Compression of QFT states using wavelets

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The wavelet basis

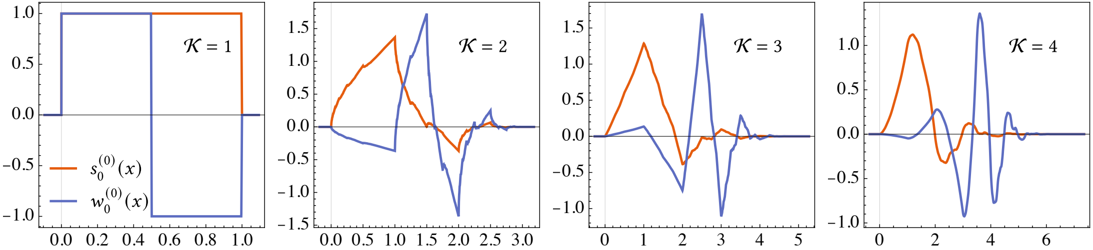
- A discrete wavelet basis is defined by a set $\{h_j\}$ of *wavelet coëfficients* from which *scaling* (or father) and *wavelet* (or mother) functions are recursively defined:

$$s(x) = \mathcal{D}\left[\sum_j h_j \mathcal{T}^j s(x)\right], \quad w(x) = \mathcal{D}\left[\sum_j (-1)^j h_{\max(j)-j} \mathcal{T}^j s(x)\right]$$

  where $\mathcal{D}f(x) = \sqrt{2}f(2x)$ and $\mathcal{T}f(x) = f(x-1)$ are dyadic scaling and translation operators.
- Scaling functions behave like **low-pass filters** and wavelet functions like **high-pass filters**.

Daubechies wavelet family

- The Daubechies db-$\mathcal{K}$ wavelet family is defined as the (unique) minimal-size set $\{h_j\}$ such that the first $\mathcal{K}$ moments of $w(x)$ vanish.

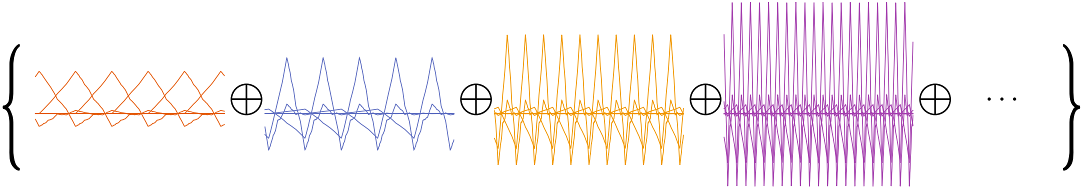


- They have particularly nice orthonormality properties (and, excepting $\mathcal{K} = 1$, no closed-form expression!):

$$\int \mathrm{d}x\, s_n^k(x) s_m^k(x) = \delta_{m,n}, \quad \int \mathrm{d}x\, w_n^k(x) w_m^l(x) = \delta_{m,n}\delta_{k,l}, \quad \int \mathrm{d}x\, s_n^k(x) w_m^{k+l}(x) = 0 \;\; (l \geq 0)$$

Countable basis for square-integrable functions

Scaling functions at some scale 0	Wavelet functions at scale 0	Wavelet functions at scale 1	Wavelet functions at scale 2	Wavelet functions at scale $r \geq 3$

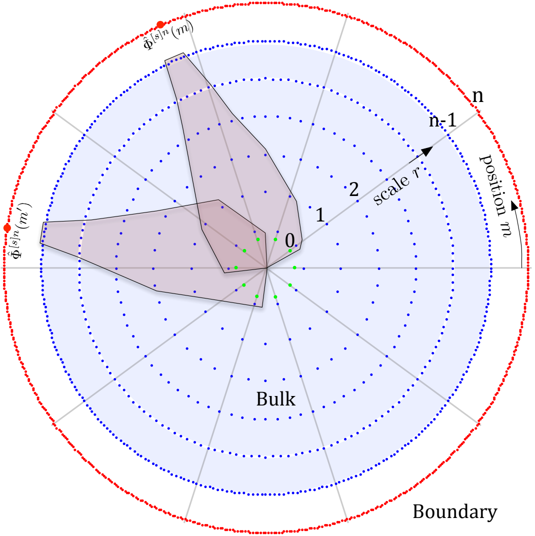


$$\mathcal{L}^2(\mathbb{R}) = \mathcal{H}_0 \bigoplus \mathcal{W}_0 \bigoplus \mathcal{W}_1 \bigoplus \mathcal{W}_2 \bigoplus_{r=3}^{\infty} \mathcal{W}_r$$

