\right] \right\}$$

- Note that the integral is *divergent* for large momenta: $\int \frac{d^dk}{(2\pi)^d} \frac{k^2}{(k^2+xy\hat{s})^3} \to \infty$ as d → 4

- In general there can exist several sources of divergences in loop integrals:

$k \to \infty$	ultraviolet
$k \to 0$	soft
$k \parallel p_i$	collinear

Ultraviolet divergences are canceled by *renormalization;* soft and collinear divergences only cancel after adding real radiation corrections! This is proven in the *Kinoshita-Lee-Nauenberg (KLN) theorem*

Dimensional regularization

- The calculation of the cross section in d-dimensions, known as *dimensional regularization*, allows all three types of divergences to be easily tracked and eventually cancelled. This is a great advantage of this method!

  - In d-dimensions the coupling constant becomes dimensionful: $g_s \to g_s \mu^{\epsilon}$

    This is how the dependence on the renormalization and factorization scales enters the calculation. For simplicity equate them to the common scale μ.

  - Necessary loop integrals from standard QFT text books:

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta)^3} = -i \frac{\Gamma(1+\epsilon)}{2(4\pi)^{2-\epsilon}} \Delta^{-1-\epsilon}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2-\Delta)^3} = i \frac{d}{4} \frac{\Gamma(1+\epsilon)}{(4\pi)^{2-\epsilon}} \frac{\Delta^{-\epsilon}}{\epsilon}$$

UV divergence appears as a pole in the parameter ε

Features of virtual corrections

- Do the integrals, plug in the result, and simplify; there are several interesting features to note in the resulting expression that teach us about important aspects of higher order corrections

$$\mathcal{M}_V = \frac{g_s^2}{(4\pi)^{2-\epsilon}} \left(-\frac{\hat{s}}{\mu^2}\right)^{-\epsilon} \Gamma(1+\epsilon)\mathrm{Tr}[T^a T^a]\mathcal{M}_{LO}$$
$$\int_0^1 dxdydz\delta(1-x-y-z)(xy)^{-\epsilon}\left\{2\frac{(1-\epsilon)^2}{\epsilon} - 2\frac{(1-x)(1-y)-\epsilon xy}{xy}\right\}$$

$$\int_0^1 dx\, x^{-1-\epsilon} = -1/\epsilon$$

and another pole from the y-integral

Divergent! Regulated by overall (xy)-ε factor in the phase space. These are the soft and collinear divergences. Lead to $1/\varepsilon^2$; one power for gluon soft in the loop, another for it collinear to p_1 or p_2

Features of virtual corrections

- Do the integrals, plug in the result, and simplify; there are several interesting features to note in the resulting expression that teach us about important aspects of higher order corrections

$$\mathcal{M}_V = \frac{g_s^2}{(4\pi)^{2-\epsilon}} \left(-\frac{\hat{s}}{\mu^2}\right)^{-\epsilon} \Gamma(1+\epsilon)\mathrm{Tr}[T^a T^a]\mathcal{M}_{LO}$$
$$\int_0^1 dxdydz\delta(1-x-y-z)(xy)^{-\epsilon}\left\{2\frac{(1-\epsilon)^2}{\epsilon} - 2\frac{(1-x)(1-y)-\epsilon xy}{xy}\right\}$$

$$\frac{(-1)^{-\epsilon}}{\epsilon^2} = \frac{1}{\epsilon^2} - \frac{i\pi}{\epsilon} - \frac{\pi^2}{2}$$

The imaginary part will cancel when we multiply the amplitude by its complex conjugate. The π^2 survives and is the source of a large NLO correction to Drell-Yan. Only occurs for s-channel processes; not in DIS, for example.

Features of virtual corrections

- The $|M_{LO}+M_V|^2=|M_{LO}|^2+2\,\mathrm{Re}[M^*_{LO}\,M_V]$+higher order in the strong coupling constant. We've already computed the squared LO amplitude, taking the interference between leading order and virtual result, doing the Feynman parameter integrals, and adding the factors needed to convert to the partonic cross section, we obtain:

$$\hat{\sigma}_V = \hat{\sigma}_{LO}\frac{\alpha_s(\mu)C_F}{2\pi}\left(\frac{\hat{s}}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}+\frac{2\pi^2}{3}-8\right)$$

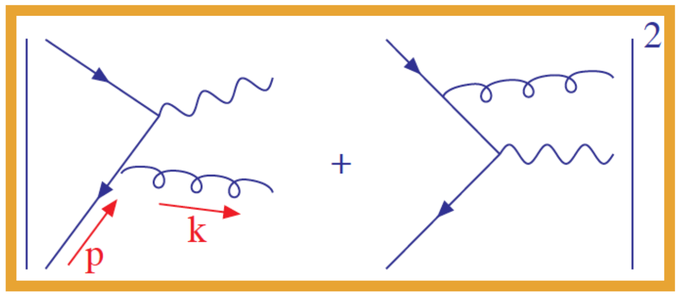
$C_F = \frac{4}{3}$ QCD Casimir constant

Further scale dependence that enters through the strong coupling constant, and also through the explicit logarithm that will occur when we expand in ε

Still divergent; we see the need for further corrections to have a physically well-defined cross section

Real radiation corrections

- Following our discussion before, we need to add the real radiation corrections. The emission of soft gluons, and gluons collinear to the beam directions, are experimentally indistinguishable from producing only a W-boson. We add these contributions at the cross section level, not the amplitude level



Use the standard definition of Mandelstam invariants:

$$\hat{s} = (p_1 + p_2)^2$$
$$\hat{t} = (p_1 - p_3)^2$$
$$\hat{u} = (p_2 - p_3)^2$$

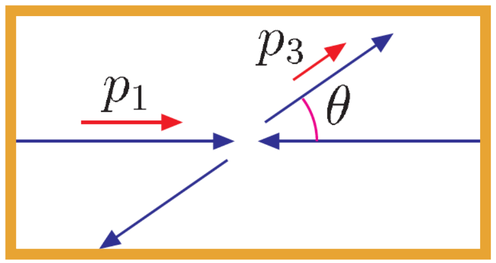
$$|\mathcal{M}_R|^2 = -\left(\frac{1}{2}\cdot\frac{1}{2}\right)\left(\frac{1}{3}\cdot\frac{1}{3}\right)\mathrm{Tr}[T^a T^a]\,2g_w^2 g_s^2 \mu^{2\epsilon}(1-\epsilon)$$
$$\left\{(1-\epsilon)\left(-\frac{\hat{u}}{\hat{t}} - \frac{\hat{t}}{\hat{u}}\right) - 2\frac{\hat{s}M_W^2}{\hat{t}\hat{u}} + 2\epsilon\right\}$$

Singularities when the Mandelstam invariants vanish! These are associated with the emitted gluon becoming soft or collinear to p_1, p_2

Real radiation corrections

- Combine the matrix elements with a parameterization of the 2→2 phase space. We will choose a parameterization that makes it easy to extract the singularities as poles in ε.

$$\hat{\sigma}_R = \frac{1}{2\hat{s}} \int \mathrm{dPS}_2 \, |\mathcal{M}_R|^2 \qquad \mathrm{dPS}_2 = \frac{1}{8\pi\Gamma(1-\epsilon)} \left(\frac{4\pi}{M_W^2}\right)^{\epsilon} z^{\epsilon}(1-z)^{1-2\epsilon} \, dy \, [y(1-y)]^{-\epsilon}$$



To derive this expression we work in the partonic CM frame, and introduce the following variables:

$$z = \frac{M_W^2}{\hat{s}}$$
$$y = \frac{1+\cos(\theta)}{2}$$
$$\hat{t} = -\hat{s}(1-z)(1-y)$$
$$\hat{u} = -\hat{s}(1-z)y$$

The singularities in the matrix elements occur when y=0,1 (collinear) and z=1 (soft)

y=0,1 occurs when cos(θ)=±1, so these are collinear singularities; when z=1, $\hat{s}=M_W^2$; no energy goes into the external gluon so this is a soft singularity

Plus distributions

- When we combine matrix elements and phase space, we get the following structure, where we've lumped all the non-singular dependence on y,z into a function *h*. Note that h contains the PDFs, since $z=M^2/(x_1x_2S)$. We can switch from an integral over x_1 or x_2 to z instead.

$$\int_0^1 dy\, y^{-1-\epsilon}(1-y)^{-1-\epsilon}(1-z)^{-1-2\epsilon}\, h(y,z)$$

- The integral over y proceeds identically to the Feynman parameter integrals. But the integral over z involves the PDFs, which have a complicated functional dependence. Can we extract the z=1 singularity without knowing the form of the PDFs?

This requires the concept of a *plus distribution*, which is a way of extracting a singularity from the endpoint of an integral before integration

$$\int_0^1 dz\, [f(z)]_+\, g(z) = \int_0^1 dz f(z)\, \{g(z) - g(1)\}$$

Plus distributions

- In our case the function is $(1-z)^{-1-2\varepsilon}$; let's see how to simplify this:

$$(1-z)^{-1-2\epsilon}h(y,z) = \underbrace{(1-z)^{-1-2\epsilon}}_{\text{can expand in } \epsilon}\underbrace{[h(y,z)-h(y,1)]}_{\text{vanishes for } z=1} + \underbrace{(1-z)^{-1-2\epsilon}h(y,1)}_{\text{simple, integrable dependence on } z}$$

$$= \left\{\frac{1}{1-z} - 2\epsilon\frac{\mathrm{Log}(1-z)}{1-z} + \mathcal{O}(\epsilon^2)\right\}[h(y,z)-h(y,1)] - \frac{1}{2\epsilon}h(y,1)$$

$$= h(y,z)\left\{-\frac{1}{2\epsilon}\delta(1-z) + \left[\frac{1}{1-z}\right]_+ - 2\epsilon\left[\frac{\mathrm{Log}(1-z)}{1-z}\right]_+ + \mathcal{O}(\epsilon^2)\right\}$$

We integrated over z from 0 to 1

- This is exactly what we need. We extracted the singularity without knowing the functional form of h. Note that the pole is proportional to the same delta-function as the leading-order cross section. As we discussed before, z=1 is the soft limit. The emission of a soft gluon shouldn't change the kinematics of the tree-level cross section. This expression makes this physical intuition manifest.

Result for the real correction

- We put all of this together and get the following result for the real-radiation correction to the cross section:

$$\hat{\sigma}_R = \sigma_0 \frac{\alpha_s C_F}{2\pi} \left(\frac{\hat{s}}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ \frac{2}{\epsilon^2}\delta(1-z) - \frac{2}{\epsilon}\frac{1+z^2}{[1-z]_+} + 4(1+z^2)\left[\frac{\mathrm{Log}(1-z)}{1-z}\right]_+ \right\}$$

(σ_0 is just the LO cross section with the $\delta(1-z)$ removed)

- Sum the real and virtual correction to obtain:

$$P_{qq}(z) = \frac{1+z^2}{[1-z]_+} + \frac{3}{2}\delta(1-z)$$

$$\hat{\sigma}_R + \hat{\sigma}_V = \sigma_0 \frac{\alpha_s C_F}{2\pi} \left(\frac{\hat{s}}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ -\frac{2}{\epsilon}P_{qq}(z) + \left(\frac{2\pi^2}{3} - 8\right)\delta(1-z) + 4(1+z^2)\left[\frac{\mathrm{Log}(1-z)}{1-z}\right]_+ \right\}$$

expand this term in epsilon, it leads to logarithmic scale dependence

We still have a singularity!

