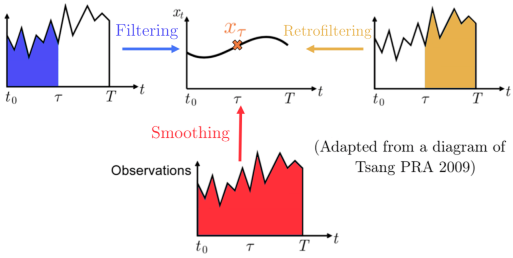
Filtering, Retrofiltering and Smoothing:

Optimal quantum state estimation using continuous-in-time measurement

Kiarn T. Laverick, Areeya Chantasri, Howard M. Wiseman

Classical State Estimation Techniques

- **Task:** Assign a value to an unknown parameter x of a system based on observations.
- If we can obtain the distribution $\wp(x)$, we can calculate many useful properties, e.g. the mean $\langle x \rangle$.
- **Filtering** (F):
  Conditioning on a past measurement records i.e.
  $\wp_{\mathrm{F}}(x) = \wp(x|\overleftarrow{Y})$
- **Retrofiltering** (R):
  Predicting future measurement records i.e.
  $E_{\mathrm{R}}(x) = \wp(\overrightarrow{Y}|x) \Leftrightarrow \wp_{\mathrm{R}}(x)$
- **Smoothing** (S):
  Conditioning on an entire measurement record $\overleftrightarrow{Y}$ i.e.
  $\wp_{\mathrm{S}}(x) \propto E_{\mathrm{R}}(x)\wp_{\mathrm{F}}(x)$



(Adapted from a diagram of Tsang PRA 2009)

Quantum State Estimation Techniques

- Transitioning to quantum systems, the problem remains, more or less, the same, where now we are concerned with estimating the quantum state ρ
- **Quantum Filtering**:
  "Quantum Trajectory Theory" $\Rightarrow \rho_{\mathrm{F}} = \rho_{\overleftarrow{O}}$
- **Quantum Retrofiltering**:
  $\hat{E}_{\mathrm{R}} \Rightarrow \mathrm{Tr}[\hat{E}_{\mathrm{R}}\rho] = \wp(\overrightarrow{O}|\rho)$, POVM describing future results
- **Weak-Value Smoothing**:
  $\varrho_{\mathrm{SWV}} \propto \hat{E}_{\mathrm{R}}\rho_{\mathrm{F}} + \rho_{\mathrm{F}}\hat{E}_{\mathrm{R}}$

$$\Lambda_W = \mathrm{Tr}[\varrho_{\mathrm{SWV}}\hat{\Lambda}] \tag{1}$$

Aharonov, Y., Albert, D.Z., Vaidman, L., Phys. Rev. Lett. 60, 1351–1354 (1988).

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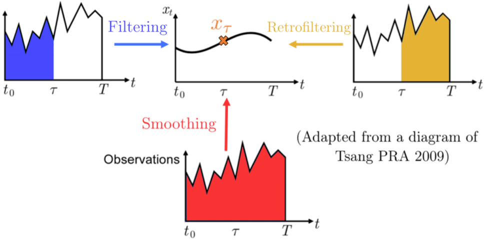

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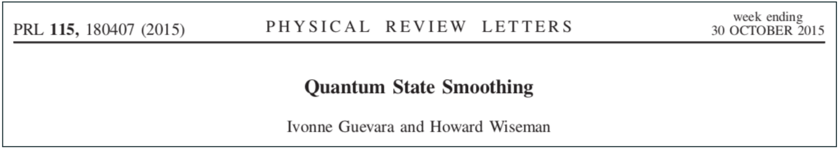
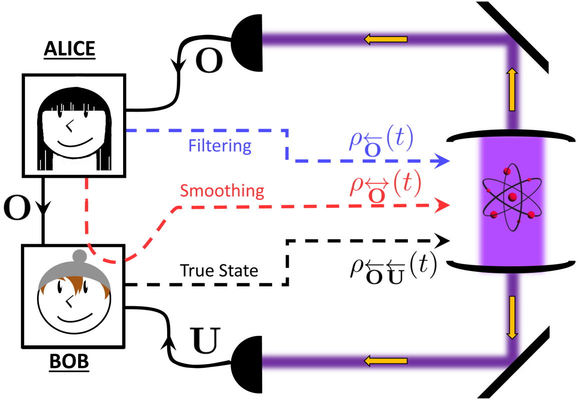
Quantum State Smoothing



