

α_s from lattice QCD

Alberto Ramos <alberto.ramos@ific.uv.es>

Mattia Dalla Brida, Roman Höllwieser, Francesco Knechtli, Tomasz Korzec, Rainer Sommer.
Based on:

- *Non-perturbative renormalization by decoupling.* [arXiv: 1912.06001]
- *Determination of $\alpha_s(m_Z)$ by the non-perturbative decoupling method.* [arXiv: 2202.XXXXX]

Luigi Del Debbio

- *Lattice determinations of α_s .* Physics Reports 920 (2021) [arXiv: 2101.04762]

FLAG review (R. Horsley, P. Petreczky, S. Sint)

- <http://flag.unibe.ch/2021/>



LATTICE DETERMINATION OF α_s

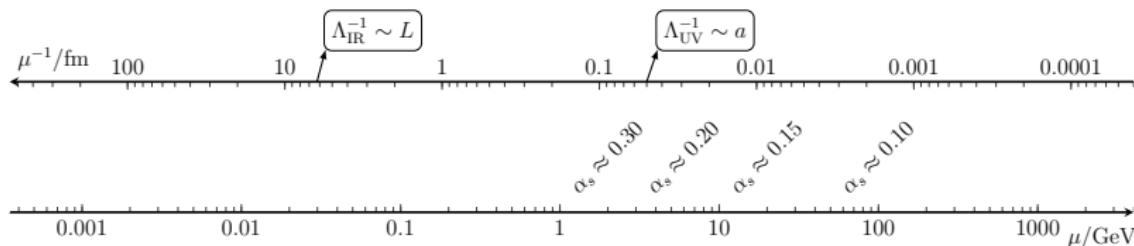
$$O(Q) \xrightarrow{Q \rightarrow \infty} \alpha_s(Q) + \sum_{n=2}^N c_n \alpha_s^n(Q) + \alpha_s^{N+1}(Q) + \left(\frac{\Lambda}{Q}\right)^p + \dots$$

Run to a convenient scale (i.e. M_Z), possibly crossing c, b quark thresholds

$$\alpha_s(Q) \longrightarrow \alpha_s(M_Z)$$

Uncertainties in $\alpha_s(M_Z)$:

- ▶ Non-perturbative uncertainties $\propto \left(\frac{\Lambda}{Q}\right)^p$
- ▶ Perturbative uncertainties $\propto \alpha_s^{N+1}(Q)$



Computational cost $\implies (L/a)^7$
 Perturbative uncertainties $\implies \log(L/a)^\#$

LATTICE DETERMINATION OF α_s

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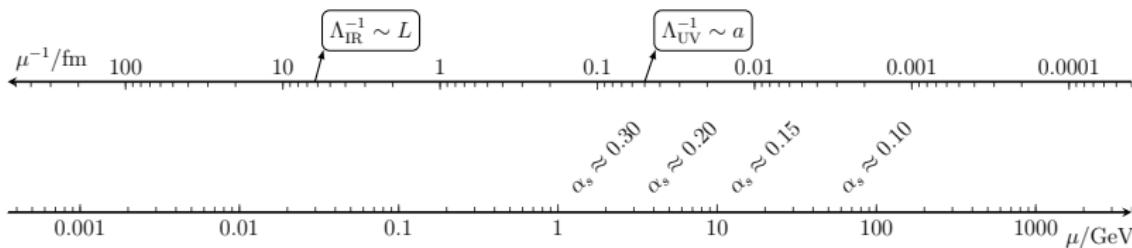
Run to a convenient scale (i.e. M_Z), possibly crossing c, b quark thresholds

$$\alpha_s(Q) \longrightarrow \alpha_s(M_Z)$$

Uncertainties in $\alpha_s(M_Z)$:

Exponentially hard problem!

