Filtering, Retrofiltering and Smoothing: Optimal quantum state estimation using continuous measurement

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In estimation theory, in particular when dealing with estimating the state of a system based on the outcomes of a continuous-in-time measurement, there are the optimal estimation techniques of *filtering*, *retrofiltering* and *smoothing*. The filtering technique constructs an estimate of the state based on the past measurement record $\overleftarrow{\mathbf{O}}$, that is, all of the measurement outcomes up until the estimation time t. In some sense the dual of filtering, the retrofiltering technique allows one to obtain the likelihood function for the future measurement record $\overrightarrow{\mathbf{O}}$, i.e., the probability of the possible outcomes of the measurement after the estimation time. From this one can easily obtain a retrofiltered state, i.e., a state conditioned on the future measurement information. Finally, the smoothing technique combines both filtering and retrofiltering to obtain an estimate conditioned on both the past and the future measurement record, i.e., the past-future record $\overleftrightarrow{\mathbf{O}}$.

While filtering and retrofiltering have direct quantum analogues, resulting in the filtered state and retrofiltered effect, the smoothing technique presented some issues. In particular, following the classical prescription lead to unphysical estimates of the density matrix. In was not until the quantum state smoothing theory of Guevara and Wiseman [1] that it became possible to define a valid smoothed quantum state, i.e. is Hermitian and positive. Although this theory provides a complete analogue between classical and quantum state estimation with continuous-in-time measurements, there is still one estimator that is missing, the retrofiltered *state*. Unlike the classical case, it is not a trivial matter to obtain a retrofiltered state from the retrofiltered effect.

In this work, using the quantum state smoothing formalism of Ref. [1] and Bayesian estimation theory [2], we derive the optimal retrofiltered quantum state, $\rho_{\rm R}(t) = \rho_{\overrightarrow{\mathbf{O}}}(t)$, for both a trace- square deviation cost function and relative entropy expected cost function. Furthermore, from this formalism, we are able to identify a handful of additional estimation problems that introduces a total of nine quantum state estimators (including the three just mentioned), of which four are equivalent. Finally, we apply all these estimators to linear Gaussian quantum systems and consider a physical system as an example, the on-threshold optical parametric oscillator.

[1] I. Guevara and H. M. Wiseman, *Phys. Rev. Lett.* **115**, 180407 (2015).

[2] K. T. Laverick, I. Guevara and H. M. Wiseman *Phys. Rev. A* **104**, 032213 (2021).