- Wavelet functions provide “refinement” to the coarse-grained scaling function representation:

$$\mathcal{H}_{r+1} = \mathcal{H}_r \oplus \mathcal{W}_r \quad \Longrightarrow \quad \mathcal{H}_n = \mathcal{H}_0 \oplus_{r=0}^{n-1} \mathcal{W}_r$$

Holographic principle and the wavelet transform



- The wavelet transform is an Exact Holographic Mapping (EHM)
- Bulk and boundary representations contain identical information:

$$\mathcal{H}_n = \mathcal{H}_0 \oplus_{r=0}^{n-1} \mathcal{W}_r$$

- The support of boundary modes widens as you decrease scale towards the center.

Wavelet transform (2D)

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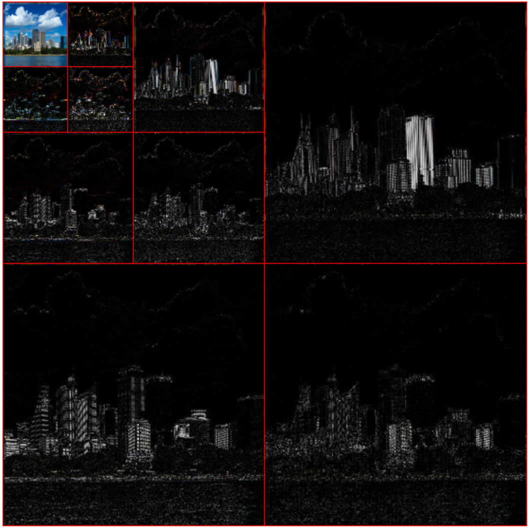


Daubechies wavelet index $\mathcal{K}$

- Left-to-right: $\mathcal{K} = \{1, 2, 3, 5\}$
- Top-to-bottom: Scale/resolution $r = \{0, 1, 2\}$
- A larger $\mathcal{K}$ (more vanishing moments) results in a sparser wavelet representation and reduced error, resulting in a more recognisable image (but only to a point).

Bosonic Hamiltonian

- Free scalar bosonic model (with periodic boundary conditions):

$$\hat{\mathcal{H}}_{\mathrm{b}}(x,t) = \frac{1}{2}\left(\hat{\Pi}^2(x,t) + \left(\nabla\hat{\Phi}(x,t)\right)^2 + m_0^2\hat{\Phi}^2(x,t)\right), \quad \hat{\Pi}(x,t) := \partial_t\hat{\Phi}(x,t)$$

Commutators: $\left[\hat{\Phi}(x,t),\hat{\Phi}(x',t)\right] = \left[\hat{\Pi}(x,t),\hat{\Pi}(x',t)\right] = 0$, $\left[\hat{\Phi}(x,t),\hat{\Pi}(x',t)\right] = \mathrm{i}\delta(x-x')\mathbb{1}$

- Massless phase is described by bosonic CFT, with long-range correlations:

$$\langle\hat{\Phi}(x)\hat{\Phi}(x')\rangle = -\frac{1}{4\pi}(\ln\left((x-x')^2\right)), \quad \langle\hat{\Pi}(x)\hat{\Pi}(x')\rangle = -\frac{1}{2\pi(x-x')^2}.$$

- Massive phase exhibits exponential decay of correlation functions for $|x-y| \gg m_0^{-1}$:

$$\langle\hat{\Phi}(x)\hat{\Phi}(x')\rangle \to -\frac{e^{-m_0|x-x'|}}{\sqrt{8\pi m_0|x-x'|}}, \quad \langle\hat{\Pi}(x)\hat{\Pi}(x')\rangle \to \sqrt{\frac{m_0}{8\pi|x-x'|^3}}e^{-m_0|x-x'|},$$