- ▶ Perturbative uncertainties $\propto \alpha_s^{N+1}(Q)$

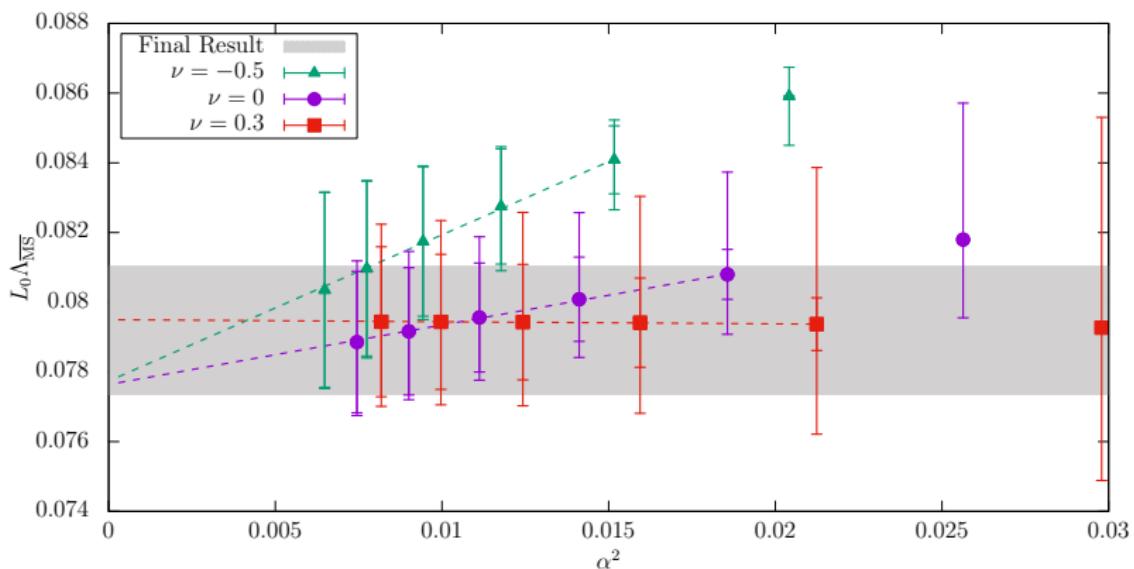


Computational cost $\implies (L/a)^7$
 Perturbative uncertainties $\implies \log(L/a)^\#$

PERTURBATIVE UNCERTAINTIES ARE DIFFICULT TO ESTIMATE

[ALPHA '16, ALPHA '18]

$$O(Q) \xrightarrow{Q \rightarrow \infty} \alpha_s(Q) + \sum_{n=2}^N c_n \alpha_s^n(Q) + \alpha_s^{N+1}(Q) + \left(\frac{\Lambda}{Q}\right)^p + \dots$$



“Non-perturbative extrapolation”

Probe different values of α , see scaling towards $\alpha \rightarrow 0$.

PERTURBATIVE UNCERTAINTIES BY SCALE VARIATION METHOD

[L. DEL DEBBIO, A. RAMOS. PHYS. REP. (2021) 970 [ARXIV:2101.04762]]

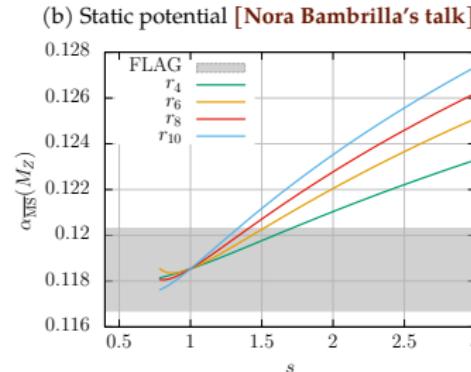
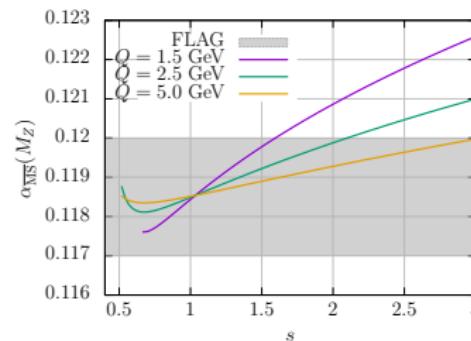
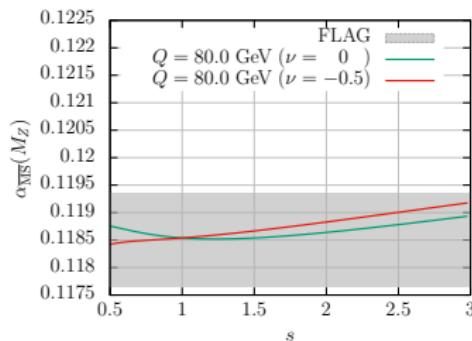
$$O(Q) \xrightarrow{Q \rightarrow \infty} \alpha_s(sQ) + \sum_{n=2}^N c_n(s) \alpha_s^n(sQ) + \alpha_s^{N+1}(sQ) + \left(\frac{\Lambda}{Q} \right)^p + \dots$$