PRL **115,** 180407 (2015) PHYSICAL REVIEW LETTERS week ending 30 OCTOBER 2015

Quantum State Smoothing

Ivonne Guevara and Howard Wiseman



$$\rho_{\mathrm{T}} \equiv \rho_{\overleftarrow{\mathbf{O}}\overleftarrow{\mathbf{U}}}$$

$$\rho_{\mathrm{F}} \equiv \rho_{\overleftarrow{\mathbf{O}}} = \mathbb{E}_{\rho_{\mathrm{T}}|\overleftarrow{\mathbf{O}}}[\rho_{\mathrm{T}}]$$

$$\rho_{\mathrm{S}} \equiv \rho_{\overleftrightarrow{\mathbf{O}}} = \mathbb{E}_{\rho_{\mathrm{T}}|\overleftrightarrow{\mathbf{O}}}[\rho_{\mathrm{T}}]$$

$$\rho_{\mathrm{R}} \equiv \rho_{\overrightarrow{\mathbf{O}}} = ?$$

Optimal Bayesian Estimation of the True State

- Objective Function:

$$\mathcal{B}_{\mathsf{c}}(\check{\rho}) = \mathbb{E}_{\rho_{\rm T}|\mathbf{O}_{\mathsf{c}}}\{\mathcal{C}[\rho_{\rm T}, \check{\rho}]\}$$

- Cost Function: Trace-Square Deviation

$$\mathcal{C}^{\rm TrSD}[\rho_{\rm T}, \check{\rho}] = \mathsf{Tr}[(\rho_{\rm T} - \check{\rho})^2]$$

- Optimal Estimator:

$$\rho_{\mathsf{c}}^{\rm opt}(t) := \arg\min_{\check{\rho}\in\mathbb{T}_t} \mathcal{B}_{\mathsf{c}}(\check{\rho}) = \mathbb{E}_{\rho_{\rm T}|\mathbf{O}_{\mathsf{c}}}\{\rho_{\rm T}\}$$

Filtered	Retrofiltered	Smoothed
$\rho_{\rm F}(t) = \mathbb{E}_{\rho_{\rm T}\vert\overleftarrow{\mathbf{O}}}\{\rho_{\rm T}\}$	$\rho_{\rm R}^{\rm Tru}(t) = \mathbb{E}_{\rho_{\rm T}\vert\overrightarrow{\mathbf{O}}}\{\rho_{\rm T}\}$	$\rho_{\rm S}(t) = \mathbb{E}_{\rho_{\rm T}\vert\overleftrightarrow{\mathbf{O}}}\{\rho_{\rm T}\}$

Laverick, K. T., Guevara, I., Wiseman, H. M., PRA **104**, 032213 (2021).

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Filtered | Retrofiltered | Smoothed

$$\rho_{\mathrm{F}}(t) = \mathbb{E}_{\rho_{\mathrm{T}}|\overleftarrow{\mathbf{O}}}\{\rho_{\mathrm{T}}\}$$

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Estimating the Filtered State and Weak-Values

- One could instead ask the question, what if one wishes to estimate the filtered state, given they only have access to the future measurement record.
- This problem is a slight change to the previous estimation problem ($\rho_{\mathrm{T}} \rightarrow \rho_{\mathrm{F}}$), with the optimal estimator:

Retrofiltered Estimate of the Observed State (REOS)

$$\rho_{\mathrm{R}}^{\mathrm{Obs}}(t) = \mathbb{E}_{\rho_{\mathrm{F}} | \overrightarrow{\mathbf{O}}}\{\rho_{\mathrm{F}}\}$$

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Estimating the Filtered State and Weak-Values cont.

