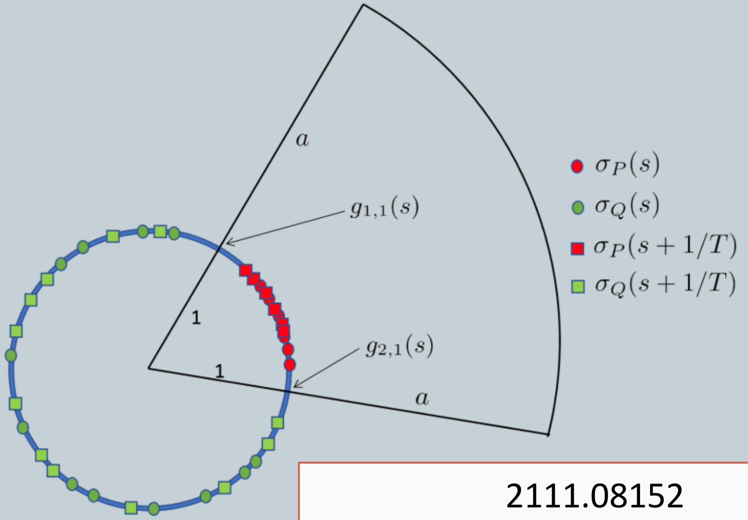
Optimal scaling quantum linear systems solver via discrete adiabatic theorem



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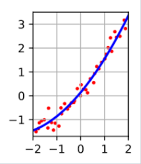
Why do we care?

- **Basis for a whole host of new quantum algorithms:**

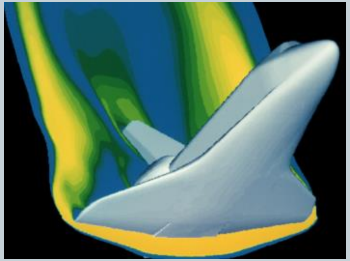
➢ Machine Learning

➢ Data analysis

➢ Least-squares fitting



➢ Solution of partial differential equation



Quantum linear systems problem

- Need to solve

$$A\vec{x} = \vec{b}$$

- Initially prepare state

$$|b\rangle = \sum_{j=1}^{N} b_j |j\rangle$$

- Apply operation A^{-1}

$$|x\rangle \sim A^{-1}|b\rangle$$

A.W. Harrow, A. Hassidim, S. Lloyd, Physical Review Letters **103**, 150502 (2008).

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- Obtain global properties from sampling $|x\rangle$ (not explicit listing).
- Exponential speedup in N.

A.W. Harrow, A. Hassidim, S. Lloyd, Physical Review Letters **103**, 150502 (2008).

Quantum linear systems problem

Optimal Quantum linear System solver

$$\mathcal{O}\left(\varkappa \, log(1/\epsilon)\right)$$

κ – condition number
ϵ – allowable error in solution

Year	Reference	Primary innovation	Query Complexity
2008	Harrow, Hassidim, Lloyd PRL 103, 150502 (2009)	first quantum approach	$\mathcal{O}(\varkappa^2/\epsilon)$
2012	Ambainis arXiv: 1010.4458	variable-time amplitude amplification	$\mathcal{O}(\varkappa(\log(\varkappa)/\epsilon)^3)$
2017	Childs, Kothari, Somma SIAM: J. Comput. 46, 1920	Fourier/Chebyshev fitting using LCU	$\mathcal{O}(\varkappa \text{ polylog}(\varkappa/\epsilon))$
2018	Subasi, Somma, Orsucci PRL 122, 060504	adiabatic randomization method	$\mathcal{O}(\varkappa \log(\varkappa)/\epsilon)$
2019	An, Lin arXiv: 1909.05500	time-optimal adiabatic method	$\mathcal{O}(\varkappa \text{ polylog}(\varkappa/\epsilon))$
2019	Lin, Tong Quantum 4, 361 (2020)	Zeno eigenstate filtering	$\mathcal{O}(\varkappa \log(\varkappa/\epsilon))$

Outline

Adiabatic Theorems

- Continuous
- Discrete

Continuous vs Discrete

Walk steps evolution + Filtering
(Overhead in $\varkappa$ removed)