- Discrete bosonic Hamiltonian:

$$\hat{H}_{\mathrm{b}}^{(n)} := \frac{1}{2}\sum_{\ell\in\mathbb{Z}}\hat{\Pi}_\ell^{(n;\mathrm{s})}\hat{\Pi}_\ell^{(n;\mathrm{s})} + \frac{1}{2}\sum_{\ell,\ell'\in\mathbb{Z}}\hat{\Phi}_\ell^{(n;\mathrm{s})}K_{\ell,n\ell'}^{(n)}\hat{\Phi}_{\ell'}^{(n;\mathrm{s})}, \quad K_{\ell,\ell'}^{(n)} := -4^n\Delta_{\ell'-\ell}^{(2)} + m_0^2\,\delta_{\ell,\ell'}$$

Fermionic Hamiltonian

- Free Ising model (Majorana spinor formulation) (Boyanovsky 1989):

$$\hat{\mathcal{H}}_{\mathrm{f}}(x,t) = \frac{1}{2}\left(-\mathrm{i}\hat{\boldsymbol{b}}^T(x,t)\boldsymbol{Z}\partial_x\hat{\boldsymbol{b}}(x,t) + m_0\hat{\boldsymbol{b}}^T(x,t)\boldsymbol{Y}\hat{\boldsymbol{b}}(x,t)\right), \quad \hat{\boldsymbol{b}}(x,t) \equiv \begin{bmatrix}\hat{b}_0(x,t)\\ \hat{b}_1(x,t)\end{bmatrix}$$

  Anticommutator: $\left\{\hat{b}_\sigma(x), \hat{b}_{\sigma'}(x')\right\} = 2\delta_{\sigma,\sigma'}\delta(x-x')$
- Massless phase correlator (continuum limit, system size X, antiperiodic boundaries):

$$\left\langle \hat{b}_0(x)\hat{b}_1(x')\right\rangle = -\frac{1}{X\sin\left(\pi(x-x')/X\right)}$$

- In a wavelet basis (showing the explicit quadratic structure):

$$\hat{H}_{\mathrm{f}}^{(n)} = -\frac{\mathrm{i}}{2}\sum_{\substack{\ell,\ell'\in\mathbb{Z}\\ \sigma,\sigma'\in\{0,1\}}} Q^{(n)}_{\ell,\sigma;\ell',\sigma'}\hat{b}^{(n;s)}_{\ell,\sigma}\hat{b}^{(n;s)}_{\ell',\sigma'}, \quad \hat{b}^{(0;s)}_{\ell,\sigma} := \int \mathrm{d}x\, s^{(0)}_\ell(x)\hat{b}_\sigma(x)$$

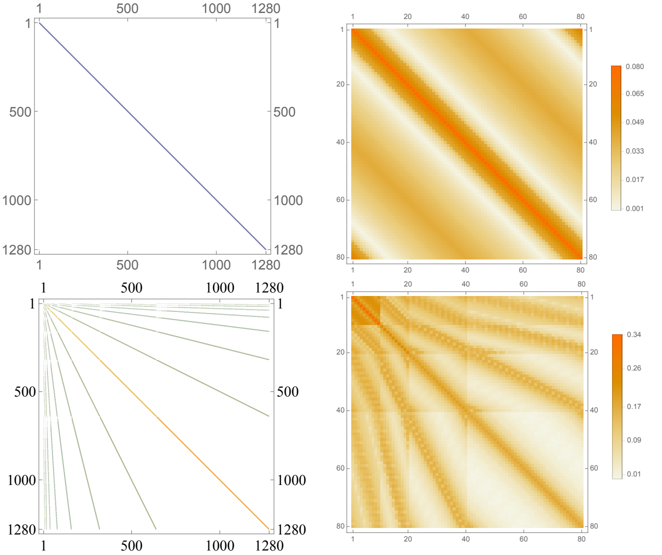
$$Q^{(n)}_{\ell,\sigma;\ell',\sigma'} := (-1)^\sigma 2^n\Delta^{(1)}_{\ell'-\ell}\delta_{\sigma,\sigma'} + m_0\delta_{\ell,\ell'}(\sigma'-\sigma)$$

- $\Delta^{(n)}_\ell$ are derivative overlap coefficients, rational values calculable using properties of the wavelet family. See Beylkin 1992.