Mass factorization

- This singularity comes from collinear emissions of a gluon from the initial quarks, which occurs at very low virtualities. Since this is a low-energy effect, it suggests that we absorb this divergence into the PDFs. This procedure is called *mass factorization*. This splitting of partons is *universal*; it does not depend upon the details of the hard scattering process. We normally derive the mass factorization counterterm in DIS, and then apply it to Drell-Yan.

one for each initial-state direction

$$\hat{\sigma}_{MF} = 2 \times \frac{\alpha_s C_F}{2\pi} \frac{P_{qq}(z)}{\epsilon} \sigma_0$$

$$\hat{\sigma}_R + \hat{\sigma}_V + \hat{\sigma}_{MF} = \sigma_0 \frac{\alpha_s C_F}{2\pi} \left\{ 2P_{qq}(z) \mathrm{Log} \frac{\hat{s}}{\mu^2} + \left(\frac{2\pi^2}{3} - 8 \right) \delta(1-z) + 4(1+z^2) \left[\frac{\mathrm{Log}(1-z)}{1-z} \right]_+ \right\}$$

Poles cancel!

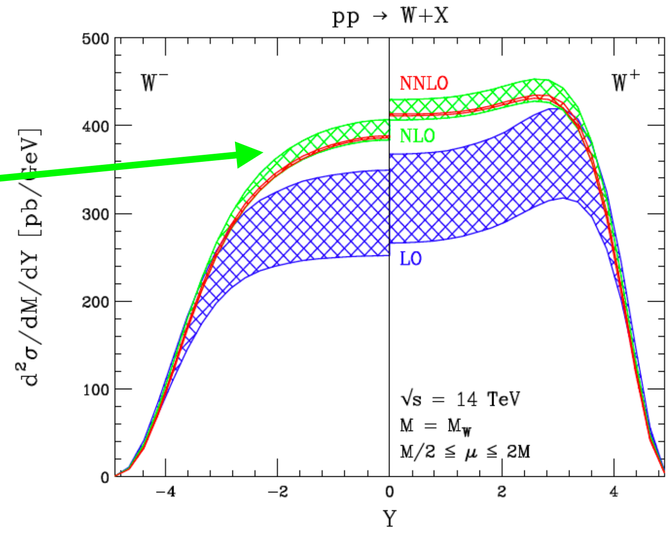
The collinear pole leads to scale dependence

Final result

- Final result for the quark-antiquark channel:

$$\hat{\sigma}_R + \hat{\sigma}_V + \hat{\sigma}_{MF} = \sigma_0 \frac{\alpha_s C_F}{2\pi} \left\{ 2P_{qq}(z) \mathrm{Log} \frac{\hat{s}}{\mu^2} + \left(\frac{2\pi^2}{3} - 8 \right) \delta(1-z) + 4(1+z^2) \left[\frac{\mathrm{Log}(1-z)}{1-z} \right]_+ \right\}$$

- The scale dependence of the PDFs is partially compensated by the explicit logarithm in the partonic cross section.



The plus distribution is peaked near $z=1$, and the corrections to the hadronic cross section are therefore enhanced for PDFs that rise when $z=M^2/(x_1x_2S)\to 1$, or equivalently for $x_{1,2}$ small. This is not the case for the quark PDFs in Drell-Yan. It is the case for the gluon PDFs in Higgs production, and the NLO corrections are much larger for Higgs production.

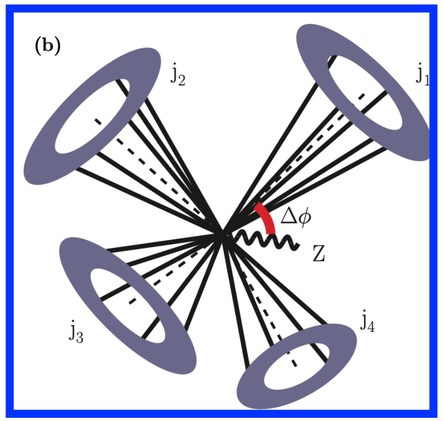
Need to add quark-gluon scattering for full NLO result; similar real-radiation calculation, but no virtual corrections

Vector Boson Production in Association With Jets

- Final states include leptons, missing transverse momentum and jets

- They provide stringent tests of the SM, as they are measured with small errors over a large energy range
- Important for improving PDFs and detector calibration as well
- Backgrounds to top production (W+jets) and Higgs studies
- They are associated with many signals of BSM physics, such as SUSY (eg. “MET+jets”) and dark matter

arXiv:2012.13967



They are also at the forefront of developing theoretical tools on the multiplicity frontier

- Lots of theoretical activity to compute amplitudes with many jets, NLO and NNLO corrections
- Systematic improvement of parton shower predictions: matching, merging, the inclusion of higher order corrections, etc

V+jets as a background to BSM searches

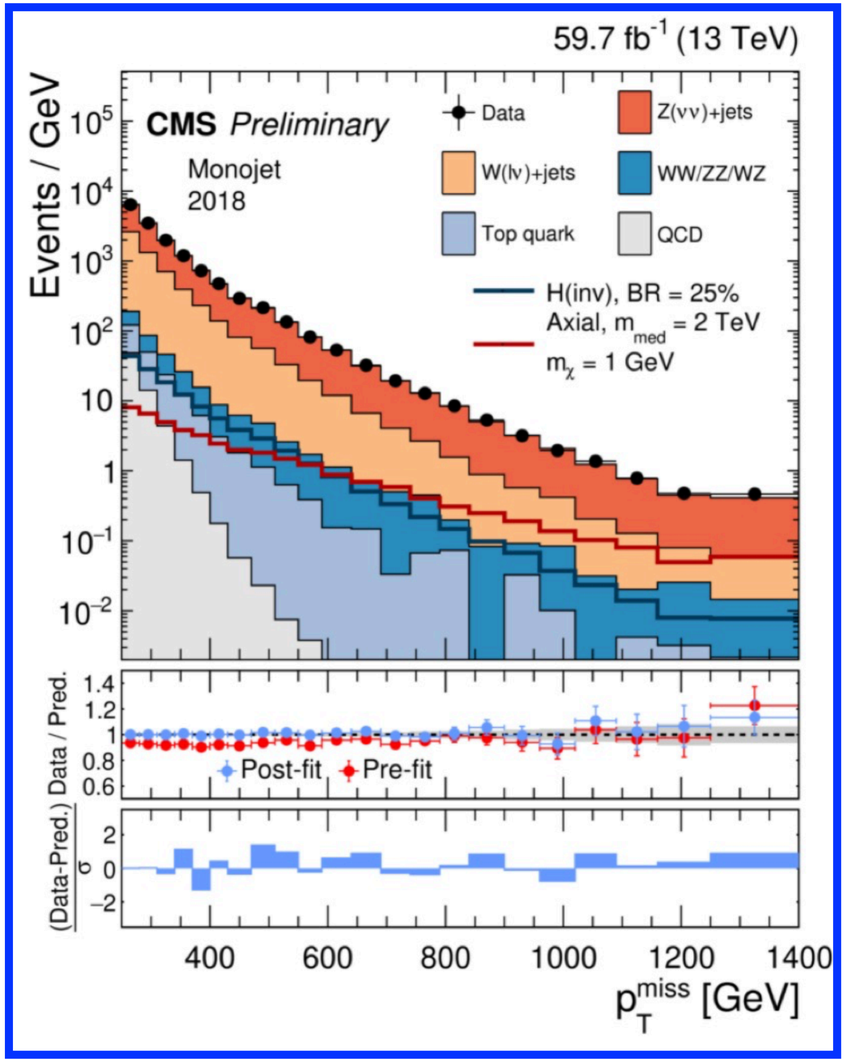
- V+jets backgrounds in MET+jets/monojet searches

irreducible backgrounds:

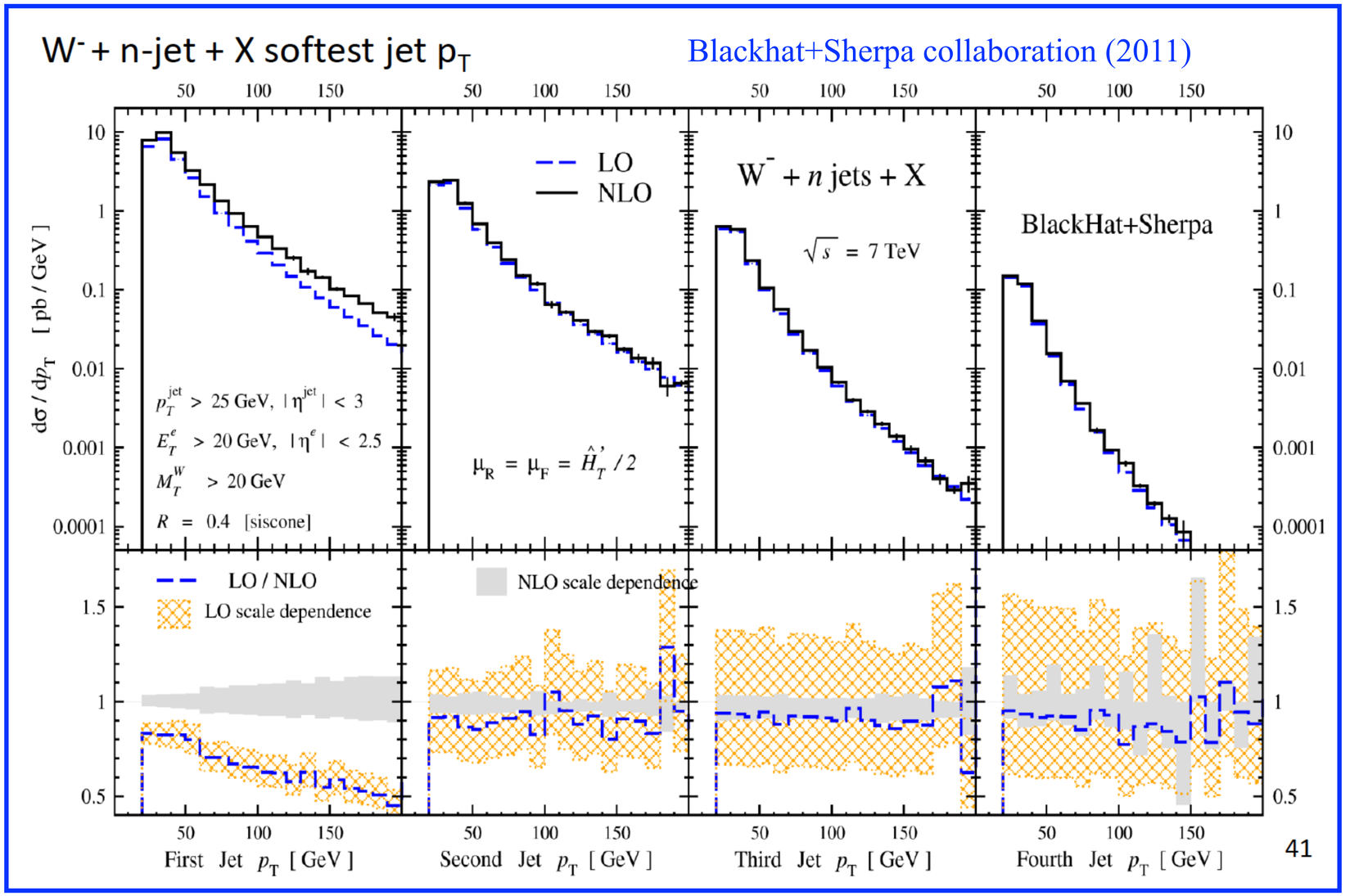
$pp \rightarrow Z(\rightarrow \nu\bar{\nu})+\text{jets} \implies \text{MET} + \text{jets}$

$pp \rightarrow W(\rightarrow l\nu)+\text{jets} \implies \text{MET} + \text{jets}$ (lepton lost)