Dependence on s is spurious

- ▶ Vary s around a reference value s^{ref} , and see how extracted $\alpha_s(M_Z)$ changes
- ▶ Some preferred values for s^{ref} : $s^{\text{ref}} = 1, s^{\text{ref}}$ s.t. $c_2(s^{\text{ref}}) = 0$
- ▶ “Easy”: Only input needed is $Q, c_n(1)$

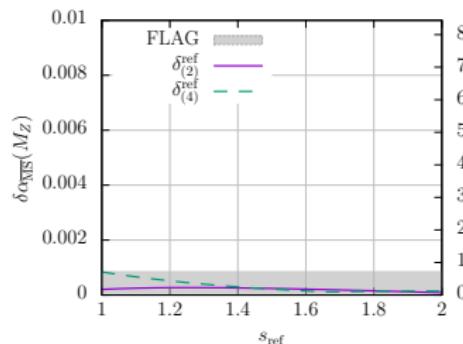
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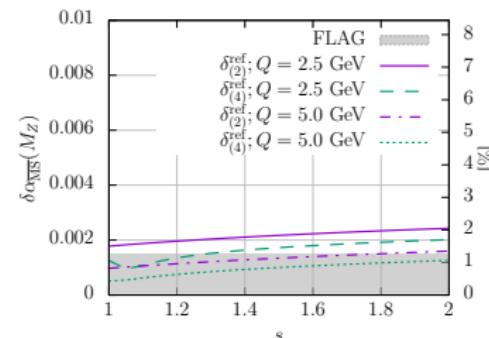


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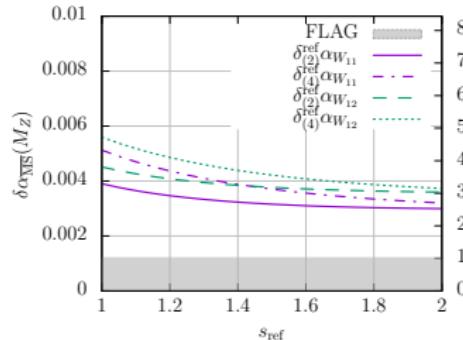
[L. DEL DEBBIO, A. RAMOS. PHYS.REP. (2021) 970 [ARXIV:2101.04762]]



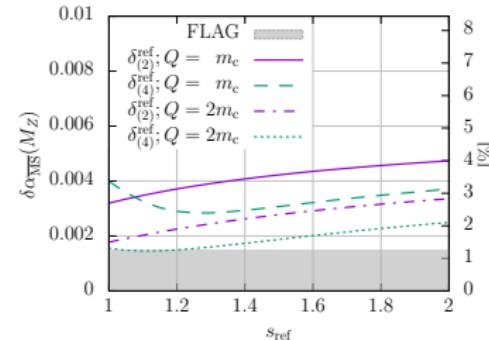
(a) Step Scaling [Mattia Dalla Brida's talk]



(b) Static potential [Nora Bambrilla's talk]



(c) Wilson Loops



(d) Heavy quark correlators [Webber's talk]

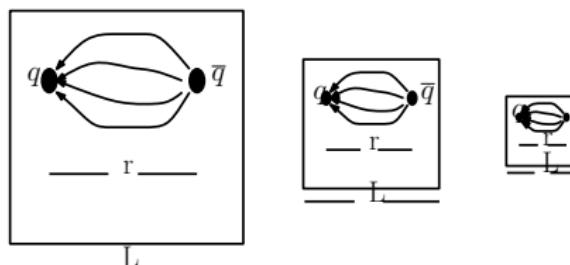
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Scale variation approach to perturbative uncertainties

- ▶ All $\alpha_s(M_Z)$ extractions can be evaluated on equal-footing.
- ▶ Maybe also non-lattice extractions??
- ▶ Freely available software:
<https://igit.ific.uv.es/alramos/scaleerrors.jl>
- ▶ This method **can fail!**: No substitute for “non-perturbative” extrapolation