- The final type of estimation we consider is weak-value estimation.
- We have already seen one of these estimators: ϱ_{SWV}
- One aims to find an estimator that minimizes the distance between its trace with a set of observables $\hat{\Lambda}_k$ (the generalized Gell-Man matrices [1]) and their weak-value Λ_k^w [2], i.e.,

$$\varrho_{\mathrm{c}} = \arg\min_{\check{\varrho}} \sum_k \mathbb{E}_{\Lambda_k|\mathbf{O}_{\mathrm{c}}}\{(\mathrm{Tr}[\check{\varrho}\hat{\Lambda}_k] - \Lambda_k^w)^2]\}$$

- Optimal Weak-valued states:

$$\varrho_{\mathrm{c}}(t) = \frac{1}{d}\hat{1} + \tfrac{1}{2}\sum_k \langle\hat{\Lambda}_k\rangle_{\mathrm{c}}\hat{\Lambda}_k\,, \qquad (2)$$

with $\langle\hat{\Lambda}_k\rangle_{\mathrm{c}} = \sum_{x_j} \wp(x_j|\mathbf{O}_{\mathrm{c}})x_j$

[1]G. Kimura, Phys. Lett. A, **314**, 339–349, (2003).
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Example: On-Threshold Optical Parametric Oscillator

Let's consider an example, the optical parametric oscillator.

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i\chi[(\hat{q}\hat{p} + \hat{p}\hat{q})/2, \rho] + \gamma\mathcal{D}[\hat{q} + i\hat{p}]\rho \,, \tag{3}$$

where $\mathcal{D}[\hat{c}]\rho = \hat{c}\rho\hat{c}^\dagger - \frac{1}{2}\{\hat{c}^\dagger\hat{c}, \rho\}$.

- We will consider the system at threshold, i.e., $\chi = \gamma$.
- Alice and Bob both use homodyne measurements with local oscillator phases θ_A and θ_B. Their respective measurement efficiencies are $\eta_\mathrm{A} = \eta_\mathrm{B} = 0.5$.
- Note: this is a linear Gaussian quantum (LGQ) system, so one only needs to know the mean and covariance matrix to specify the state.

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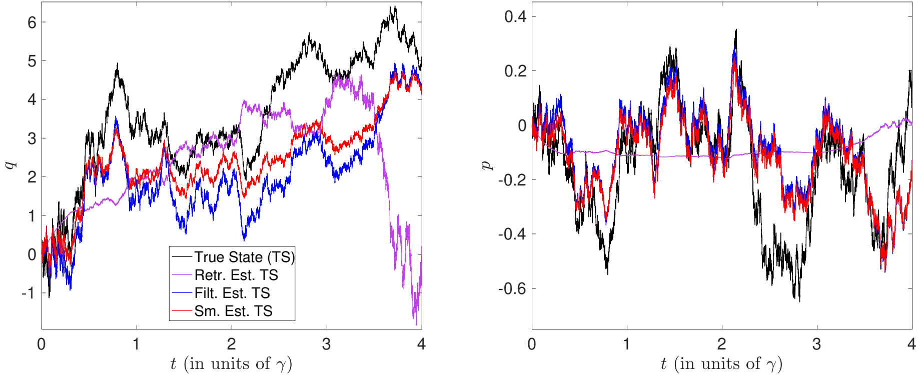
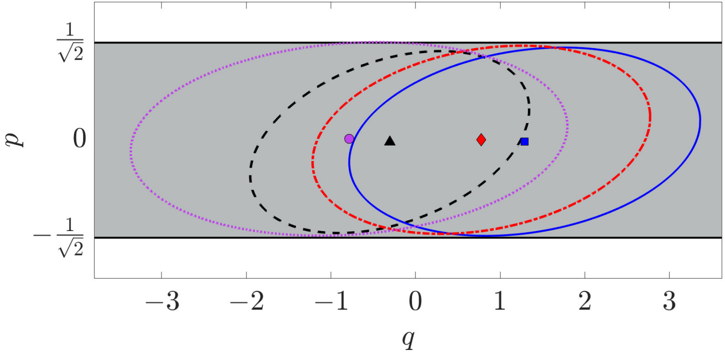
Example: OPO Trajectories