- Major background to SUSY and dark matter searches
- The determination of backgrounds relies on theory predictions for cross sections

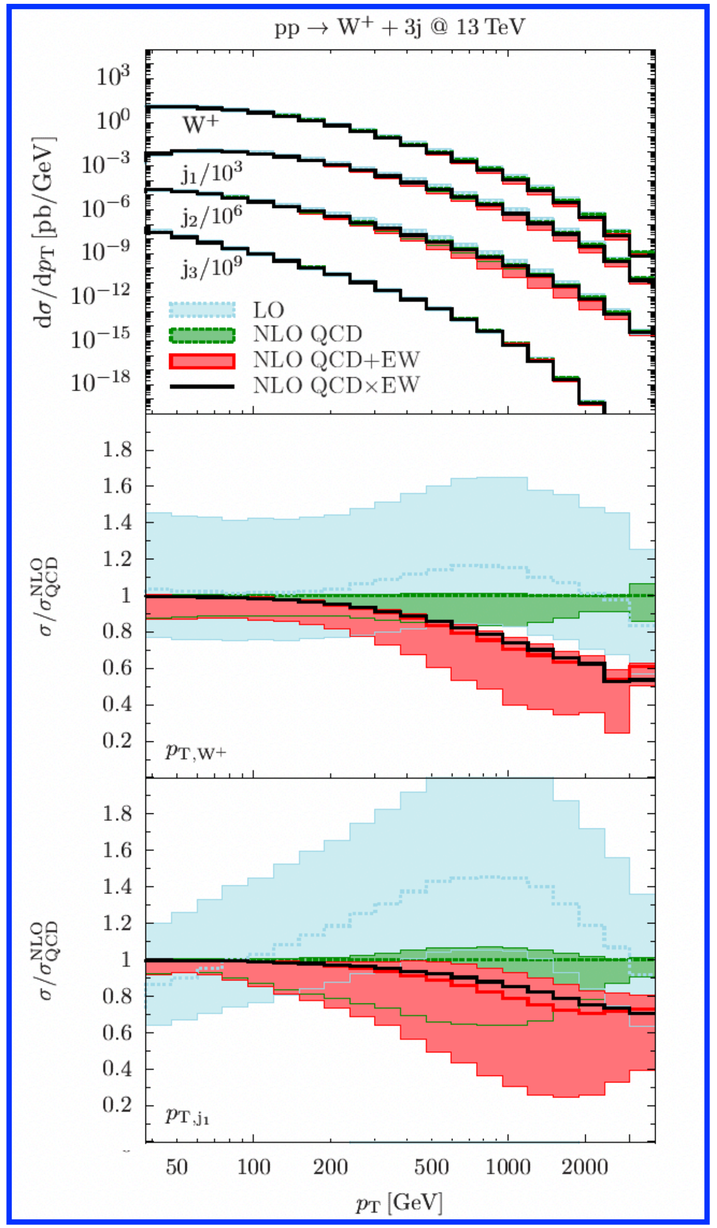


V+ higher jet multiplicity @ NLO QCD

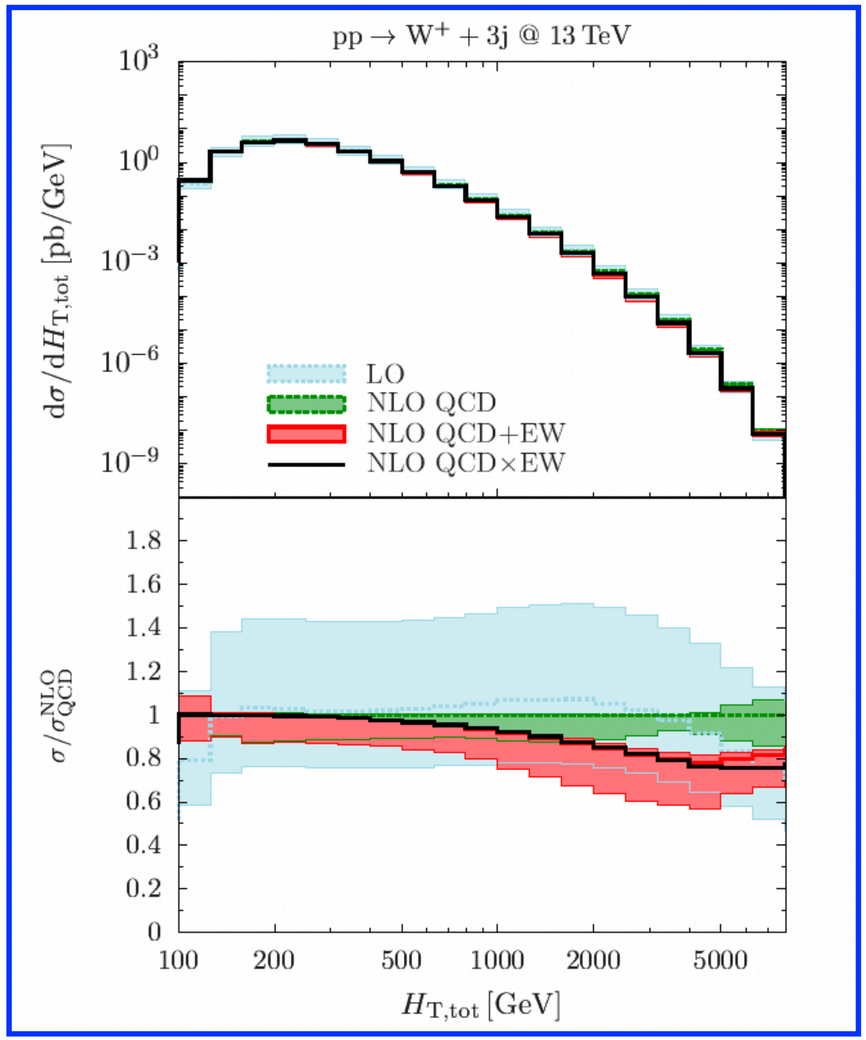


- Better perturbative stability as we go to higher jet multiplicity. The large NLO corrections for V+1jet shows the need for higher order corrections

V+ higher jet multiplicity @ NLO QCD+EW



Kallweit, Lindert, Maierhoefer, Pozzorini, Schoenherr, 2014



- normalization to $\sigma^{\mathrm{NLO}}_{\mathrm{QCD}}$
- $\mu_{\mathrm{ren}} = \mu_{\mathrm{fact}} = \hat{H}_{\mathrm{T}} = \sum E_{\mathrm{T}}$
- $H_{\mathrm{T}}^{\mathrm{tot}} = p_{\mathrm{T,W}} + \sum p_{\mathrm{T},j_k}$

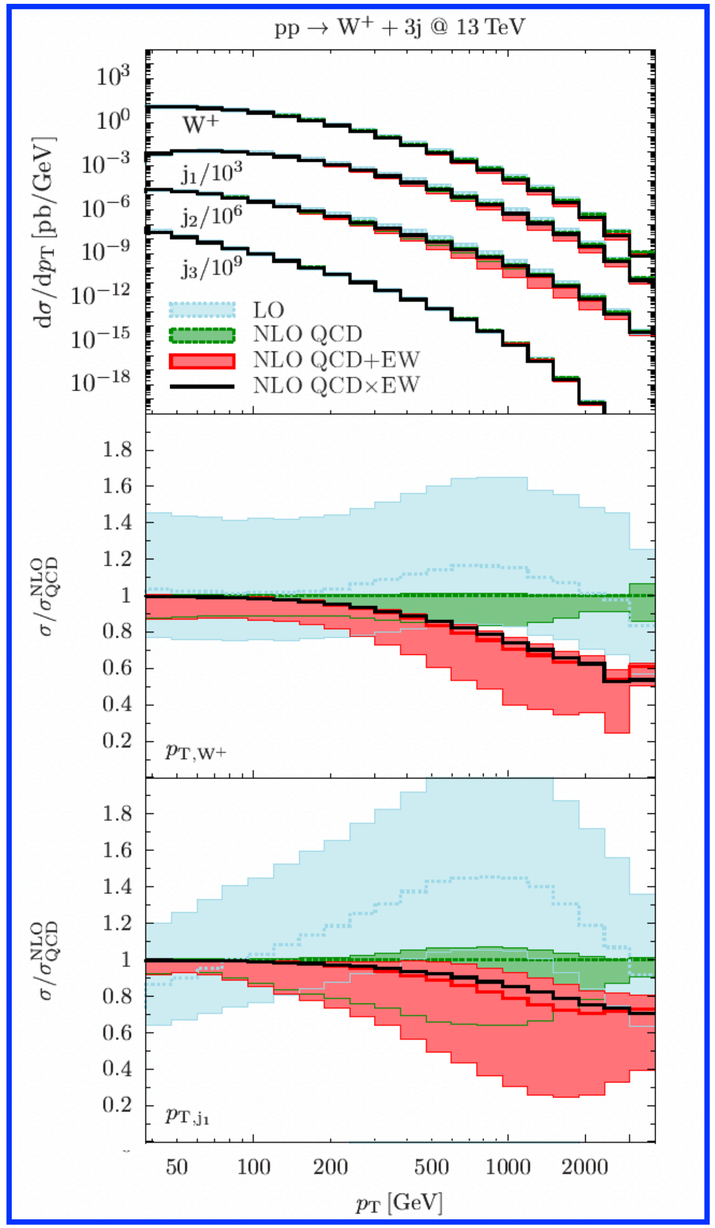
Note the difference between the two combination schemes for QCD and EW effects (additive vs multiplicative):

$$(1 + \delta^{\mathrm{NLO}}_{\mathrm{QCD}} + \delta^{\mathrm{NLO}}_{\mathrm{EW}})$$

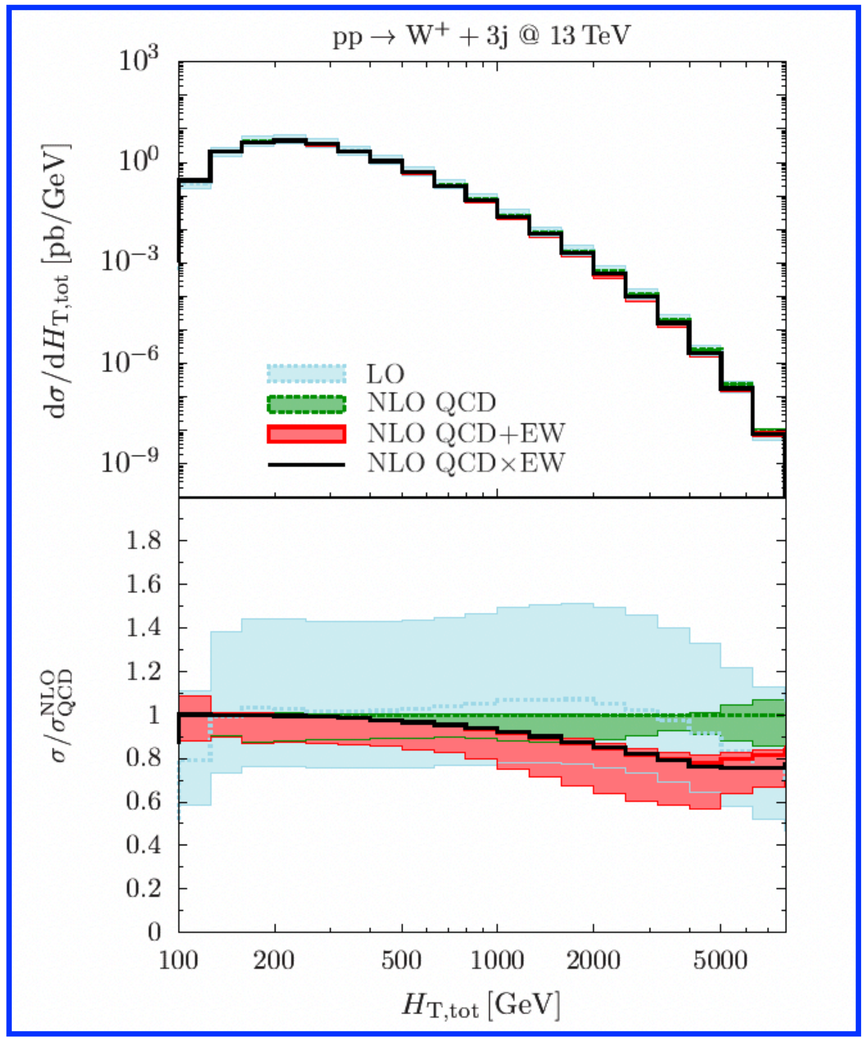
versus

$$(1 + \delta^{\mathrm{NLO}}_{\mathrm{QCD}}) \times (1 + \delta^{\mathrm{NLO}}_{\mathrm{EW}})$$

