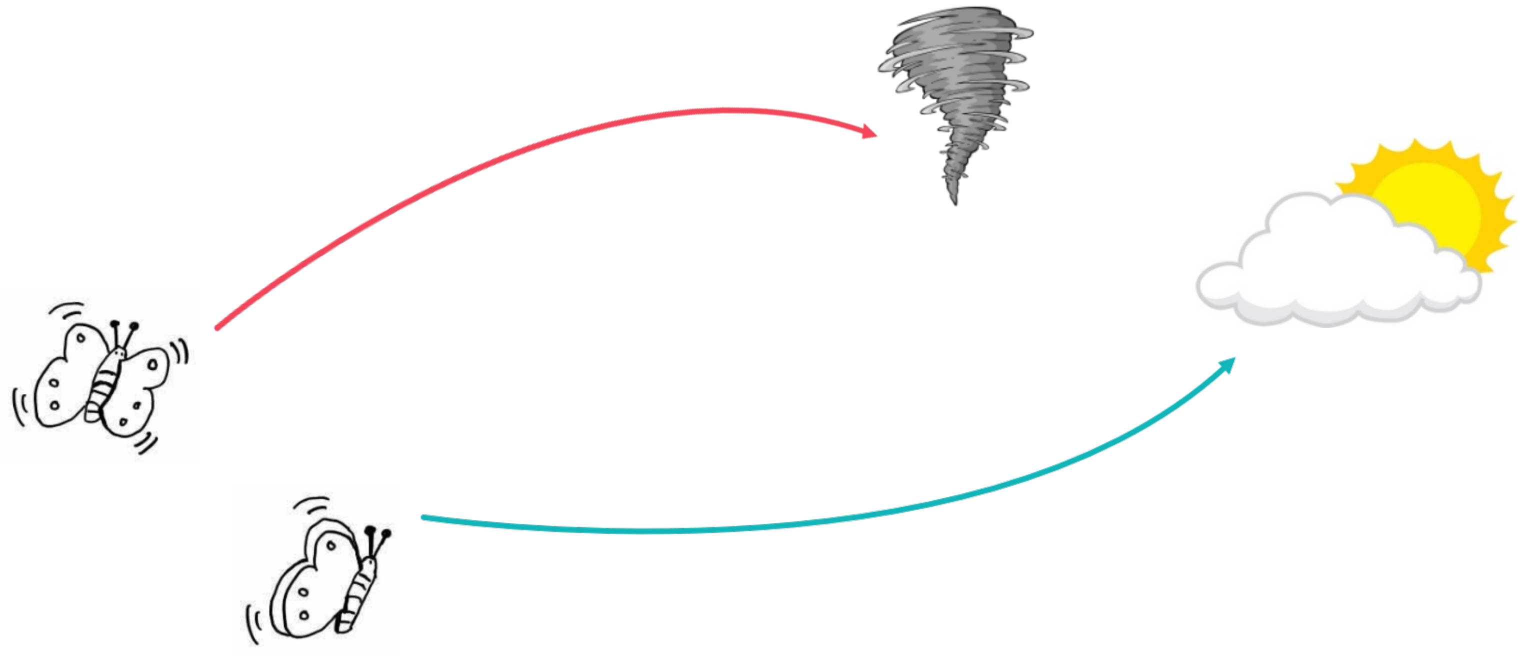
Quantum Chaos = Volume-Law Spatiotemporal Entanglement