THE SOLUTION: FINITE SIZE SCALING [LÜSCHER, WEISZ, WOLFF '91]



Finite volume renormalization schemes: fix $QL = \text{constant}$

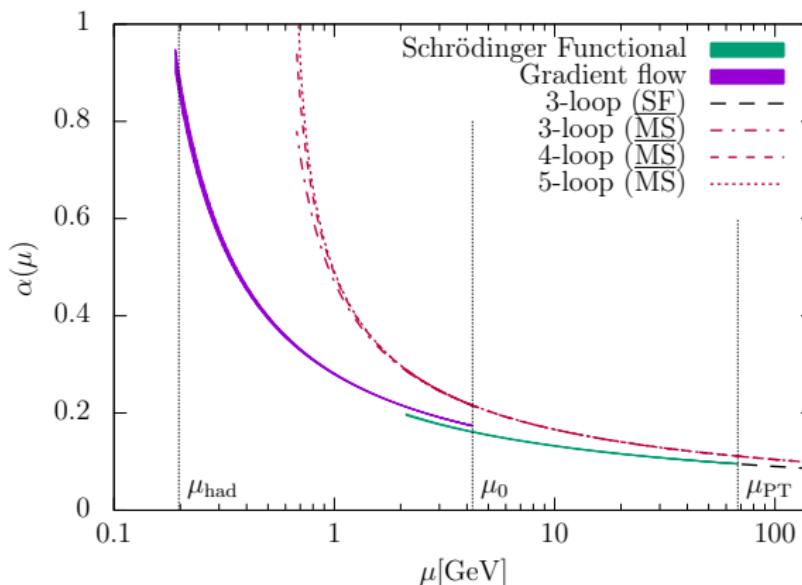
- ▶ Coupling $\alpha(Q)$ depends on no other scale but L (Notation: $\alpha(L), \alpha(1/L)$).
- ▶ Small $L \implies$ small $\alpha(L)$
- ▶ $a \ll 1/Q$ easily achieved: $L/a \sim 10 - 40$
- ▶ Step scaling function: How much changes the coupling when we change the renormalization scale:

$$\sigma(u) = g^2(Q/2) \Big|_{g^2(Q)=u}$$

achieved by simple changing $L/a \rightarrow 2L/a!$

- ▶ $1/L$ is a IR cutoff \Rightarrow simulate directly $m_q = 0$
- ▶ We need dedicated simulations of the **femto-universe**

RESULTS FOR $\alpha_s(M_Z)$ [ALPHA '17. PHYS.REV.LETT (2017) 119. [ARXIV:1706.03821]]



- ▶ Non-perturbative running from 200 MeV to 140 GeV
- ▶ Many technical improvements:
 - ▶ Gradient flow couplings
 - ▶ Symanzik analysis of cutoff effects
 - ▶ ...

$$\alpha_s(M_Z) = 0.11852(84) [0.7\%].$$

3M: A UNIVERSE WITH THREE HEAVY DEGENERATE QUARKS ($M \gg \Lambda$)

Alice uses fundamental theory

$$S_{\text{fund}}[A_\mu, \psi, \bar{\psi}] = \int d^4x \left\{ -\frac{1}{2g^2} \text{Tr}(F_{\mu\nu}F_{\mu\nu}) + \sum_{i=1}^3 \bar{\psi}_i(\gamma_\mu D_\mu + M)\psi_i \right\}$$

Bob uses effective theory

$$S_{\text{eff}}[A_\mu] = -\frac{1}{2g_{\text{eff}}^2} \int d^4x \{\text{Tr}(F_{\mu\nu}F_{\mu\nu})\} + \frac{1}{M^2} \sum_k \omega_k \int d^4x \mathcal{L}_k^{(6)} + \dots$$

3M: A UNIVERSE WITH THREE HEAVY DEGENERATE QUARKS ($M \gg \Lambda$)

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Decoupling

- Dimensionless “low energy quantities” $\sqrt{t_0}/r_0, w_0/\sqrt{8t_0}, r_0/w_0, \dots$ from effective theory

$$\frac{\mu_1^{\text{fund}}(M)}{\mu_2^{\text{fund}}(M)} = \frac{\mu_1}{\mu_2} + \mathcal{O}\left(\frac{\mu^2}{M^2}\right)$$