V+ higher jet multiplicity @ NLO QCD+EW



Kallweit, Lindert, Maierhoefer, Pozzorini, Schoenherr, 2014



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- $H_{\mathrm{T}}^{\mathrm{tot}} = p_{\mathrm{T,W}} + \sum p_{\mathrm{T},j_k}$

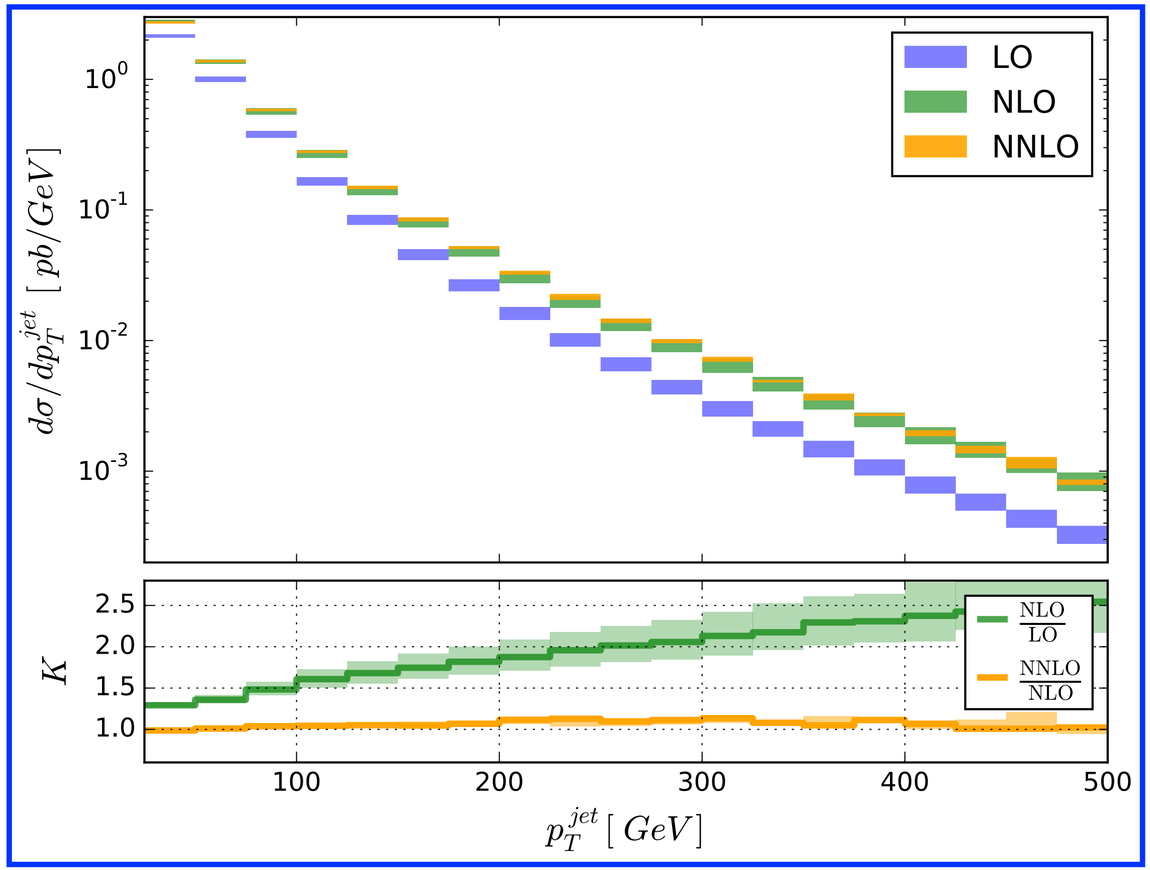
- This is an example where QCD+EW result is different from QCDxEW
- Can ultimately be resolved through an actual QCD-EW calculation

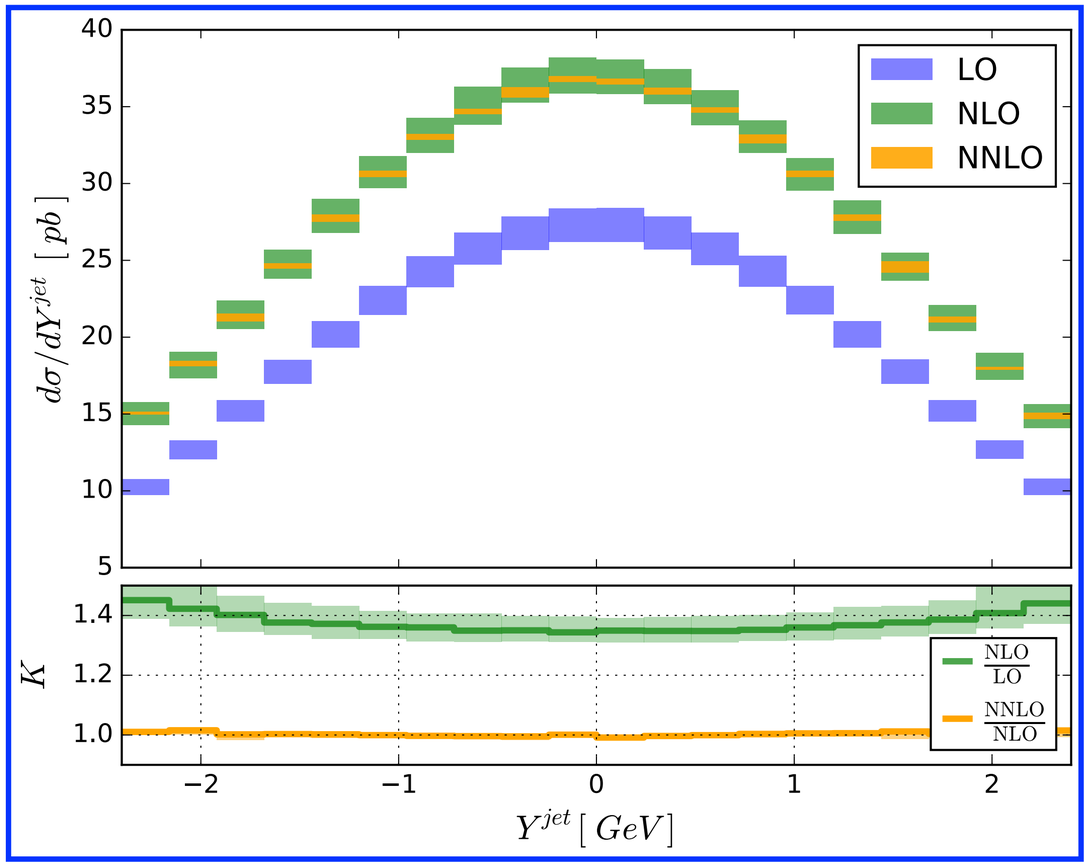
V+ single jet @ NNLO QCD

- Example: Z+jet. NNLO jet distribution corrections small and stable under perturbative QCD. NLO corrections are large and very dependent on kinematics

Z+jet

RB, Liu, Petriello (2016)



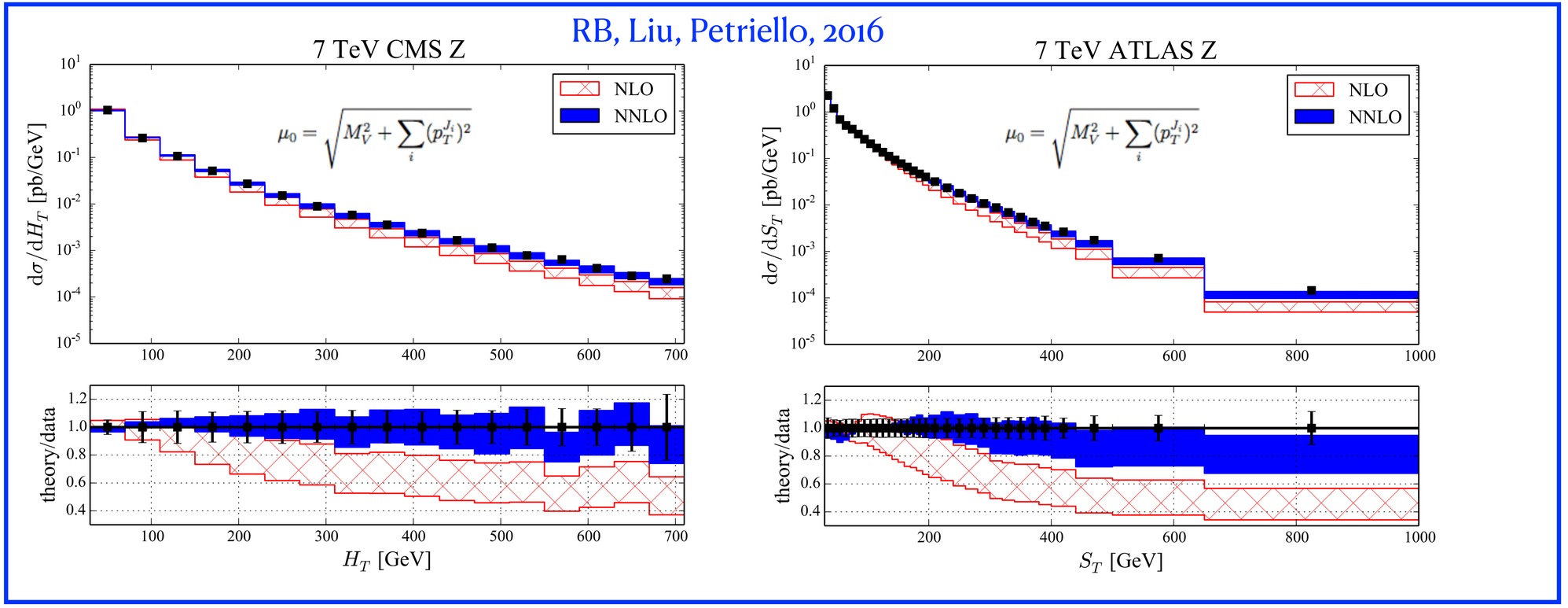


V+ single jet @ NNLO QCD

- H_T: the scalar sum of the transverse momenta of all reconstructed jets, and is called S_T by ATLAS. Important for BSM searches.