Neil Dowling

Neil Dowling and Kavan Modi (2022), arXiv:2210.14926

Classical Chaos: The Butterfly Effect

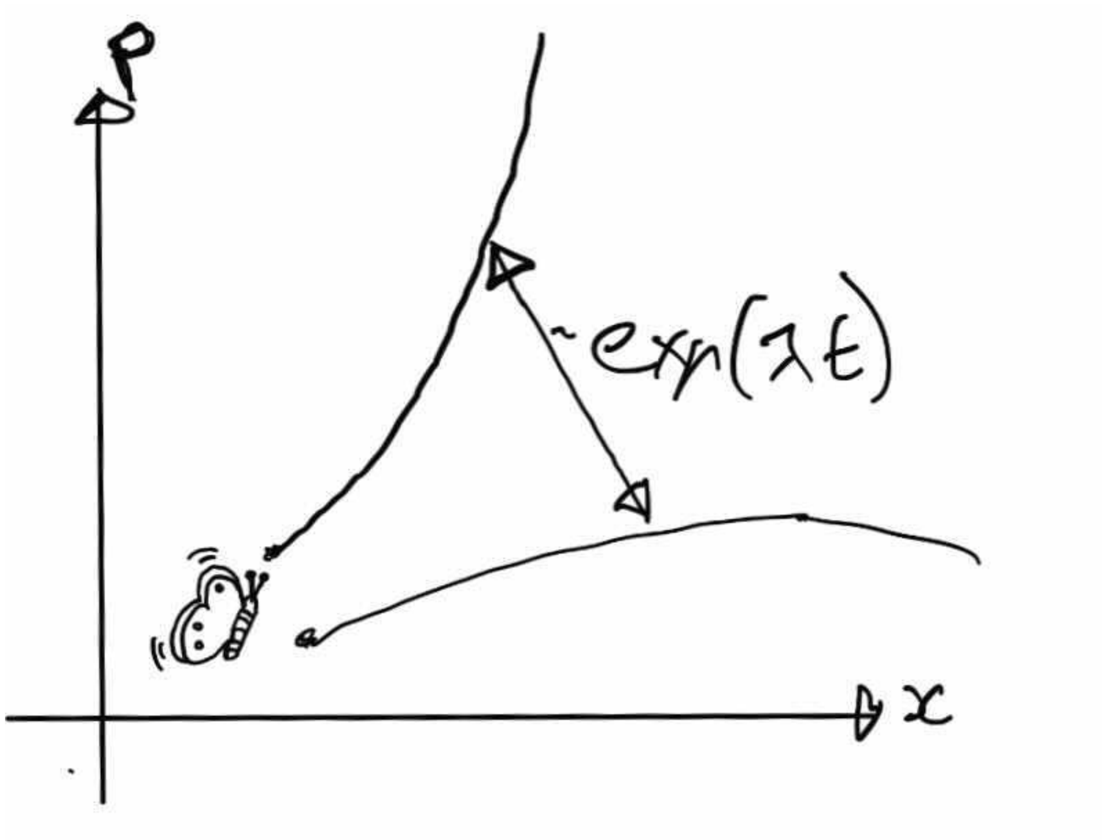
"The phenomenon that a small alteration in the state of a dynamical system will cause subsequent states to differ greatly from the states that would have followed without the alteration"



Lorenz, Edward (1993). The Essence of Chaos.

Classical Chaos: Lyapunov Exponents

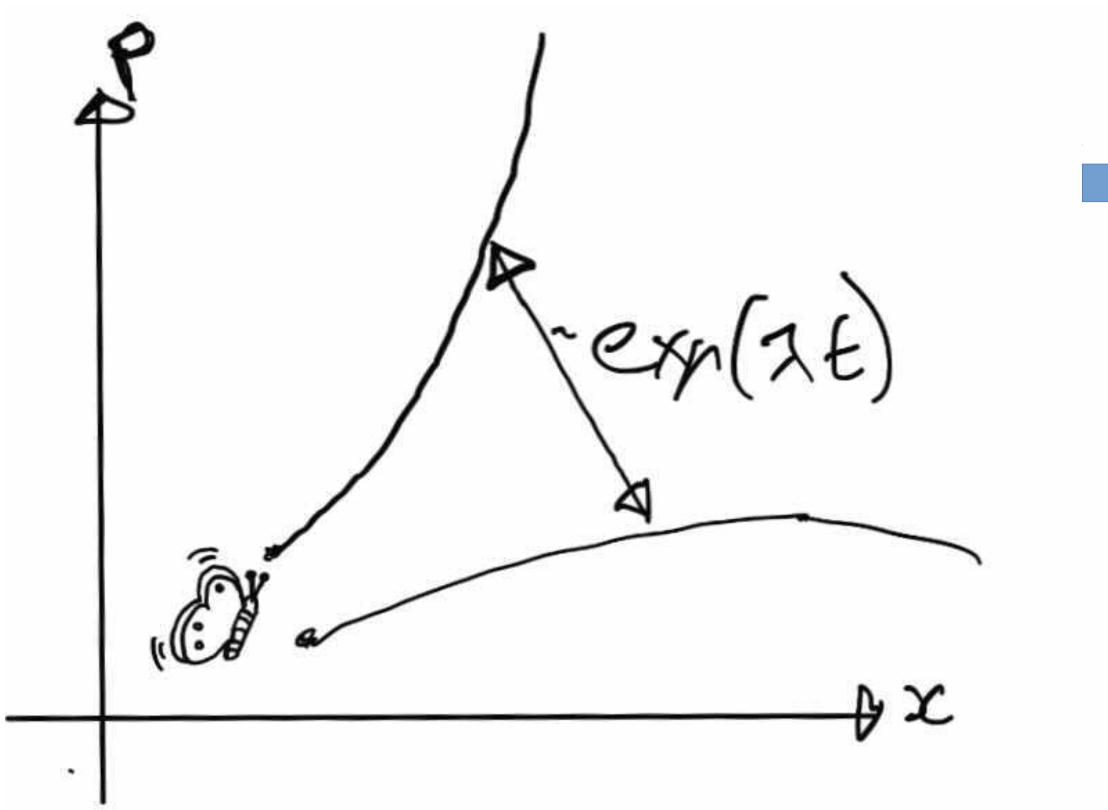
Classical chaos determined by a exponential sensitivity to perturbation