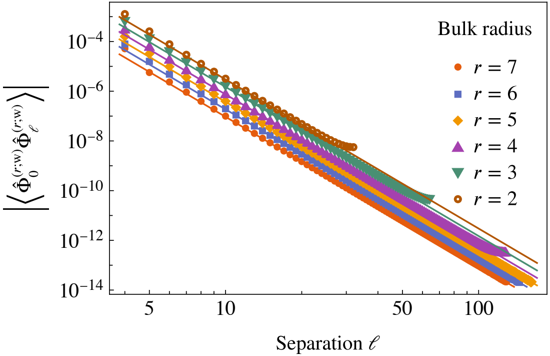
Example: Bosonic coupling matrix K and GS covariance matrix Γ

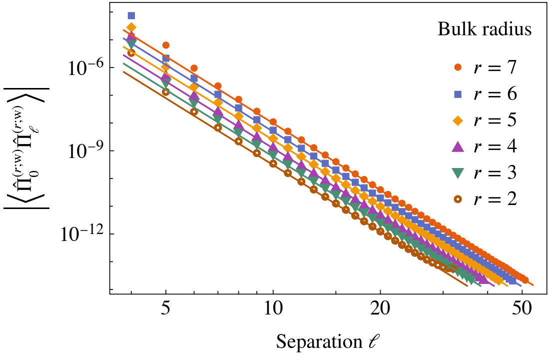
Top: Boundary Hamiltonian coupling and covariance matrices showing near-neighbour coupling only.

Bottom: Bulk Hamiltonian coupling and covariance matrices showing near-neighbour coupling and coupling across scales. Note also dominance of coarse scale fields in top-left of covariance matrix.



Bosonic correlators (critical phase)



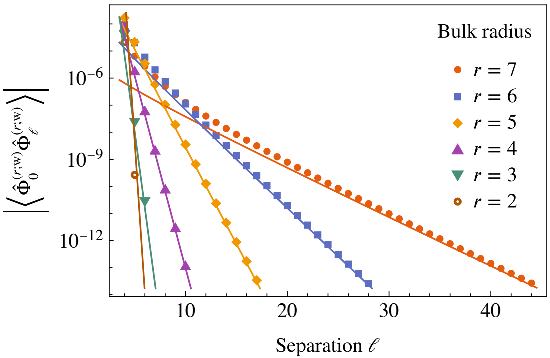


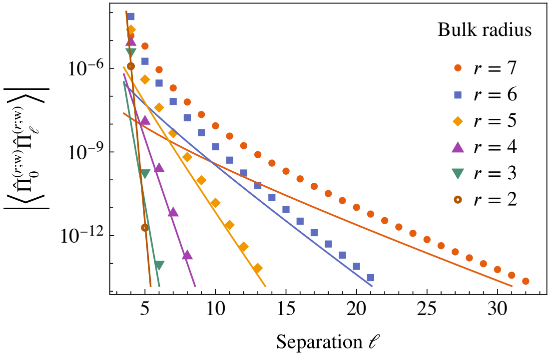
$$\left\langle \hat{\Phi}_0^{(r;\mathrm{w})} \hat{\Phi}_\ell^{(r;\mathrm{w})} \right\rangle \approx -\frac{2^{n-r}}{4\pi\, \ell^{2\mathcal{K}} \mathcal{K}} \times \binom{2\mathcal{K}}{\mathcal{K}} \left\langle x^{\mathcal{K}} \right\rangle_w^2 \qquad \left\langle \hat{\Pi}_0^{(r;\mathrm{w})} \hat{\Pi}_\ell^{(r;\mathrm{w})} \right\rangle \approx \frac{2^{r-n}(2\mathcal{K}+1)}{2\pi\, \ell^{2\mathcal{K}+2}} \times \binom{2\mathcal{K}}{\mathcal{K}} \left\langle x^{\mathcal{K}} \right\rangle_w^2$$

Polynomial decay with exponent proportional to $\mathcal{K}$

(analytic expressions: Singh and Brennen 2016)

Bosonic correlators (massive phase)