Z+jet

RB, Liu, Petriello, 2016

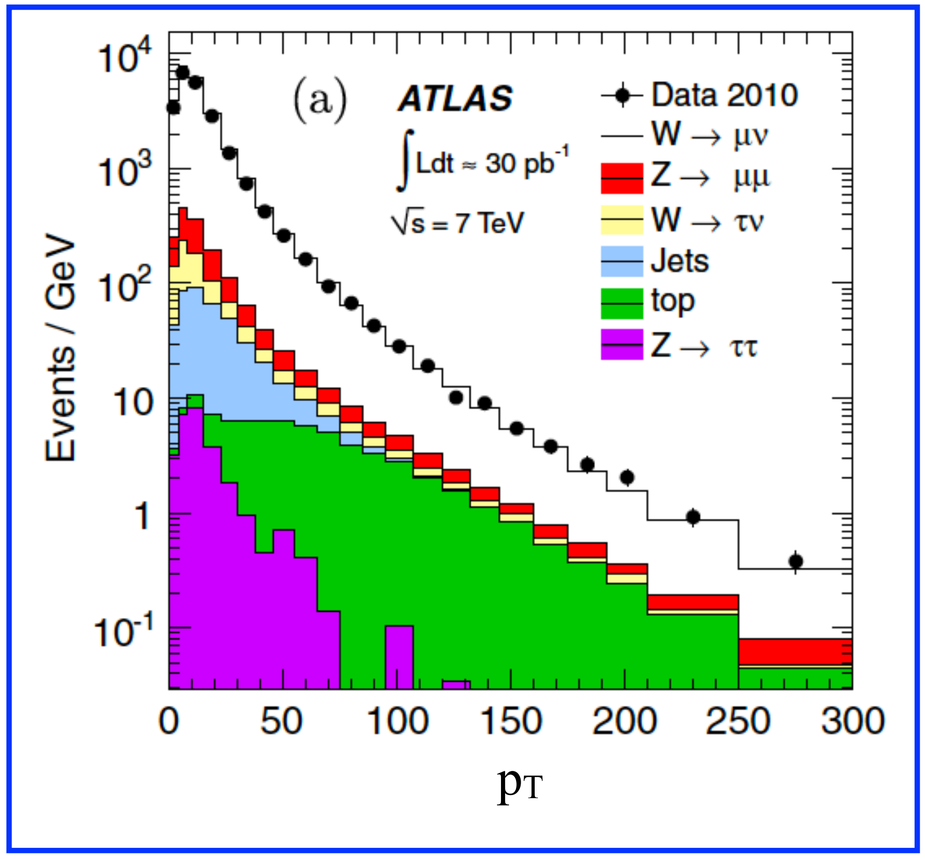


- While NLO QCD results significantly underestimate the cross section at intermediate and high H_T, the ATLAS and CMS data for the entire H_T/S_T range are well described with the NNLO QCD corrections

Transverse momentum of W/Z bosons

- The transverse momentum distribution is related to the W/Z+jet process, since the W/Z p_T is generated by emission of jets. However, important differences occur when we don't require the jet to be observed, especially in the low p_T region. Since the low-p_T region is important for measurements like the W mass, we need to understand it

ATLAS, 1108.6308



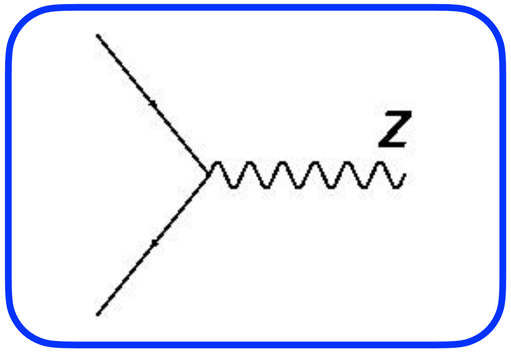
Features of the distribution:

- Steeply falling at for p_T>30 GeV. This high-p_T region is governed by fixed-order perturbative QCD calculations
- Peaked near p_T≈10 GeV. The description of this region requires the resummation of large logarithms to all orders in QCD perturbation theory
- Drops again for p_T<10 GeV. This region receives large non-perturbative corrections, and requires model parameters extracted from data (ResBos: Nadolsky, Yuan et al)

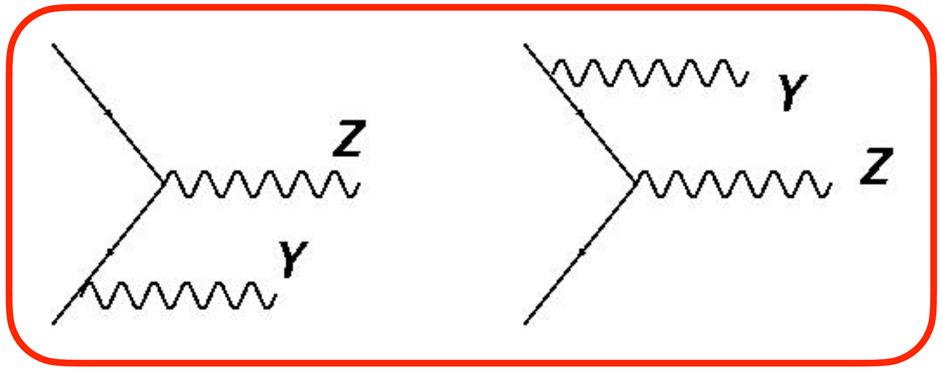
Calculation of low p_T logarithms

- The study of the low transverse momentum region, and the need for resummation of large logarithms, has been extensively discussed in the literature (Pariso, Petronzio (1979); Collins, Soper, Sterman (1985); Becher, Neubert (2010))
- We will study a very simple example of this in the related process $e^+e^- \to Z$, where the transverse momentum is generated by emission of photons.

Leading order diagram

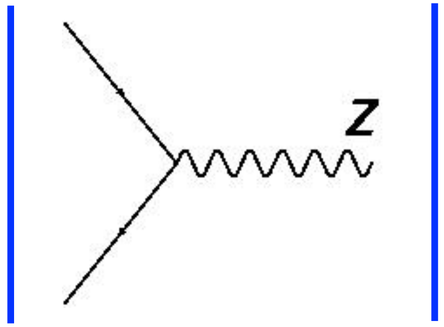


Real emission corrections



Leading order cross section

- We will make a few simplifying assumptions. Give the Z boson an invariant mass M, then integrate over that invariant mass. This mimics the decay of the Z to fermions without all the algebra. Only consider the vector coupling of the Z for simplicity, again to make the algebra simpler.



$$^2 = |\bar{\mathcal{M}}|^2 = \underbrace{\frac{1}{4}}_{\text{spin average}} g_Z^2 \mathrm{Tr}[\not{p}_1 \not{\epsilon}_Z \not{p}_2 \not{\epsilon}_Z] = g_Z^2 s$$

All the energy goes into producing the Z boson

Phase space: $\mathrm{LIPS} = \underbrace{\frac{1}{2s}}_{\text{flux factor}} \int dM^2 \int \frac{d^d p_Z}{(2\pi)^d} (2\pi)\delta(p_Z^2 - M^2)(2\pi)^d \delta(p_1 + p_2 - p_Z) = \frac{\pi}{s} \int dM^2 \delta(s - M^2)$

Integrate the matrix elements over the phase space to obtain the result:

$$\sigma_0 = \pi g_Z^2$$

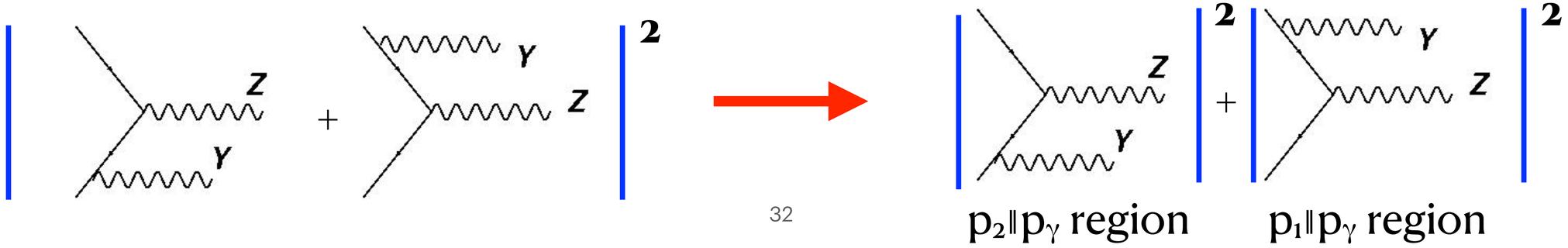
Real emission

- Our goal is to calculate the leading term in the low-p_T expansion of the NLO cross section. This allows us to make two simplifying assumptions.

  1. The final-state photon must be collinear to an initial beam direction: $p_1 \| p_\gamma$ or $p_2 \| p_\gamma$. This is physically clear: low-p_T for the Z-boson requires the photon to be low-p_T as well by momentum conservation, and low-p_T means close to the beam axis.

  2. The photon must also be soft in the final state, so that $s \approx M^2$. We know from our previous calculation that soft emissions lead to large logarithms in the cross section, making this a reasonable (and it turns out correct) assumption.

- An immediate consequence of the first assumption is that we can use a simplifying feature of gauge theories: collinear singularities only arise from squares of diagrams Ellis, Georgi, Machacek, Politzer, Ross (1978)



Phase space in the collinear limit

- We'll first simplify the phase space in the $p_2 \| p_\gamma$ limit, then the matrix elements. The $p_1 \| p_\gamma$ limit gives an identical result, so we'll just multiply by two at the end.

$$\begin{aligned}\text{LIPS} &= \frac{1}{2s}\int dM^2 \int \frac{d^4p_Z}{(2\pi)^4}\frac{d^4p_\gamma}{(2\pi)^4}(2\pi)\delta(p_Z^2 - M^2)(2\pi)\delta(p_\gamma^2)(2\pi)^4\delta(p_1 + p_2 - p_Z - p_\gamma) \\ &= \frac{1}{2s(2\pi)^2}\int dM^2 \int d^4p_\gamma \delta(p_\gamma^2)\delta((p_1 + p_2 - p_\gamma)^2 - M^2)\end{aligned}$$

soft photon when $s \approx M^2$

Choose the negative root for $p_2 \| p_\gamma$

- Parameterize the momenta as:

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1)$$
$$p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1)$$
$$p_\gamma = (E, p_T, 0, k)$$

- Solve the delta-functions to find:

$$E = \frac{s - M^2}{2\sqrt{s}}$$
$$k = -\sqrt{\frac{(s-M^2)^2}{4s} - p_T^2}$$

$$\frac{d\text{LIPS}}{dp_T^2} = \frac{1}{16\pi s}\int dM^2 \frac{1}{\sqrt{(s-M^2)^2 - 4sp_T^2}}$$

Amplitude in the collinear limit

- Now calculate the matrix elements, focusing on just the diagram where the photon is emitted from p_2 for the $p_2 \| p_\gamma$ limit.