Classical Chaos: Lyapunov Exponents

Classical chaos determined by a exponential sensitivity to perturbation

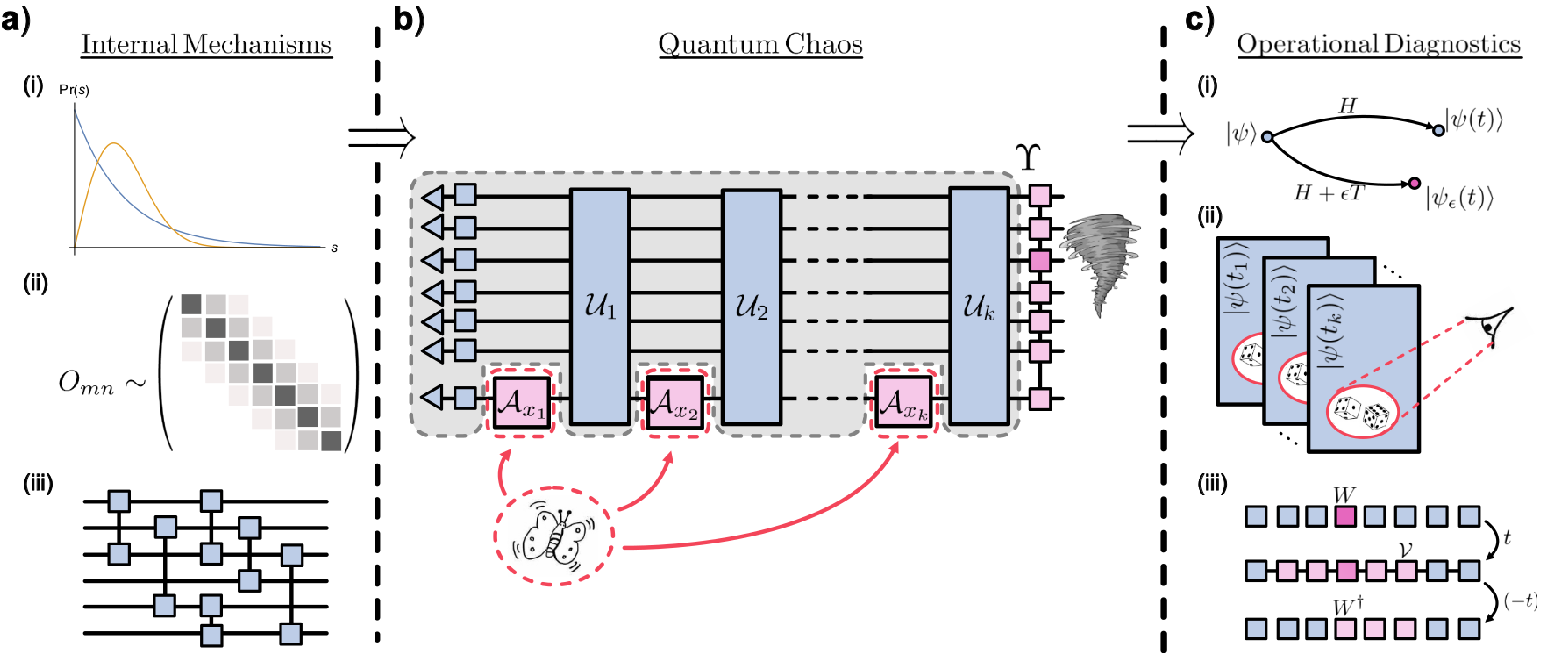
Linearity of isolated quantum mechanics – how can there be chaos?!



?