Adiabatic approach to QLSP

Adiabatic Theorems

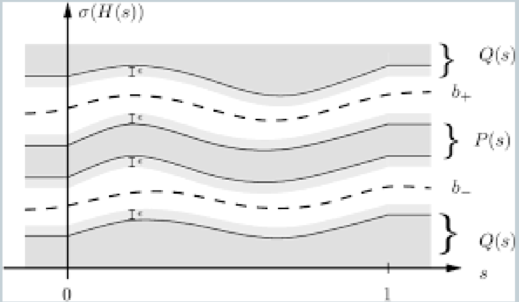
Continuous Adiabatic Theorem

- **Adiabatic Hamiltonian**

$$H_T^A(s) := H(s) + \frac{i}{T}[\dot{P}(s), P(s)], \qquad s \in [0,1]$$

Associated with

$$P(s) = \frac{1}{2\pi i}\oint_{\Gamma(s)} (H(s) - z)^{-1} dz$$



$\sigma(H(s)) \cap [b_-(s), b_+(s)]$

$$Q(s) = I - P(s)$$

Discrete Adiabatic Theorem

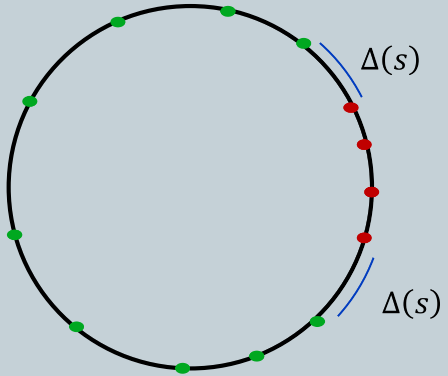
- **Adiabatic Walk**

$$W_T^A(s) := V_T(s)W_T(s)$$

where V_T is associated with

$$P_T(s) = \frac{1}{2\pi i}\oint_{\Gamma_T(s)} (W_T(s) - zI)^{-1} dz$$

- Eigenvalues on a circle in complex plane



- $\sigma_P(s)$ set of eigenvalues of $W_T(s)$ *associated with* $P_T(s)$
- $\sigma_Q(s)$ set of eigenvalues of $W_T(s)$ *associated with* $Q_T(s)$

Discrete Adiabatic Theorem

- **The goal is**

$$\left\| U_T(s) - U_T^A(s) \right\| = \left\| U_T^{A\dagger}(s) U_T(s) - I \right\| = \| \Omega_T(s) - I \|$$

where

$$U_T(s) = \prod_{n=0}^{sT-1} W_T(n/T)$$

- **Adiabatic Theorem** (DKS), Given $T \in \mathbb{N}$, let $W_T(n)$ be operators satisfying conditions $\mathrm{D}W_T(s) = W_T(s + 1/T) - W_T(s) \approx \mathcal{O}(1/T)$, then for any time s, s.t., $sT \in \mathbb{N}$ we have

$$\| \Omega_T(s) - I \| = \mathcal{O}(1/T)$$

A. Dranov, J. Kellendonk, R. Seiler, Journal of Mathematical Physics **39**, 1340 (1998).

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$$\|\Omega_T(s) - I\| = \mathcal{O}(1/T)$$

They showed that the walk is asymptotically accurate,
But nothing about the gap dependence!

A. Dranov, J. Kellendonk, R. Seiler, Journal of Mathematical Physics **39**, 1340 (1998).

Discrete Adiabatic Theorem

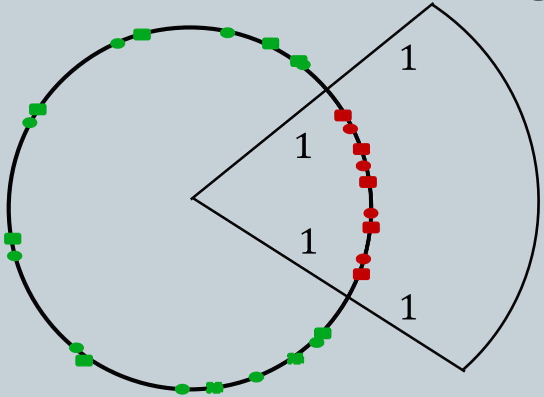
- In order to get a discrete adiabatic theorem with **strict bounds** we replace the general order scaling $DW_T(s)$ with an upper bound with explicit schedule dependence

$$\|DW_T(s)\| \leq \frac{c(s)}{T}$$

More general

$$\left\|D^{(k)}W_T(s)\right\| \leq \frac{c_k(s)}{T^k}$$

- Bound on difference of projectors needs contour threaded between eigenvalues for multiple steps.