  - Lets define the invariants: $s_{1\gamma} = (p_1 - p_\gamma)^2$, $s_{2\gamma} = (p_2 - p_\gamma)^2$ → $$|\bar{\mathcal{M}}|^2 = \frac{e^2 g_Z^2}{4 s_{2\gamma}} \left\{ \frac{4 s s_{2\gamma}}{M^2} + 8 s_{1\gamma} + 16 s + 16 \frac{s^2}{s_{1\gamma}} \right\}$$

  It is important to sum over only physical photon polarizations: $$\sum_{\text{pol}} \epsilon_\gamma^\mu \epsilon_\gamma^{*\nu} = -g^{\mu\nu} + \frac{p_1^\mu p_\gamma^\nu + p_1^\nu p_\gamma^\mu}{p_1 \cdot p_\gamma}$$

  - We can expand the invariants in the collinear limit (small p_T) using the phase space parameterization on the previous slide:

$$s_{1\gamma} \approx -(s - M^2)$$
$$s_{2\gamma} \approx -\frac{s}{s - M^2} p_T^2$$

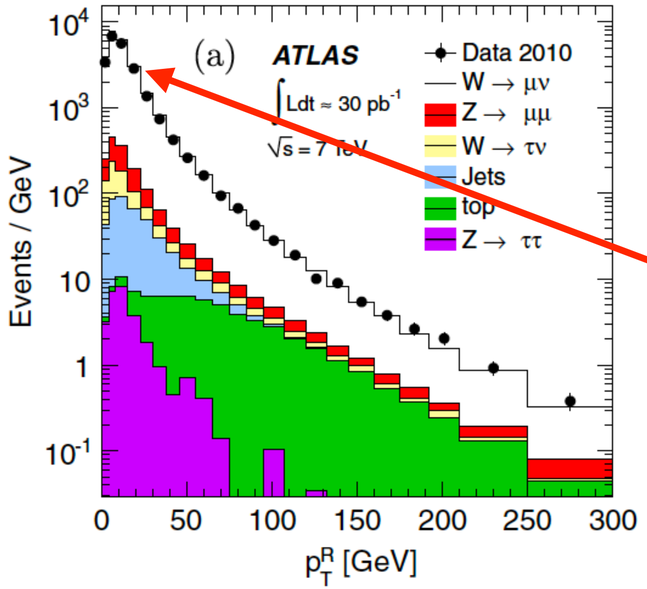
  →

  - The red term in the matrix element gives the leading contribution; all other terms are suppressed by p_T^2 or $s\text{-}M^2$

$$|\bar{\mathcal{M}}|^2 \approx \frac{4 e^2 g_Z^2 s}{p_T^2}$$

Cross section at low p_T

- Combine phase space and matrix elements, integrate over the off-shell Z-boson mass M^2, multiply by two for the other collinear region, and obtain:



$$\frac{d\sigma_R}{dp_T^2} = \sigma_0 \frac{\alpha}{\pi} \frac{1}{p_T^2} \mathrm{Log} \frac{s}{p_T^2} + \cdots$$

where the ellipsis denotes sub-leading terms in the low pT limit

- Single photon emission correction gets the falling tail of the p_T distribution. The logarithmic enhancement comes from the radiated photon being soft as well as collinear
- How do we get the turnover of the distribution at low p_T?

Resummation of soft photons

- To solve this problem, we will consider the total NLO correction to the cross section. It is a sum of the real emission corrections integrated over p_T, and the virtual corrections. We will split the integral of the real-emission corrections into a low transverse momentum region, and a high one. We will derive the low region, which is the contribution of soft and collinear photons to the cross section

$$\Delta\sigma = \underbrace{\int_0^{q_T^2} dp_T^2 \frac{d\sigma_R}{dp_T^2} + \overbrace{\sigma_V}^{\text{virtual}}}_{\text{finite; poles cancel}} + \int_{q_T^2}^{s} dp_T^2 \frac{d\sigma_R}{dp_T^2}$$

Δσ does not have a log(q_T) dependence but the individual real radiation pieces do

- We will approximate Δσ=0; it has no logarithm of q_T, which is the functional dependence we're interested in. We can then derive the total correction to the cross section of soft photons up to q_T, making sure to add back in the LO result that has no photon:

$$\sigma(0 \le p_T \le q_T) = \sigma_0 + \int_0^{q_T^2} dp_T^2 \frac{d\sigma_R}{dp_T^2} + \sigma_V = \sigma_0 - \int_{q_T^2}^{s} dp_T^2 \frac{d\sigma_R}{dp_T^2} = \sigma_0 \left\{ 1 - \frac{\alpha}{2\pi} \mathrm{Log}^2 \frac{s}{q_T^2} \right\}$$

Exponentiation

- This is a very useful expression because the contribution of soft photons to a cross section *exponentiate*, as covered in QFT textbooks. Defining the correction from a single soft photon as Δ , the emission of n soft photons is given as:

$$\Delta^{(n)} = \frac{\Delta^n}{n!}$$ (see Peskin and Schroder, chapter 6)

- We can use this to resum the contribution of soft photons to all orders:

$$\sigma(0 \leq p_T \leq q_T) = \sigma_0 \sum_{n=0}^{\infty} \frac{\Delta^n}{n!} = \sigma_0 e^{\Delta} = \sigma_0 \,\mathrm{Exp}\left[-\frac{\alpha}{2\pi}\mathrm{Log}^2\frac{s}{q_T^2}\right]$$

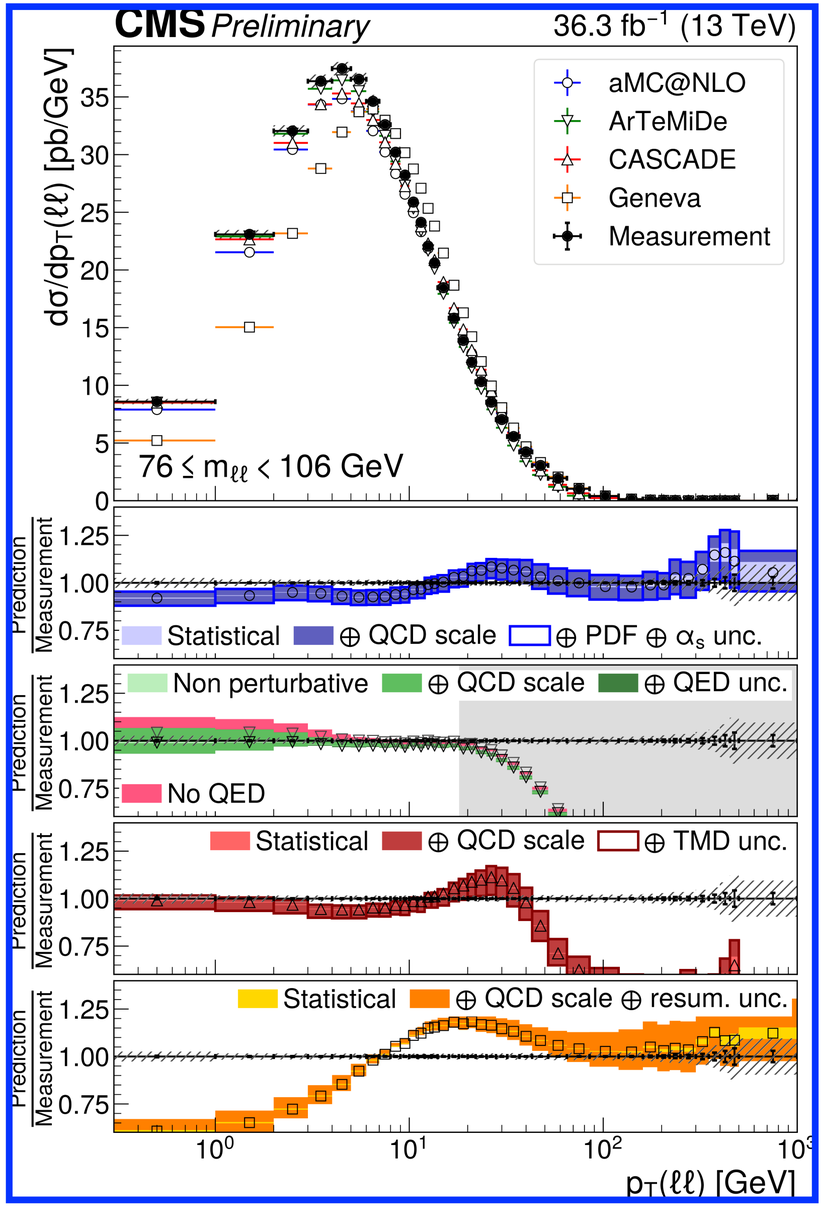
Sudakov form factor: no emissions harder than q_T

- Differentiate this with respect to q_T^2 to get the differential cross section, with the leading effect of soft photons resummed to all orders (*double leading-log approximation*):

$$\frac{d\sigma_{DLLA}}{dq_T^2} = \sigma_0 \frac{\alpha}{2\pi}\mathrm{Log}\frac{s}{q_T^2}\mathrm{Exp}\left[-\frac{\alpha}{2\pi}\mathrm{Log}^2\frac{s}{q_T^2}\right]$$

The p_T distribution in QCD

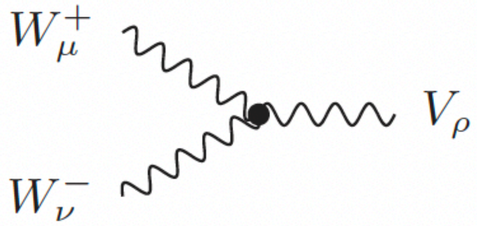
- This expression reproduces the peak at low q_T observed in data; simple to plot in Mathematica and check
- There is some work to convert this to QCD: change $\alpha \rightarrow \alpha_s C_F$; need to deal with the strong coupling becoming large at low q_T; sub-leading logarithms important. The original literature covers these points *Collins, Soper, Sterman (1985)*
- The state-of-the-art is the resummation of these logarithms in QCD to the next-to-next-to-next-to-leading logarithmic order Bizon et al (2019)



Weak-boson self interactions

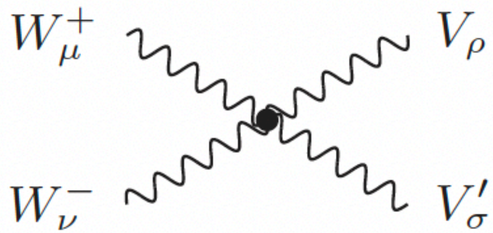
- Consider now the production of multiple gauge bosons with at least one W or Z
- They differ from di-photon production through their self-interactions

Feynman rules for gauge-boson self-interactions



$$\mathrm{i}eC_{WWV}\left[g_{\mu\nu}(k_+ - k_-)_\rho + g_{\nu\rho}(k_- - k_V)_\mu + g_{\rho\mu}(k_V - k_+)_\nu\right]$$

with $C_{WW\gamma} = 1, \quad C_{WWZ} = -\frac{c_W}{s_W}$



$$\mathrm{i}e^2 C_{WWVV'}\left[2g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\nu\rho}\right]$$

with $C_{W^2\gamma^2} = -1, C_{W^2\gamma Z} = \frac{c_W}{s_W}, C_{W^2Z^2} = -\frac{c_W^2}{s_W^2}, C_{W^4} = \frac{1}{s_W^2}$

Weak-boson self interactions

- Consider now multiple production of gauge bosons with at least one W or Z
- They differ from di-photon production through their self-interactions
- Their self-interactions allow multiple probes:

triple gauge couplings through:

- di-boson production (ee/pp →VV)
- Single production through VBF

quartic gauge couplings through:

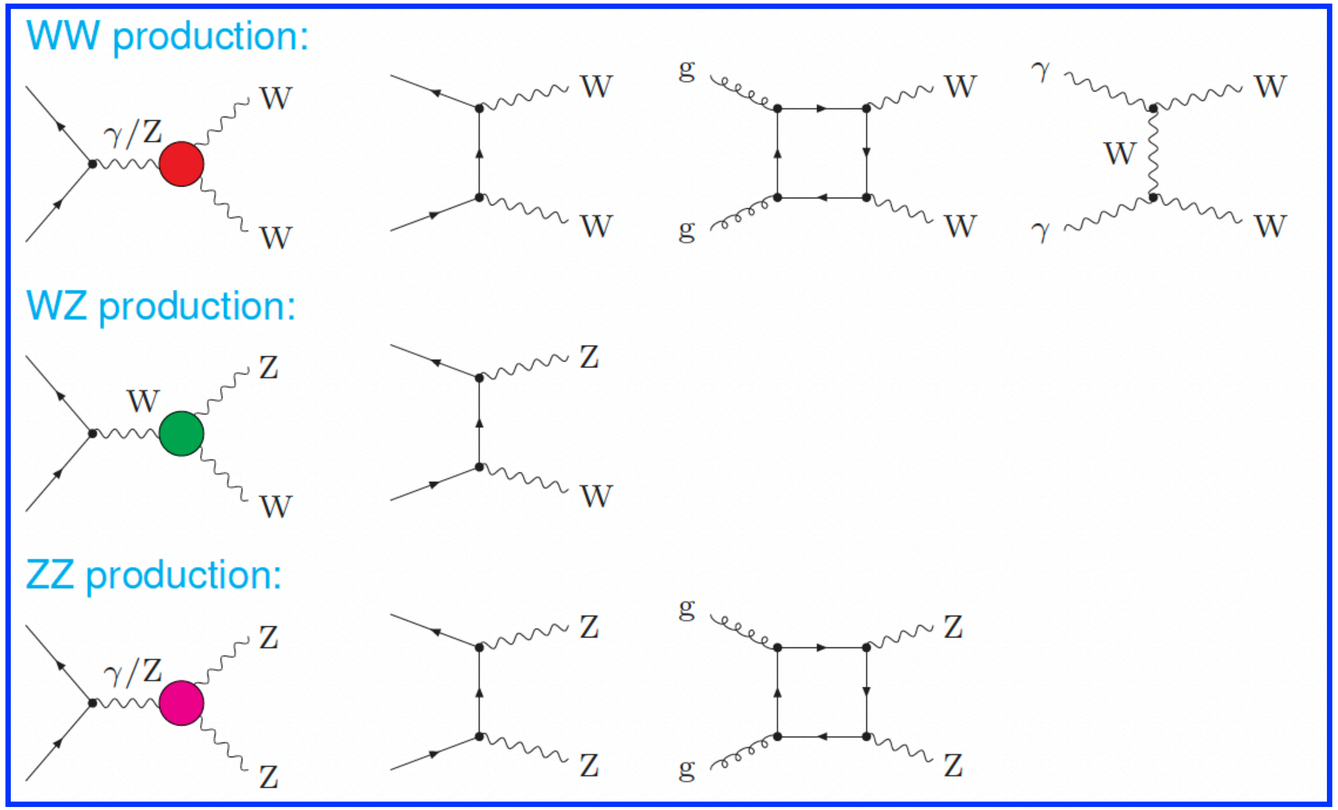
- tri-boson production (ee/pp→VVV)
- di-boson production in VBS

A rich structure predicted by the SM Lagrangian
that can be tested in all these processes

WW/WZ/ZZ production

- The WW/WZ/ZZ production provide important complementary information in various aspects of physics

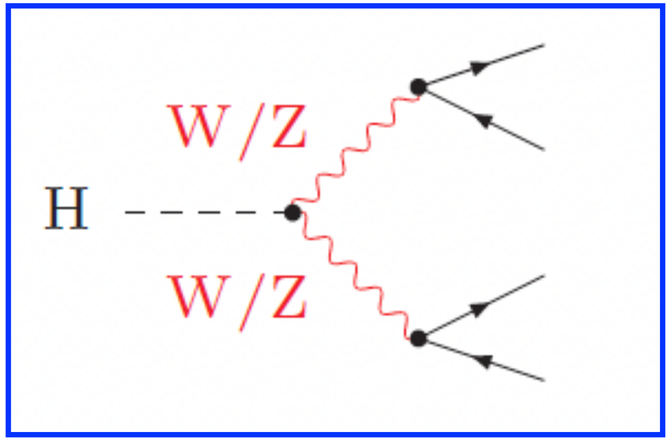
- Sensitivity to different anomalous triple-gauge-couplings (aTGCs)
  - Mixture of γWW/ZWW in WW
  - Only ZWW in WZ
  - γZZ/ZZZ in ZZ



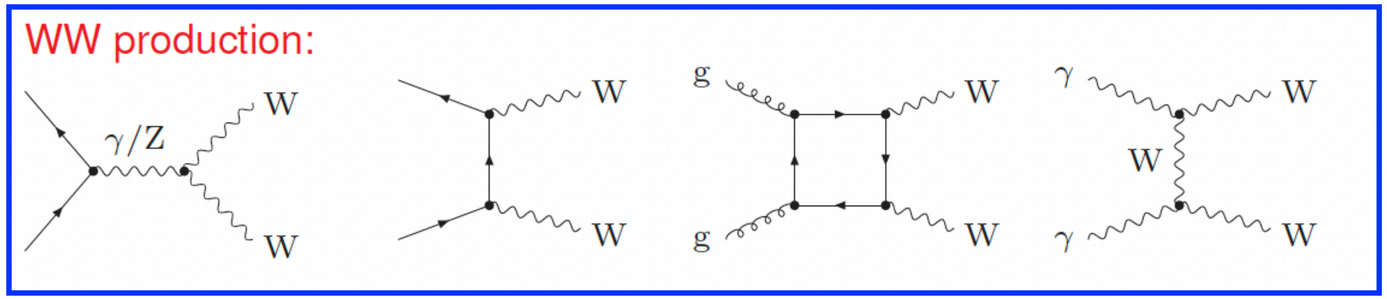
WW/WZ/ZZ production

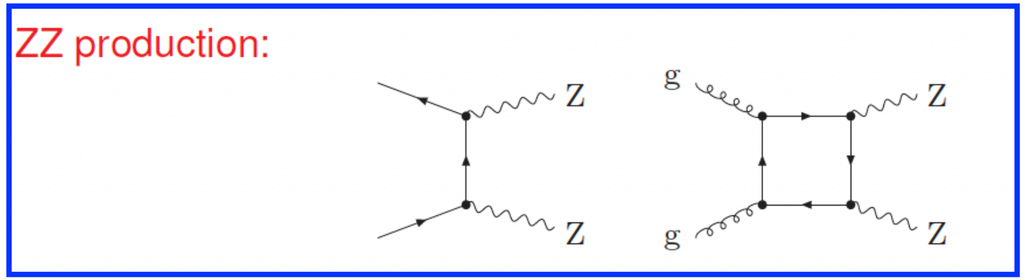
- The WW/WZ/ZZ production provide important complementary information in various aspects of physics

- Background to Higgs production in the channel: H→WW*/ZZ*→4f

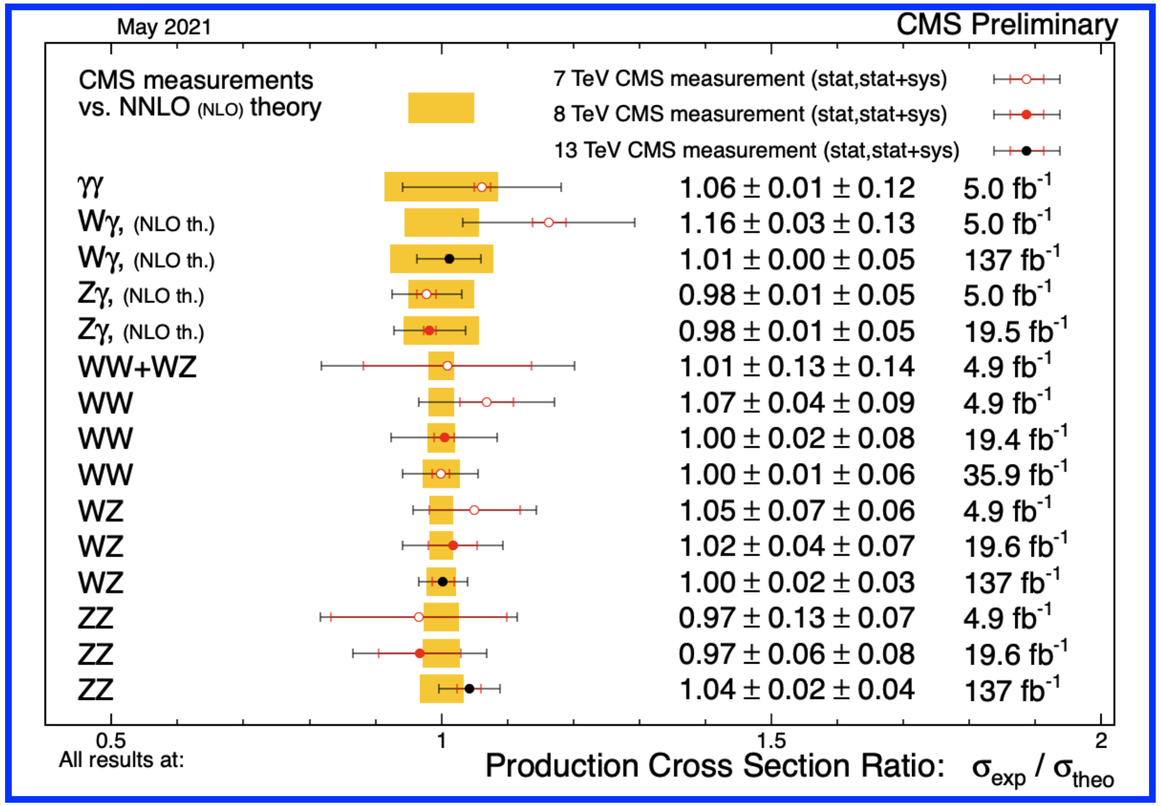


Off-shell calculation particularly important in this case!

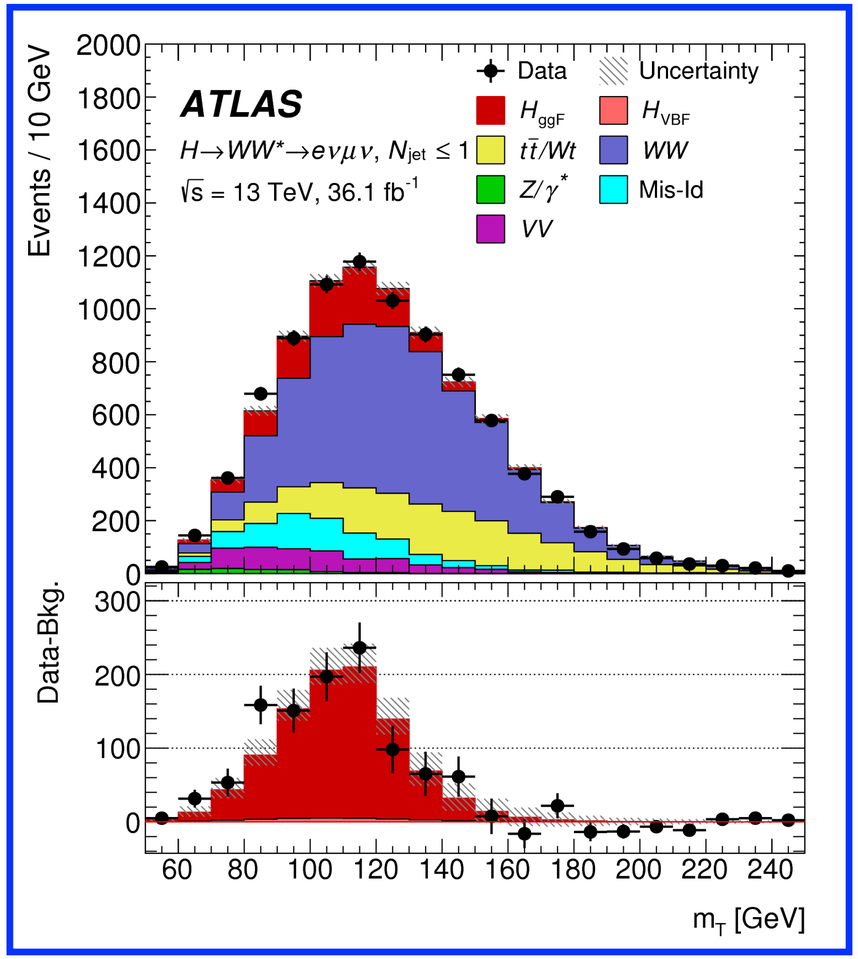




Di-boson experimental summary



- Good agreement between experiment and SM, which is computed to the NLO and often the NNLO in QCD

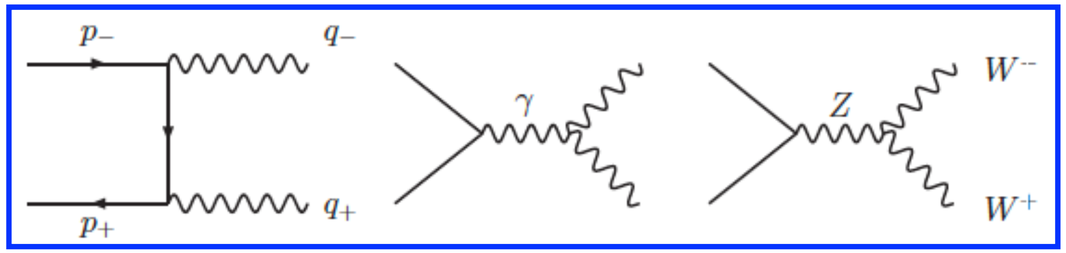


- This precision understanding has been pivotal in controlling backgrounds to Higgs production at the LHC

Di-boson production and unitarity

- Let's test the importance of gauge-boson self interactions by looking at the unitarity of gauge-boson scattering. It turns out that the production of longitudinal vector bosons is extremely sensitive to the gauge structure of the SM. Deviations in the couplings from the SM predictions can disturb delicate cancellations between diagrams in the SM and indicate the presence of heavy new physics.