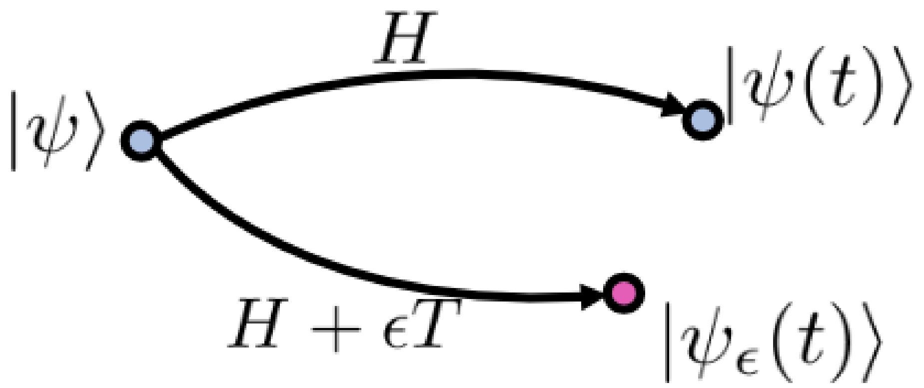
$$\langle \psi_t | \phi_t \rangle = \langle \psi | U_t^\dagger U_t | \phi \rangle$$
$$= \langle \psi | \phi \rangle$$

Quantum Chaos and its Causes and Effects



Example: Peres-Loschmidt Echo

- How does a perturbation to the Hamiltonian affect pure state dynamics?

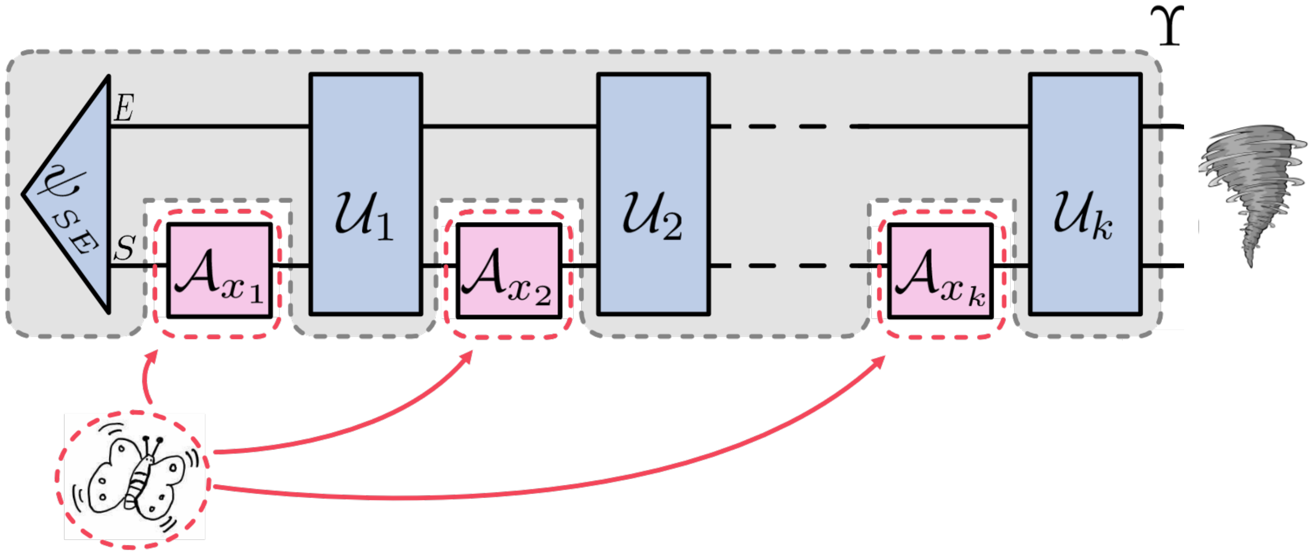


$$|\langle\psi_t|\psi_t^\epsilon\rangle|^2 = |\langle\psi|e^{iHt}e^{-it(H+\epsilon T)}|\psi\rangle|^2$$

- Conjectured that it decays exponentially for chaotic systems.
- But can show this is not sufficient for chaos.