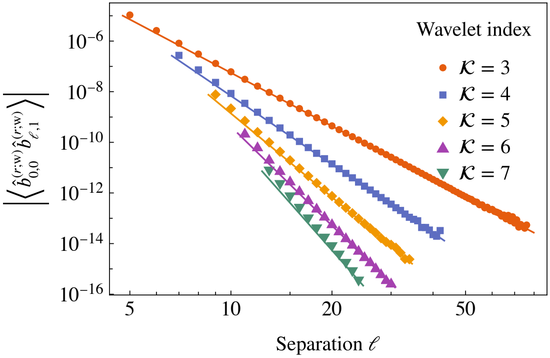


$$\left\langle \hat{\Phi}_0^{(r;\mathrm{w})} \hat{\Phi}_\ell^{(r;\mathrm{w})} \right\rangle \approx -\frac{2^{n-r} e^{-\ell \tilde{m}}}{\sqrt{8\pi \ell \tilde{m}}} \left\langle e^{-\tilde{m}x} \right\rangle_w \left\langle e^{\tilde{m}x} \right\rangle_w \qquad \left\langle \hat{\Pi}_0^{(r;\mathrm{w})} \hat{\Pi}_\ell^{(r;\mathrm{w})} \right\rangle \approx 2^{r-n} e^{-\ell \tilde{m}} \sqrt{\frac{\tilde{m}}{8\pi \ell^3}} \left\langle e^{-\tilde{m}x} \right\rangle_w \left\langle e^{\tilde{m}x} \right\rangle_w$$

where $\tilde{m} = 2^{n-r} m_0 \implies$ scale-dependent mass renormalisation!

(analytic expressions: Singh and Brennen 2016)

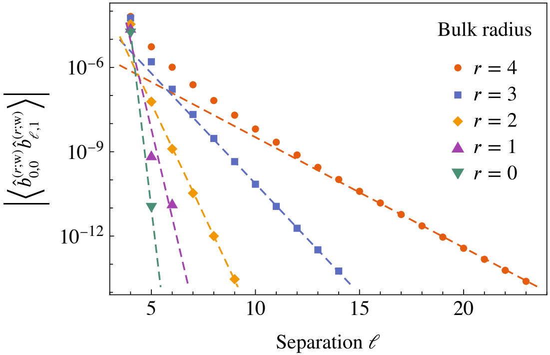
Fermionic correlators



Critical phase (analytic):

$$\left\langle \hat{b}^{(r;\mathrm{w})}_{0,0}\, \hat{b}^{(r;\mathrm{w})}_{\ell,1} \right\rangle \approx \frac{\mathrm{i}\,(-1)^{\mathcal{K}}}{\pi \ell^{2\mathcal{K}+1}} \times \binom{2\mathcal{K}}{\mathcal{K}} \left\langle x^{\mathcal{K}} \right\rangle_w^2$$

Polynomial decay with exponent linear in $\mathcal{K}$

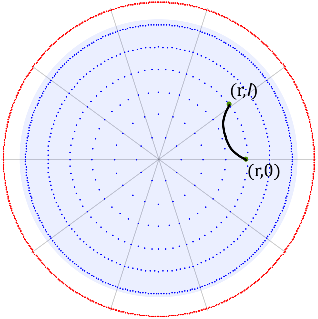


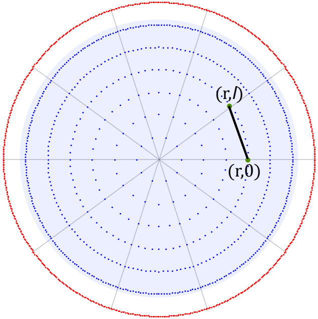
Massive phase (least-squares):

$$\left\langle \hat{b}^{(r;\mathrm{w})}_{0,0}\, \hat{b}^{(r;\mathrm{w})}_{\ell,1} \right\rangle \sim \mathrm{e}^{-1.13\ell\tilde{m}}, \quad \tilde{m} = 2^{(n-r)} m_0$$

Exponential decay with mass renormalisation!

Holographic picture - critical vs massive





- Same-scale correlators in the critical phase correspond to a negatively curved AdS(2+1) geodesic distance in the bulk.
- Cross-scale correlators can be shown to decay exponentially.

- In the massive phase, exponential decay corresponds to Euclidean geometry (i.e. flat space) in the bulk.
- Expected given that the massive theory is not conformal.