RENORMALIZATION IN 3M: ALICE DETERMINES THE STRONG COUPLING

$$\frac{\Lambda}{\mu} = \left[b_0 \bar{g}^2(\mu) \right]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0 \bar{g}^2(\mu)}} \exp \left\{ - \int_0^{\bar{g}(\mu)} dx \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\}.$$

- ▶ Determine non-perturbatively the β -function in the fundamental ($N_f = 3$) theory, mass-less scheme.
- ▶ Integral up to $\bar{g}^{(3)}(\mu_{\text{massless}})$ (in a mass-less scheme!) gives:

$$\frac{\Lambda^{(3)}}{\mu_{\text{massless}}}$$

- ▶ Only needs to compute a dimensionless ratio with a physical scale

$$\frac{\mu_{\text{massless}}}{\mu_{\text{ref}}(M)}$$

- ▶ Result

$$\frac{\Lambda^{(3)}}{\mu_{\text{ref}}(M)} = \frac{\Lambda^{(3)}}{\mu_{\text{massless}}} \times \frac{\mu_{\text{massless}}}{\mu_{\text{ref}}(M)}$$

RENORMALIZATION IN 3M: BOB DETERMINES THE STRONG COUPLING

$$\frac{\Lambda}{\mu} = \left[b_0 \bar{g}^2(\mu) \right]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0 \bar{g}^2(\mu)}} \exp \left\{ - \int_0^{\bar{g}(\mu)} dx \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\}.$$

- ▶ Determine non-perturbatively the β -function in the effective ($N_f = 0$) theory.
- ▶ Integral up to $\bar{g}^{(0)}(\mu_{\text{ref}})$ gives:

$$\frac{\Lambda^{(0)}}{\mu_{\text{ref}}}$$

- ▶ Match across quark threshold to convert to $\Lambda^{(3)}$ (using perturbation theory)

$$\frac{\Lambda^{(3)}}{\mu_{\text{ref}}} = \frac{\Lambda^{(0)}}{\mu_{\text{ref}}} \times \frac{1}{P(\Lambda/M)}.$$

- ▶ Matching factor $P(\Lambda/M)$ [ALPHA 1809.03383]:
 - ▶ Known in perturbation theory up to three-loops. Power series in $\alpha(m^*)$
 - ▶ “Good” PT: corrections very small even at m_c^* .
 - ▶ Perspective: we simulate $M \in [5 - 10]$ GeV

RELATION BETWEEN ALICE AND BOB COMPUTATION

Relation between Alice and Bob results:

$$\frac{\Lambda^{(3)}}{\mu_{\text{ref}}(M)} = \frac{\Lambda^{(0)}}{\mu_{\text{ref}}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^3(m^*)) + \mathcal{O}\left(\frac{\mu_{\text{dec}}^2}{M^2}\right)$$

Bob is telling us that $\Lambda^{(3)}$ can be computed from $\Lambda^{(0)}$

$$\Lambda^{(3)} = \lim_{M \rightarrow \infty} \mu_{\text{ref}}(M) \times \frac{\Lambda^{(0)}}{\mu_{\text{ref}}} \times \frac{1}{P(\Lambda/M)}$$

We need

- ▶ Running in pure gauge: $\frac{\Lambda^{(0)}}{\mu_{\text{ref}}}$
- ▶ A scale in a world with degenerate massive quarks: $\mu_{\text{ref}}(M)$ in fm/MeV.

Lattice QCD can simulate *unphysical* worlds

$$\mu_{\text{ref}}(M) = \mu_{\text{ref}}^{\text{phys}} \times \frac{\mu_{\text{ref}}(M)}{\mu_{\text{ref}}^{\text{phys}}}$$

with, for example: $\mu_{\text{ref}}^{\text{phys}} = \frac{1}{\sqrt{8t_0^{\text{phys}}}} = 0.415(4)(2)$ fm from [Bruno et al. '17]

NON-PERTURBATIVE RENORMALIZATION BY DECOUPLING

Master relation

$$\frac{\Lambda^{(N_f)}}{\mu_{\text{dec}}(M)} = \frac{\Lambda^{(0)}}{\mu_{\text{dec}}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^3(m^\star)) + \mathcal{O}\left(\frac{\mu_{\text{dec}}^2}{M^2}\right)$$