The Quantum Butterfly Effect

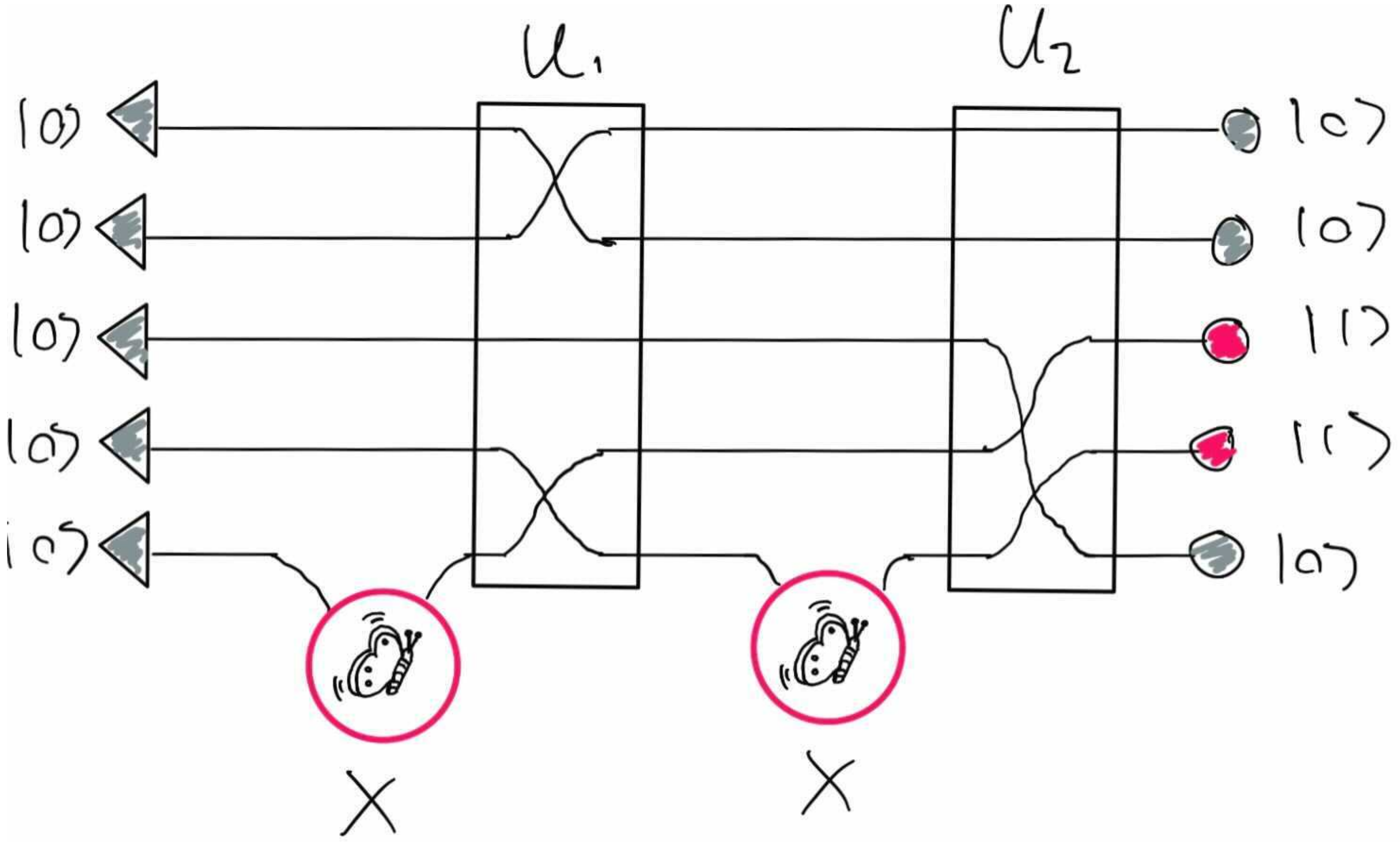
How well does a sequence of local operations orthogonalise the global pure state of a system?



$$|\Upsilon_{R|\vec{w}}\rangle := \frac{A_{w_k} U_k \cdots A_{w_2} U_2 A_{w_1} U_1 |\psi_{SE}\rangle}{\sqrt{\langle \psi_{SE}| U_1^\dagger \cdots A_{w_k}^\dagger A_{w_k} \cdots U_1 |\psi_{SE}\rangle}}$$