Mutual information and bulk radius of curvature

- Mutual information is a useful basis-independent quantity for measuring correlations:

$$I(A:B) = S(\rho_{\mathrm{A}}) + S(\rho_{\mathrm{B}}) - S(\rho_{AB}).$$

- To find the bulk radius of curvature R in the critical phases, adopt the ansatz:

$$I((r,0),(r,\ell)) = S_0 \mathrm{e}^{-d_g((r,0),(r,\ell))/\xi}$$

  with geodesic distance $d_g((r,0),(r,\ell)) = 2R\ln(\ell/R)$ and ξ the correlation length.
- Then for the scalar bosonic CFT, for large $\mathcal{K}$, (Singh and Brennen 2016):

$$R(\mathcal{K}) \approx 0.32\mathcal{K} - 0.88/\mathcal{K} + 0.43$$

- And for the critical Ising model, for large $\mathcal{K}$, (Brennen, unpublished):

$$R(\mathcal{K}) \approx 0.32\mathcal{K} + 0.66$$

- Linear dependence on $\mathcal{K}$ can be linked to the Daubechies wavelets coupling modes within a neighbourhood of $2\mathcal{K}$, or equivalently being simulable by a circuit of nearest-neighbour gates of depth $\mathcal{K}$.

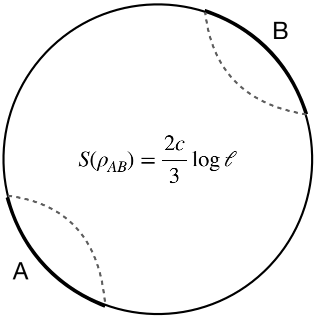
Holographic subsystem entropy (bosonic CFT)

- The subsystem entanglement entropy $S(\rho_A)$ of a bosonic $1+1$D CFT is equal to the length of the geodesic joining the boundary points in the bulk AdS3 slice geometry.

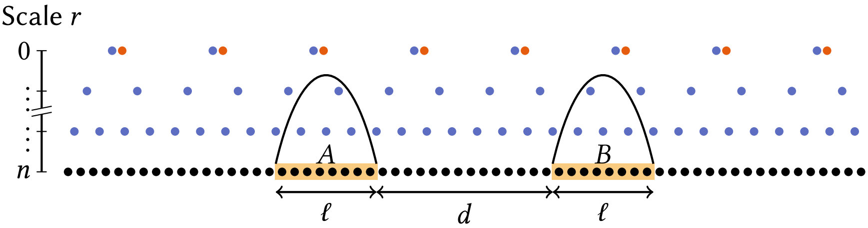
$$S(\rho_A) = \frac{c}{3} \log \frac{|A|}{\epsilon}$$

- Combined entropy $S(\rho_{AB})$ is equal to the length of the geodesics joining the boundary points of the two subregions.
- MI between two subregions of a bosonic conformal field theory:

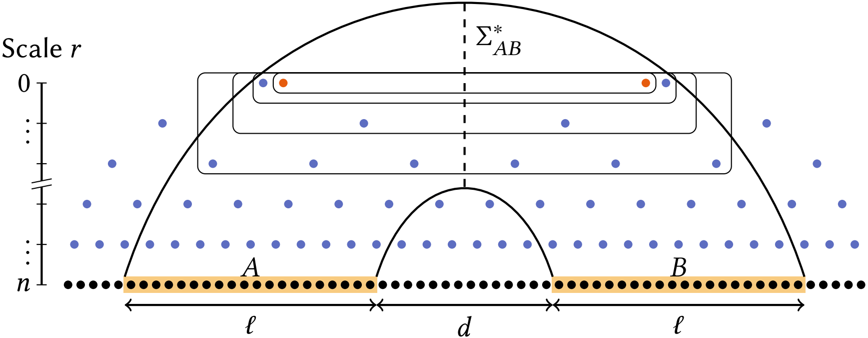
$$I(A:B) = \begin{cases} 0 & d/\ell \geq \sqrt{2}-1 \\ -\frac{c}{3} \log\left((d/\ell)^2 + 2d/\ell\right) & d/\ell < \sqrt{2}-1 \end{cases}.$$



Mutual information and the entanglement wedge



$I(\rho_{AB}) = 0$



$I(\rho_{AB}) > 0$

Entanglement wedge cross-section

- Geometric quantity derivable from the properties of the relevant CFT

$$E_W(\rho_{AB}) = \frac{\left|\Sigma^*_{AB}\right|}{4G_N} = \frac{c}{6}\log(1+2\ell/d)$$