With a proper limit:

$$\Lambda^{(N_f)} = \lim_{M \rightarrow \infty} \mu_{\text{dec}}(M) \times \frac{\Lambda^{(0)}}{\mu_{\text{dec}}} \times \frac{1}{P(\Lambda/M)}$$

Where

- ▶ Pure gauge running: $\Lambda^{(0)} / \mu_{\text{dec}}$
- ▶ A scale with N_f massive degenerate quarks: $\mu_{\text{dec}}(M)$

NOTE: this is not completely trivial: still a multi-scale problem

$$\frac{\mu_{\text{dec}}(M)}{\mu_{\text{dec}}^{\text{phys}}} = \lim_{a \rightarrow 0} \frac{a \mu_{\text{dec}}(M)}{a \mu_{\text{dec}}^{\text{phys}}}$$

is a difficult extrapolation (M wants to be large, aM wants to be small).

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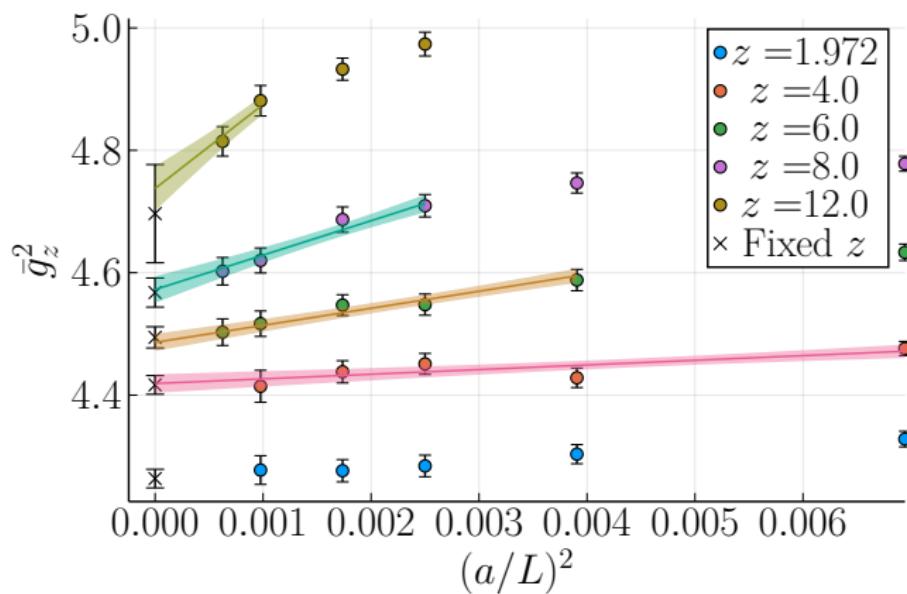
Why is this better?

- ▶ We will use $M \approx 5 - 10 \text{ GeV}$
- ▶ Corrections $\mathcal{O}(\alpha^3(m^\star))$ completely negligible
- ▶ “Exponential improvement”:
 - ▶ Computational cost $\Rightarrow (L/a)^7$
 - ▶ Uncertainties decrease as $\Rightarrow (L/a)^2$

RESULTS FOR $\alpha_s(M_Z)$: CONTINUUM EXTRAPOLATION

$$\bar{g}_{z_i}^2 = c_i + p_1[\alpha_s(a^{-1})]^{\Gamma_1} \left(\frac{a}{L}\right)^2 + p_2[\alpha_s(a^{-1})]^{\Gamma_1} (aM)^2$$