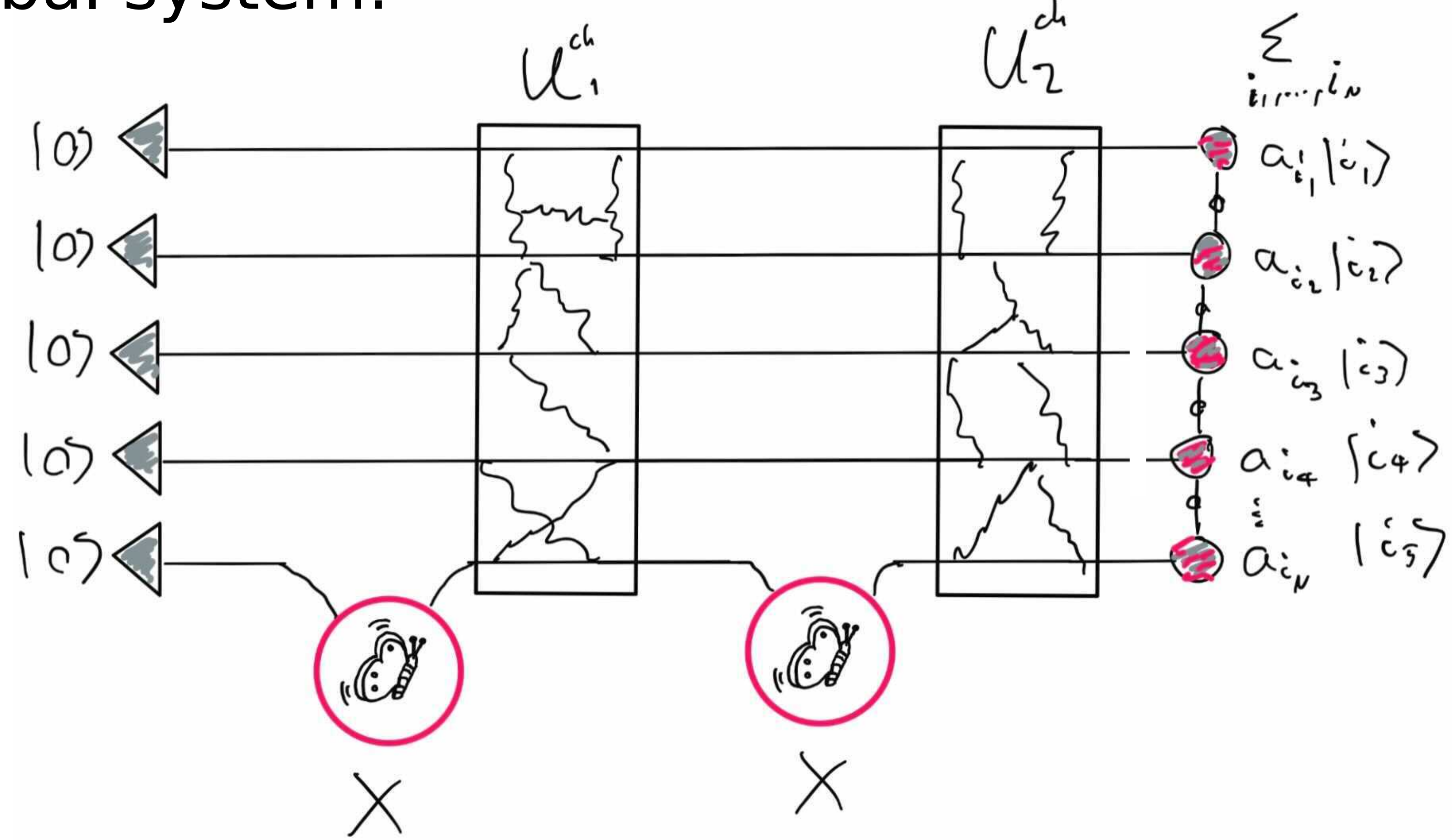
The Quantum Butterfly Effect

But, a circuit of swaps should not be chaotic:



The Quantum Butterfly Effect

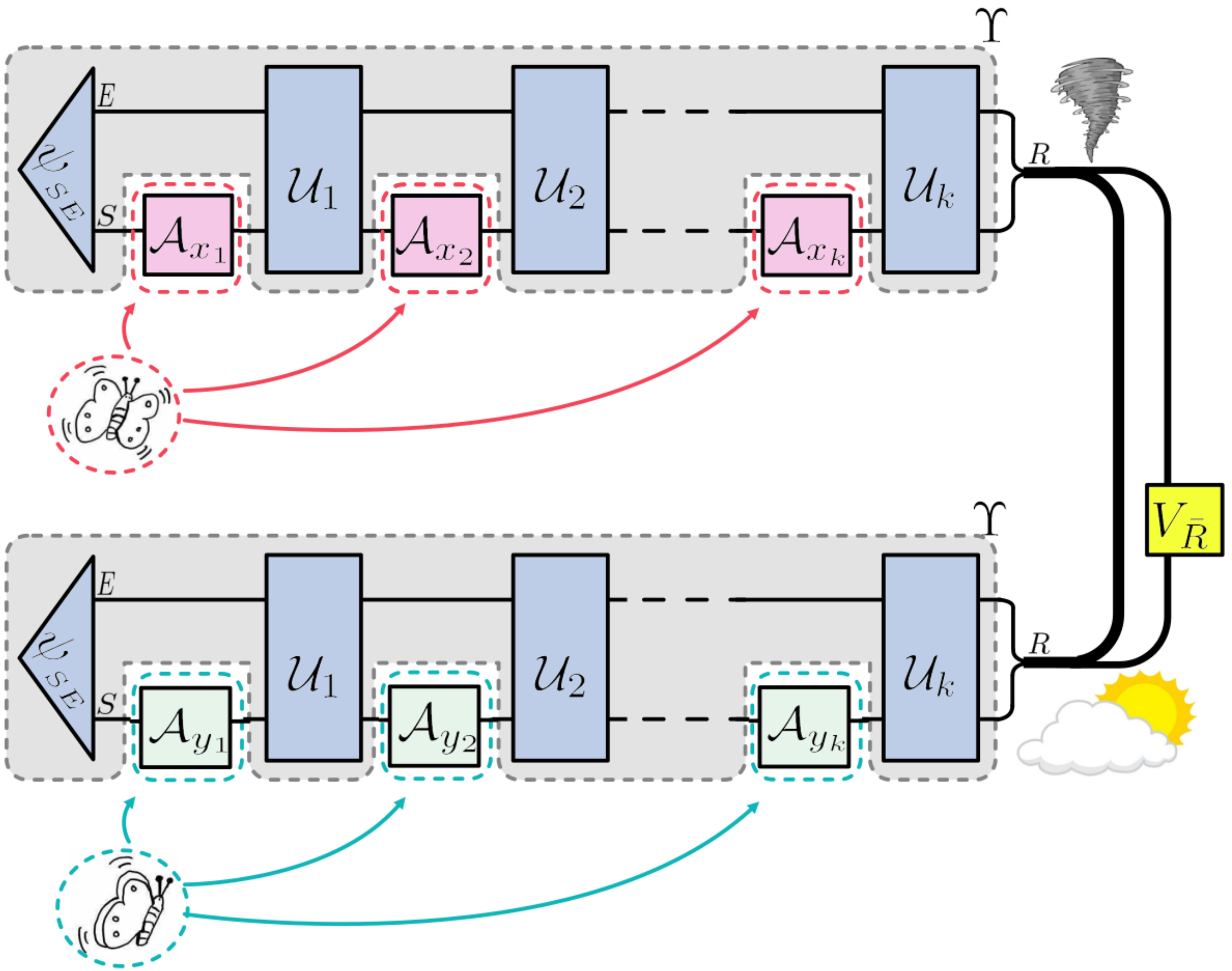
The orthogonality of the perturbation should spread through the global system:



The Quantum Butterfly Effect

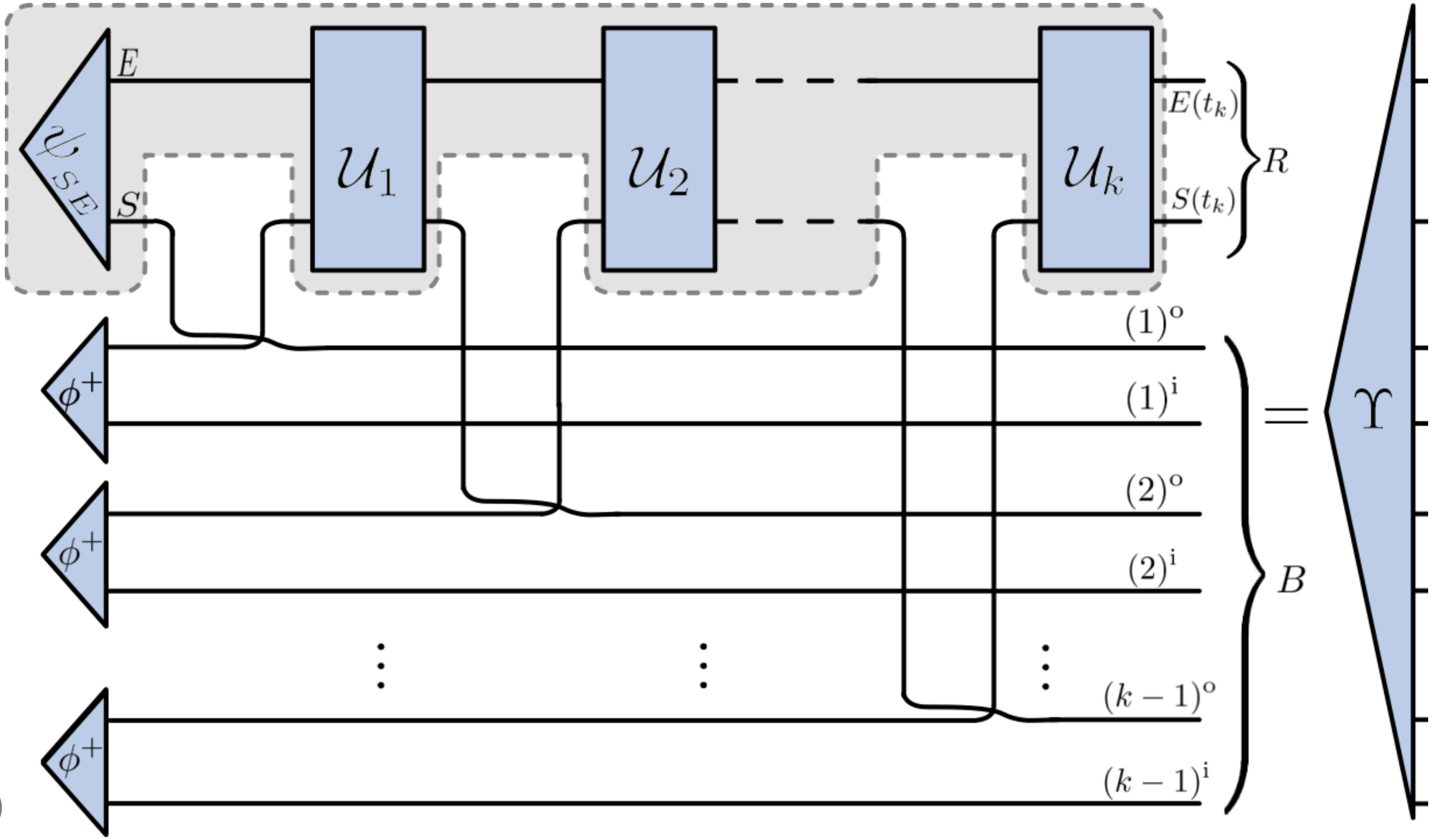
So enforce that this effect is highly non-local – that the process is *scrambling.*