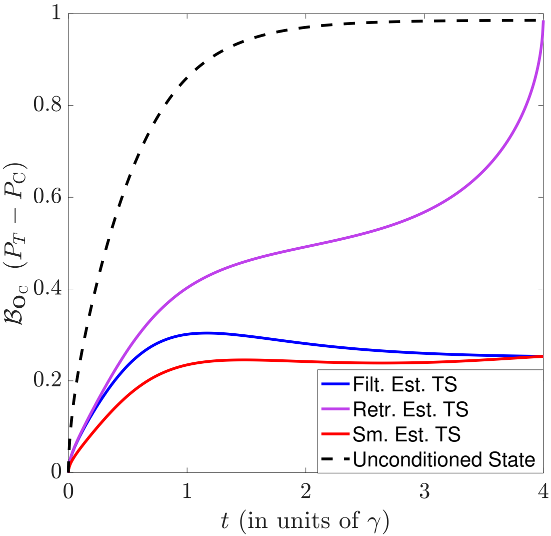


$$\theta_{\mathrm{A}} = 3\pi/8 \qquad \theta_{\mathrm{B}} = 3\pi/8 \tag{4}$$

Example: OPO Phase-Space Diagram



Example: Expected Cost Function



Conclusion

	Filtered ($\overleftarrow{\mathbf{O}}$)	Retrofiltered ($\overrightarrow{\mathbf{O}}$)	Smoothed ($\overleftrightarrow{\mathbf{O}}$)
True (ρ_{T})	ρ_{F}	$\rho_{\mathrm{R}}^{\mathrm{Tru}}$	ρ_{S}
Observed (ρ_{F})	ρ_{F}	$\rho_{\mathrm{R}}^{\mathrm{Obs}}$	ρ_{F}
Weak-Valued (ϱ)	ρ_{F}	ϱ_{RWV}	ϱ_{SWV}

Conclusion

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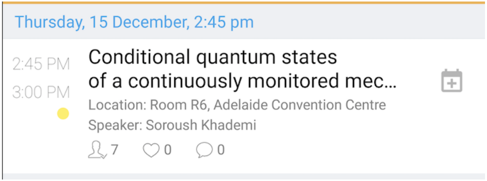
True ($\rho_{\rm T}$)	$\rho_{\rm F}$	$\rho_{\rm R}^{\rm Tru}$	$\rho_{\rm S}$
Observed ($\rho_{\rm F}$)	$\rho_{\rm F}$	$\rho_{\rm R}^{\rm Obs}$	$\rho_{\rm F}$
Weak-Valued (ϱ)	$\rho_{\rm F}$	$\varrho_{\rm RWV}$	$\varrho_{\rm SWV}$



Extra Slide: LGQ Equations

True Retrofiltered State

$$\langle \hat{\mathbf{x}} \rangle_{\mathrm{R}}^{\mathrm{Tru}} = (V_{\mathrm{R}}^{\mathrm{Tru}} - V_{\mathrm{T}}) \left[(V - V_{\mathrm{T}})^{-1} \langle \mathbf{x} \rangle + (V_{\mathrm{R}} + V_{\mathrm{T}})^{-1} \langle \hat{\mathbf{x}} \rangle_{\mathrm{R}} \right] \quad (5)$$

$$V_{\mathrm{R}}^{\mathrm{Tru}} = \left[(V - V_{\mathrm{T}})^{-1} + (V_{\mathrm{R}} + V_{\mathrm{T}})^{-1} \right]^{-1} + V_{\mathrm{T}} \quad (6)$$

Observed Retrofiltered State

$$\langle \hat{\mathbf{x}} \rangle_{\mathrm{R}}^{\mathrm{Obs}} = (V_{\mathrm{R}}^{\mathrm{Obs}} - V_{\mathrm{F}}) \left[(V - V_{\mathrm{F}})^{-1} \langle \mathbf{x} \rangle + (V_{\mathrm{R}} + V_{\mathrm{F}})^{-1} \langle \hat{\mathbf{x}} \rangle_{\mathrm{R}} \right] \quad (7)$$

$$V_{\mathrm{R}}^{\mathrm{Obs}} = \left[(V - V_{\mathrm{F}})^{-1} + (V_{\mathrm{R}} + V_{\mathrm{F}})^{-1} \right]^{-1} + V_{\mathrm{F}} \quad (8)$$

Retrofiltered Weak-Valued State

$$\langle \mathbf{x} \rangle_{\mathrm{RWV}} = V_{\mathrm{RWV}} V_{\mathrm{R}}^{-1} \langle \hat{\mathbf{x}} \rangle_{\mathrm{R}} \quad (9)$$

$$V_{\mathrm{RWV}} = (V_{\mathrm{R}}^{-1} + V^{-1})^{-1} \quad (10)$$