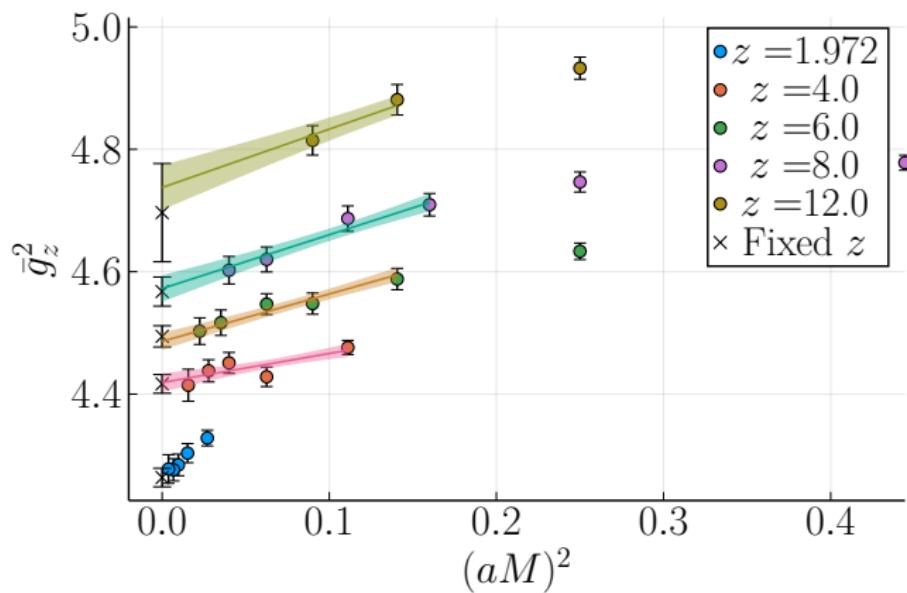
- ▶ No (aM/L^2) terms [Mattia Dalla Brida's talk]
- ▶ $\log a$ corrections to a^2 scaling [Husung, Mardquat, Sommer, '21]



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WHERE WE STAND?

$$\Lambda^{(3)} = \mu_{\text{dec}}(M) \times \frac{\Lambda^{(0)}}{\mu_{\text{dec}}} \times \frac{1}{P(\Lambda/M)} + \cancel{\mathcal{O}(\alpha^3(m^*))} + \cancel{\mathcal{O}\left(\frac{\mu_{\text{dec}}}{M}\right)} + \mathcal{O}\left(\frac{\mu_{\text{dec}}^2}{M^2}\right)$$

- ▶ We have $\mu_{\text{dec}}(M) = 788(15)$ MeV.
- ▶ We have continuum values of the massive coupling (3-degenerate quarks):

$$\bar{g}_{z_i}^2 \quad \text{for } z_i = 1.972, 4, 6, 8, 12 \quad (M \approx 1.6 - 9.5 \text{ GeV}).$$

- ▶ Decoupling tells us:

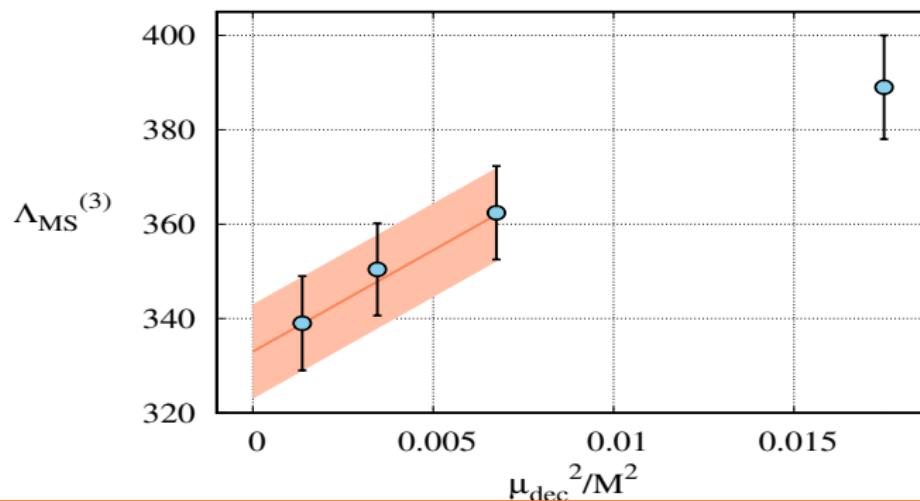
$$\bar{g}_{z_i}^2 \sim \text{Pure gauge coupling!}$$

(i.e. **up to heavy mass corrections** we have the value of the pure gauge coupling

- ▶ Thanks to [M. Dalla Brida, A. Ramos '19] we have $\frac{\Lambda^{(0)}}{\mu_{\text{dec}}}$
- ▶ $P(\Lambda/M)$ easily evaluated in P.T.

