Quantum algorithm for time-dependent differential equations using Dyson series

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Quantum computers offer tremendous potential advantages in computing speed, but it is a long-standing challenge to find speedups to important computational tasks. A major advance for quantum algorithms was the development of a way of solving systems of linear equations by Harrow, Hassidim, and Lloyd [1]. This algorithm provides an exponential improvement in complexity in terms of the dimension as compared to classical solution, with some caveats. The matrix needs to be given in an oracular way (rather than as classical data), and the solution is encoded in amplitudes of a quantum state.

There has therefore been a great deal of follow-up work on applications of solving linear equations where the result is useful despite these limitations. For example, in discretised partial differential equations (PDEs), the discretisation yields a large set of simultaneous equations. Then one may aim to obtain some global feature of the solution which may be obtained by sampling from the quantum state [2].

There is also the question of how to solve an *ordinary* differential equation (ODE), where the task is to solve the time evolution. ODEs are very important as they are produced by the spatial discretisation of PDEs. The original algorithm for this task used linear multistep methods [3], and a further improvement was to use a series encoded in a larger system of linear equations to obtain complexity logarithmic in the allowable precision [4]. That then leaves open the question of how to solve time-dependent differential equations.

Here we provide an algorithm for time-dependent differential equations with logarithmic dependence on the error and derivative. Moreover, our algorithm only requires conditions on the first derivative, not any higher-order derivatives. We combine methods from [4] with those used for Hamiltonian evolution in [5, 6]. We use a block matrix similar to [4], but we do *not* use extra lines of the block matrix to implement terms in the series as in that work. Instead we construct this matrix via a block encoding using a Dyson series, in an analogous way as the block encodings in [5, 6]. Our method can also be used to provide a simplified approach for the time-independent case.

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