$$\|DP_T(s)\| \leq \frac{2c_1(s)}{T\Delta_1(s)}$$

Discrete Adiabatic Theorem

- **Adiabatic Theorem**

$$\left\|U_T(s) - U_T^A(s)\right\| \le \frac{12\hat{c}_1(0)}{T\check{\Delta}(0)^2} + \frac{12\hat{c}_1(s)}{T\check{\Delta}(s)^2} + \frac{6\hat{c}_1(s)}{T\check{\Delta}(s)}$$

$$+305\sum_{n=1}^{sT-1}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^3} + 44\sum_{n=0}^{sT-1}\frac{\hat{c}_1(n/T)^2}{T^2\check{\Delta}(n/T)^2} + 32\sum_{n=1}^{sT-1}\frac{\hat{c}_2(n/T)}{T^2\check{\Delta}(n/T)^2}$$

$\check{\Delta}(s) = \min(\Delta(s - 1/T), \Delta(s), \Delta(s + 1/T))$ $\hat{c}_k(s) = max_{s' \in \{s-1/T, s, s+1/T\} \cup [0, 1-k/T]}\, c_k(s')$

- $\hat{c}_1$ – analogous to first derivative in continuous adiabatic theorem
- $\hat{c}_2$ – analogous to second derivative in continuous adiabatic theorem

Discrete Adiabatic Theorem

- **Adiabatic Theorem**

$$\left\|U_T(s) - U_T^A(s)\right\| \le \frac{12\hat{c}_1(0)}{T\breve{\Delta}(0)^2} + \frac{12\hat{c}_1(s)}{T\breve{\Delta}(s)^2} + \frac{6\hat{c}_1(s)}{T\breve{\Delta}(s)}$$

$$+305\sum_{n=1}^{sT-1}\frac{\hat{c}_1(n/T)^2}{T^2\breve{\Delta}(n/T)^3} + 44\sum_{n=0}^{sT-1}\frac{\hat{c}_1(n/T)^2}{T^2\breve{\Delta}(n/T)^2} + 32\sum_{n=1}^{sT-1}\frac{\hat{c}_2(n/T)}{T^2\breve{\Delta}(n/T)^2}$$

$\breve{\Delta}(s) = \min(\Delta(s-1/T), \Delta(s), \Delta(s+1/T))$ $\quad \hat{c}_k(s) = max_{s' \in \{s-1/T, s, s+1/T\} \cup [0, 1-k/T]}\, c_k(s')$

- $\hat{c}_1$ – analogous to first derivative in continuous adiabatic theorem
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$$\frac{1}{T}\frac{\left\|H^{(1)}(0)\right\|}{g^2(0)} + \frac{1}{T}\frac{\left\|H^{(1)}(s)\right\|}{g^2(s)} + \frac{1}{T}\int_0^s ds' \left(\frac{\left\|H^{(1)}(s')\right\|^2}{g^3(s')} + \frac{\left\|H^{(2)}(s')\right\|}{g^2(s')}\right)$$

J. Math. Phys. 48, 102111 (2007)

Continuous vs Discrete

Continuous vs Discrete

- **Continuous**

$$U(t_0, t_0 + T) = \boldsymbol{\tau} exp\left(-i\int_{t_0}^{t_0+T} dt' H(t')\right)$$

M. Kieferová, A. Scherer, and D. W. Berry, PRA **99**, 042314 (2019); 1805.00582.
G. H. Low and N. Wiebe, arXiv:1805.00675 (2018).