- Conjectured equality with the Entanglement of Purification (Umemoto and Takayanagi 2018):

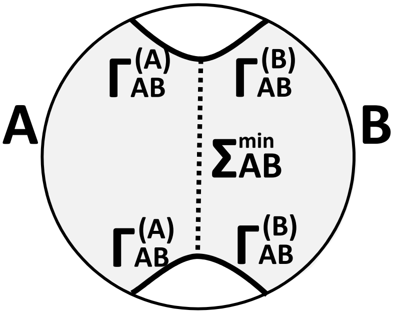
$$E_p(\rho_{AB}) \stackrel{?}{=} E_W(\rho_{AB})$$

- In the context of CFT, it shares several inequalities with the EoP:

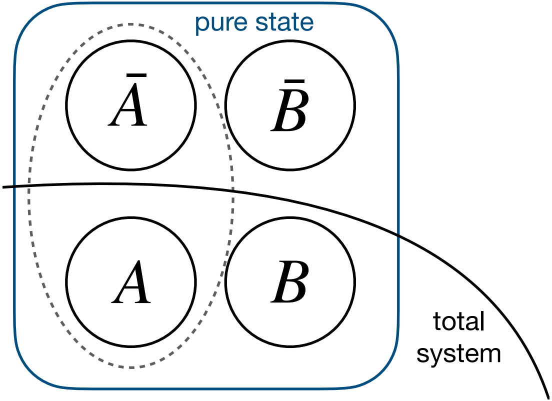
$$I(A:B)/2 \leq E_W(\rho_{AB}) \leq \min(S(\rho_{\mathrm{A}}), S(\rho_{\mathrm{B}}))$$

$$E_W(\rho_{AB}) \leq E_W(\rho_{A(BC)}) \leq E_W(\rho_{AB}) + E_W(\rho_{BC})$$

$$E_W(\rho_{(AA')(BB')}) \geq E_W(\rho_{AB}) + E_W(\rho_{A'B'})$$



Entanglement of purification



$$E_p(\rho_{AB}) = \min_{|\psi\rangle_{A\bar{A}B\bar{B}};\mathrm{Tr}_{\bar{A}\bar{B}}[|\psi\rangle\langle\psi|]=\rho_{AB}} S(\rho_{A\bar{A}})$$

- Given two subsystems A and B of an overall pure state, minimise the joint entropy $S(\rho_{A\bar{A}})$ over all possible pure states $|\psi\rangle_{A\bar{A}B\bar{B}}$ with ancillary systems $\bar{A}, \bar{B}$.
- Generalises entanglement entropy to a measure of correlation (classical and quantum) for mixed states.
- Unsurprisingly, this minimisation is extremely difficult in the general case

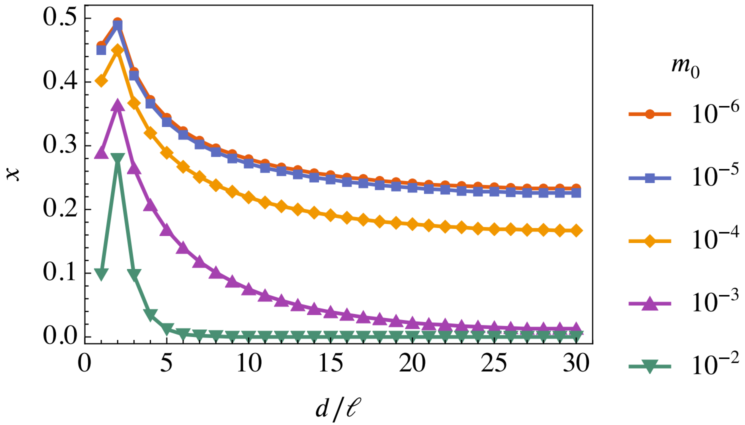
Solving for the entanglement of purification

- Consider a pure state on a total system $A\bar{A}B\bar{B}$ described by covariance matrix:

$$\Gamma^{\Pi\Pi}_{AB\bar{A}\bar{B}} = \frac{1}{2}\begin{pmatrix} J & K \\ K^T & L \end{pmatrix}, \quad \Gamma^{\Phi\Phi}_{AB\bar{A}\bar{B}} = \frac{1}{2}\begin{pmatrix} D & E \\ E^T & F \end{pmatrix} \quad \text{such that} \quad \begin{pmatrix} J & K \\ K^T & L \end{pmatrix}^{-1} = \begin{pmatrix} D & E \\ E^T & F \end{pmatrix}$$