Example: uubar→W^+W^-



W± momenta:

$$q^{\pm} = (E, 0, 0, \pm q) \quad ; E^2 - q^2 = M_W^2$$

Polarization vectors satisfy:

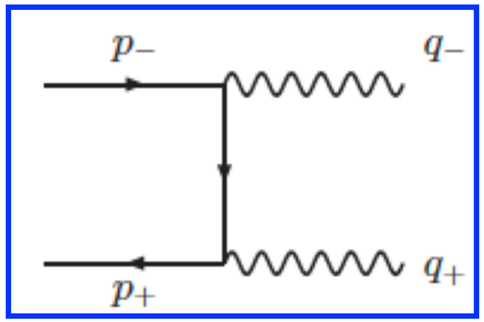
$$\epsilon \cdot q = 0$$

Transverse polarizations: $\epsilon = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$

Longitudinal polarizations: $\epsilon_{\pm} = \frac{1}{m_W}(q, 0, 0, \pm E) \approx \frac{q^{\pm}}{M_W}$ in the high-energy limit where E>>M_W

Longitudinal matrix elements

- Let's calculate each diagram using the Feynman rules presented earlier



$$\mathcal{M}_1 = -i\bar{v}(p_+)\left(\frac{-ig_w}{\sqrt{2}}\right)\not{\epsilon}_+\frac{i(\not{p}_- - \not{q}_-)}{(p_- - q_-)^2}\left(\frac{-ig_w}{\sqrt{2}}\right)\not{\epsilon}_-\frac{1-\gamma_5}{2}u(p_-)$$

Replace the polarization vectors with the longitudinal ones:

$$\epsilon_\pm = \frac{q^\pm}{M_W}$$

$$\mathcal{M}_1 = -\frac{g_w^2}{4M_W^2}\bar{v}(p_+)\not{q}_+\frac{\not{p}_- - \not{q}_-}{(p_- - q_-)^2}\not{q}_-(1-\gamma_5)u(p_-)$$

In the high-energy limit we can replace:

$$\not{q}_-\not{q}_- = q_-^2 \approx 0$$

we're keeping only the leading term in E^2/M_W^2

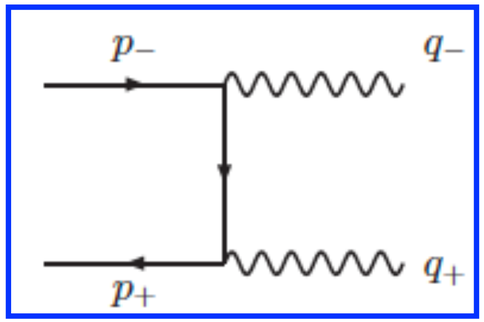
We can also use the massless Dirac equation:

$$\not{p}_- u(p_-) = 0$$

$$\mathcal{M}_1 = \frac{g_w^2}{4M_W^2}\bar{v}(p_+)\not{q}_+(1-\gamma_5)u(p_-)$$

Longitudinal matrix elements

- Let's calculate each diagram using the Feynman rules presented earlier

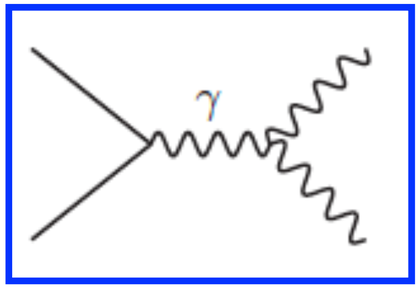


$$\mathcal{M}_1 = \frac{g_w^2}{4M_W^2}\bar{v}(p_+)\not{\epsilon}_+(1-\gamma_5)u(p_-)$$

- This factor grows as E^2; can see this from dimensional analysis (spinors grow as √Energy) or explicit calculation
- The E^2/M_W^2 growth leads to *unitarity violation*
- The cross section from this amplitude would grow too large at high energy; probability conservation would be violated

Must find that this behavior is cancelled by the other diagrams

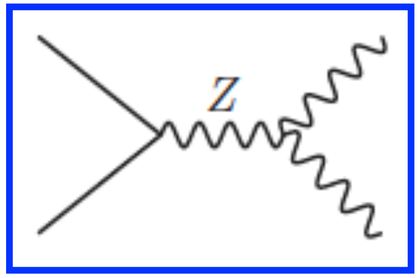
Longitudinal matrix elements



$$\mathcal{M}_2 = -i\bar{v}(p_+)(-ieQ_u\gamma^\mu u(p_-)\frac{-ig_{\mu\nu}}{(q_- + q_+)^2}$$
$$\times(ie)\left\{(q_- - q_+)^\mu \epsilon_+ \cdot \epsilon_- + 2q_+ \cdot \epsilon_- \epsilon_+^\mu - 2q_- \cdot \epsilon_+ \epsilon_-^\mu\right\}$$

Replace the polarization vectors with the longitudinal ones; use momentum conservation and also the following SM relation: $e = g_w \sin(\theta_w)$

$$\mathcal{M}_2 = -\frac{g_w^2 \sin^2\theta_w Q_u}{M_W^2}\bar{v}(p_+)\not{q}_+ u(p_-)$$



$$\mathcal{M}_3 = -\frac{g_w^2}{2M_W^2}\bar{v}(p_+)\not{q}_+(V_u - A_u\gamma_5)u(p_-)$$

Cancellation of E^2 growth

- The sum of these three diagrams is derived to be:

$$\mathcal{M} = -\frac{g_w^2}{2M_W^2}\bar{v}(p_+)\left\{-\frac{1}{2} + V_u + 2Q_u \sin^2\theta_w + \gamma_5\left[\frac{1}{2} - A_u\right]\right\}u(p_-)$$

The SM requires the following couplings for gauge invariance:

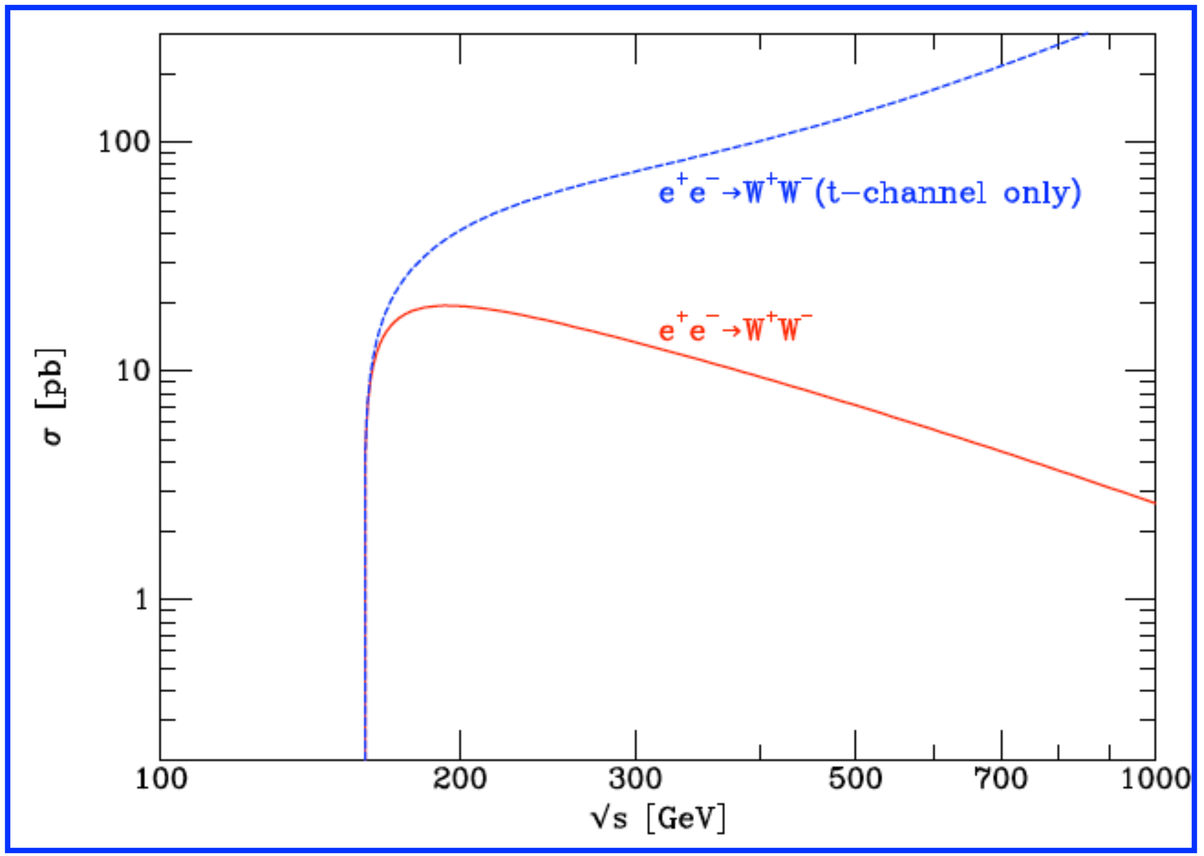
$$V_u = \frac{1}{2} - 2Q_u \sin^2\theta_w$$

$$A_u = \frac{1}{2}$$

- The E^2 growth cancels due to the relations between triple-gauge and fermion-gauge couplings required by the SM
- These relations between couplings can be tested in high-energy measurements of gauge boson production at the LHC

Longitudinal matrix elements

- Numerical results with only the first diagram vs with all the diagrams for the similar process $e^+e^- \to W^+W^-$



J.Campbell

Summary of lecture II

- What we covered in this lecture:
  - A look under the hood of Drell-Yan production at NLO in QCD: soft, collinear and UV divergences, dimensional regularization, and mass factorization.
  - The phenomenology and status of vector boson plus jet production at hadron colliders: V+up to five jets at NLO, and V+1-jet to NNLO in QCD; the impact of electroweak corrections
  - The transverse momentum distribution of W/Z bosons: exponentiation of soft photons and the resummation of low-p_T logarithms.
  - Multi-gauge boson production and unitarity constraints on gauge-boson couplings.