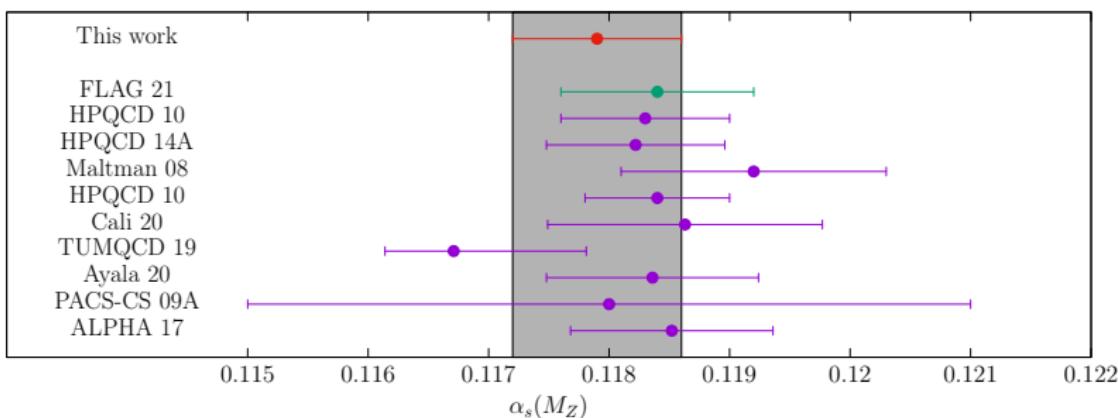
$M \rightarrow \infty$ EXTRAPOLATION: $\Lambda_{est}^{(3)} = \Lambda^{(3)} + \frac{1}{z^2} [\alpha_s(m^*)]^{\Gamma'}$

c	$z \geq 4$				$z \geq 6$			
	$\Lambda_{\text{MS}}^{(3)}$	χ^2	$\langle \chi^2 \rangle$	$\chi^2/\langle \chi^2 \rangle$	$\Lambda_{\text{MS}}^{(3)}$	χ^2	$\langle \chi^2 \rangle$	$\chi^2/\langle \chi^2 \rangle$
0.30	339.2(9.9)	0.22	0.03	7.33	335(10)	0.025	0.013	1.88
0.33	337.2(9.9)	0.18	0.04	4.78	333(10)	0.020	0.017	1.15
0.36	336(10)	0.15	0.05	3.17	332(10)	0.015	0.022	0.68
0.39	335(10)	0.12	0.06	2.09	331(11)	0.010	0.028	0.36
0.42	334(10)	0.10	0.07	1.45	331(11)	0.007	0.035	0.20



FINAL RESULT

$$\Lambda_{\overline{\text{MS}}}^{(3)} = 332(10) \text{ MeV}; \quad \alpha_s(M_Z) = 0.1179(7).$$



CONCLUSIONS

- ▶ Extraction of α_s on the lattice is a **very hard** multi-scale problem
 - ▶ Computational cost $\Rightarrow (L/a)^7$
 - ▶ Perturbative uncertainties $\Rightarrow \log(L/a)^\#$
- ▶ Perturbative uncertainties hard to estimated with data in a limited range of scales
- ▶ Still, several determinations (with compromises) achieve/will achieve $\delta\alpha_s \approx 1 - 2\%$
- ▶ Perturbative uncertainties using scale variation are a guide: Common framework to all approaches? [L. Del Debbio, A. Ramos Phys.Rep.(2021)190]
- ▶ One **real solution**: Step scaling
 - ▶ Non-perturbative running from 200 MeV to 140 GeV: $\alpha_s(M_Z) = 0.1185(8)$ [0.7%]
- ▶ *Exponential improvement* (still a multi-scale problem): Decoupling of heavy quarks
 - ▶ Perturbative uncertainties negligible ($M \approx 10$ GeV)
 - ▶ Non-perturbative corrections can be extrapolated
 - ▶ Relies on *pure gauge determinations* of $\Lambda^{(0)}$
 - ▶ Precise result: $\alpha_s(M_Z) = 0.1179(7)$ [0.6%]
- ▶ $\delta\alpha_s(M_Z) \approx 0.4\%$ certainly possible (uncertainties dominated by pure gauge (!!)) and low energy running (!)).
- ▶ $\delta\alpha_s(M_Z) < 0.3\%$ requires serious thinking.

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Personal opinion

Future belongs to **dedicated** approaches, **not** to beating an exponentially hard problem with your machines

Many thanks!

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- ▶ Precise result: $\alpha_s(M_Z) = 0.1179(7)$ [0.6%]
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