- When $|A| = |B| = 1$ it can be shown that $\left|\bar{A}\right|, \left|\bar{B}\right| > 1$ provides minimal additional accuracy for the EoP.
- Canonical form of $K = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}$ reduces the minimisation to a single physical parameter x (a momenta-momenta correlation). (Battacharyya, Takayanagi and Umemoto 2018)
- Can we identify any phase transitions when minimising over a single coarse-grained mode in a wavelet basis, corresponding to a much larger subsystem?

Phase transition in bosonic CFT



Peak in x at $d/\ell = 2$ indicative of a quantum phase transition in the neighbourhood of the geodesic crossing!

Phase transitions in fermionic QFT

- One witness of a quantum phase transition is a sudden drop in overlap fidelity between ground states $|\Psi(g)\rangle$ adjacent in some parameter g.
- For pure states:

$$F(m) = |\langle\Psi(g_-)|\Psi(g_+)\rangle|$$

- For reduced states:

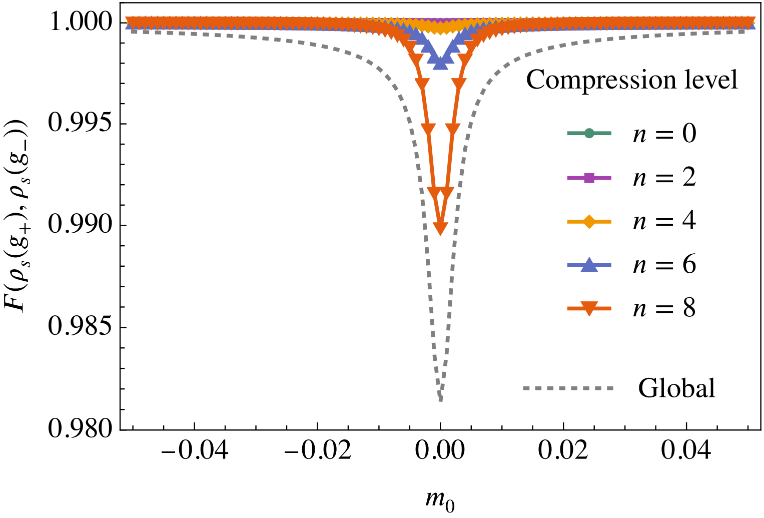
$$F(\rho_s(g_+), \rho_s(g_-)) = \mathrm{Tr}\left[\sqrt{\sqrt{\rho_s(g_+)}\rho_s(g_-)\sqrt{\rho_s(g_+)}}\right],$$

  where $g_\pm = g \pm \delta/2$, and δ is small.
- For the fermionic Ising model, a QPT is evident at $m_0 = 0$ and the overlap fidelity can be shown to drop from approximately 1 to

$$F(m_0 = 0) \approx 1 - \frac{\delta^2 V^2}{8\pi^2}$$

- Can we see this in a coarse-grained wavelet basis state?

Identifying quantum phase transitions



- Yes, the phase transition is clearly evident.
- Suggests potential of wavelet compression for e.g. experimental observations of phase transitions
- Fast wavelet transform in $\mathcal{O}(V \log(V))$ in the number of modes

Conclusion

- Wavelets provide a natural basis for describing multi-scale properties of QFTs
- In particular, features like renormalisation are readily apparent - and the wavelet index $\mathcal{K}$ allows tuning the bulk geometry in the holographic picture.
- Wavelet state compression can work - we can use reduced states coarse-grained in a wavelet basis to identify phase transitions, and without needing to fine-tune e.g. tensor network descriptions for the specifics of the QFT
- Future research directions:
  - How do **excitations** behave in the bulk?
  - How do correlations scale for **thermal states**?
  - Can we replicate the **black-hole bulk geometry** of Qi (2013) and relate the behaviour of the metric to the wavelet index $\mathcal{K}$?
  - Can we use **continuous wavelets** to better understand entanglement in continuous QFTs?
  - Do wavelets offer any advantages when describing **bandlimited QFT** and/or **interacting theories** (e.g. ϕ^4)