Continuous vs Discrete

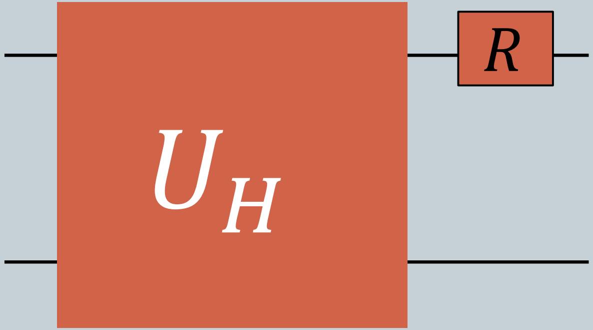
- **Continuous**

$$U(t_0, t_0 + T) = \boldsymbol{\tau} exp\left(-i\int_{t_0}^{t_0+T} dt' H(t')\right)$$

- **Discrete:** Qubitisation

Construct quantum walk using reflection

$W =$



$\equiv e^{i \arcsin H/\lambda}$

G. H. Low, I. L. Chuang, Quantum **3**, 163 (2019).

Adiabatic approach to QLSP

Adiabatic approach to QLSP

- Initial and final Hamiltonians with $Q_b = I_N - |b\rangle\langle b|$

$$H_0 = \begin{pmatrix} 0 & Q_b \\ Q_b & 0 \end{pmatrix} \qquad H_1 = \begin{pmatrix} 0 & AQ_b \\ Q_b A & 0 \end{pmatrix}$$

$$\begin{pmatrix} b \\ 0 \end{pmatrix} \qquad \begin{pmatrix} A^{-1}b \\ 0 \end{pmatrix}$$

D. An, L. Lin, arXiv:1909.05500 (2019).

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- For $H(s) = \big(1 - f(s)\big)H_0 + f(s)H_1$ gap is

$$\Delta_0(s) = 1 - f(s) + f(s)/\kappa$$

- Schedule is chosen as

$$\dot{f}(s) = d_p \Delta_0^p(s), \qquad d_p = \int_0^1 \Delta_0^{-p}(u)\, du$$

- Solution is

$$f(s) = \frac{\kappa}{\kappa - 1}\left[1 - \big(1 + s(\kappa^{p-1} - 1)\big)^{\frac{1}{1-p}}\right]$$

$$1 < p \le 2$$

D. An, L. Lin, arXiv:1909.05500 (2019).

Adiabatic approach to QLSP

- **Theorem for $p = 1.5$**

  Overall inequality

$$\left\| U_T(s) - U_T^A(s) \right\| \leq 5632 \frac{\kappa}{T} + O\left(\frac{\sqrt{\kappa}}{T}\right)$$

The discrete adiabatic evolution removes the $\mathbf{log}(\boldsymbol{\kappa}/\boldsymbol{\epsilon})$ overhead for adiabatic solution of QLSP.

Adiabatic approach to QLSP

- **Theorem for $p = 1.5$**

The discrete adiabatic evolution removes the **log(κ/ϵ)** overhead for adiabatic solution of QLSP.

Overall inequality

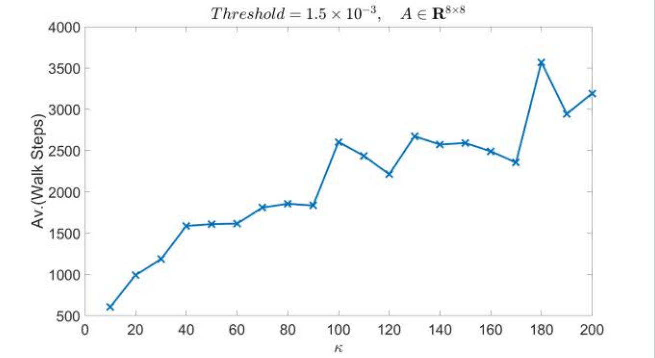
$$\left\| U_T(s) - U_T^A(s) \right\| \leq 5632\frac{\kappa}{T} + O\left(\frac{\sqrt{\kappa}}{T}\right)$$

- **Numerical testing for constant factor**

Numerically test with the complete discrete adiabatic theorem.

About 9 times better than theorem.

Numerical testing for examples matrices A for p=1.4



Conclusions

- Fastest possible quantum algorithm for the quantum linear systems problem in terms of κ and ϵ (we don't consider sparsity d).
- Lower bound of Kothari and Harrow is $\Omega\left(\sqrt{d}\kappa \log(1/\epsilon)\right)$.
- Our discrete adiabatic algorithm with qubitisation gives speedup for all adiabatic algorithms too.
- Method for LCU with two ancilla qubits can be used generally.

Future work:

- There could be scope for improving constant factors.
- How to achieve optimal scaling in d at the same time is open problem.

P. C. S. Costa, D. An, Y. R. Sanders, Y. Su, R. Babbush, D. W. Berry, PRX Quantum 3, 040303 (2022).