$$\zeta(\Upsilon) := \sup_{\bar{R},V,\langle x|y\rangle=0} \left(|\langle \Upsilon_{R|x}|V_{\bar{R}}|\Upsilon_{R|y}\rangle|^2\right)$$



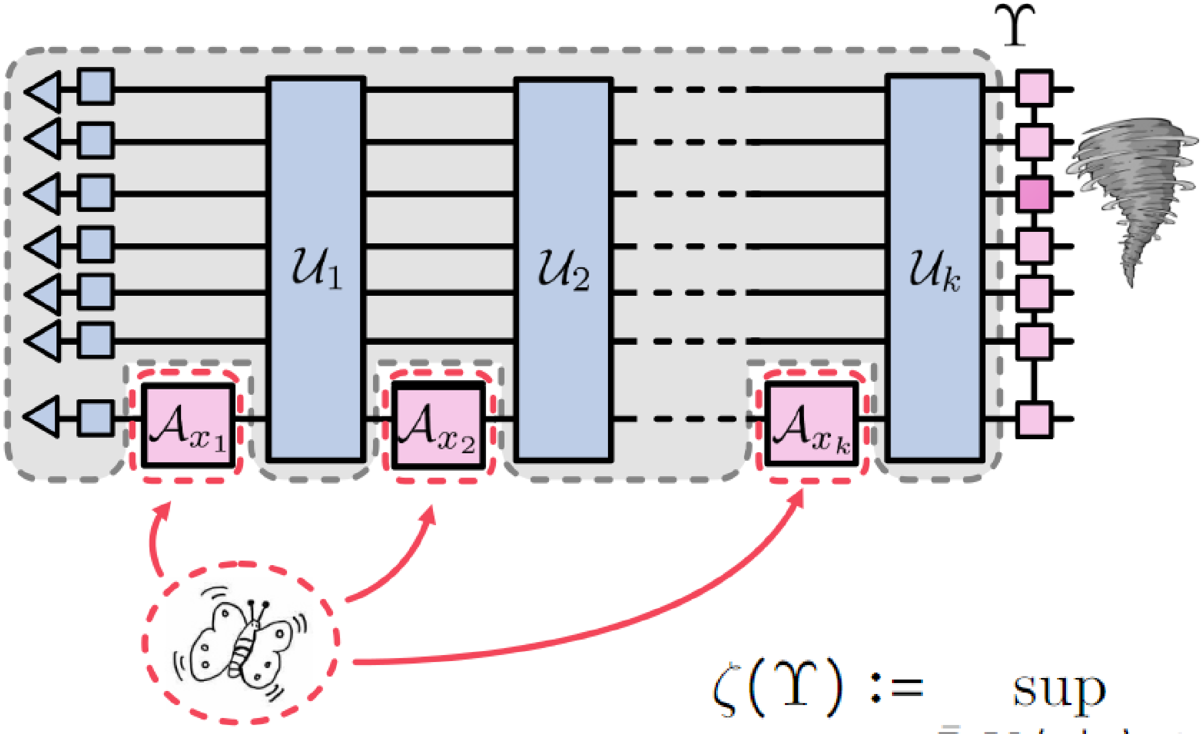
Tool: Process-State Duality

Choi–Jamiołkowski Isomorphism allows one to map any process one-to-one to a quantum state.



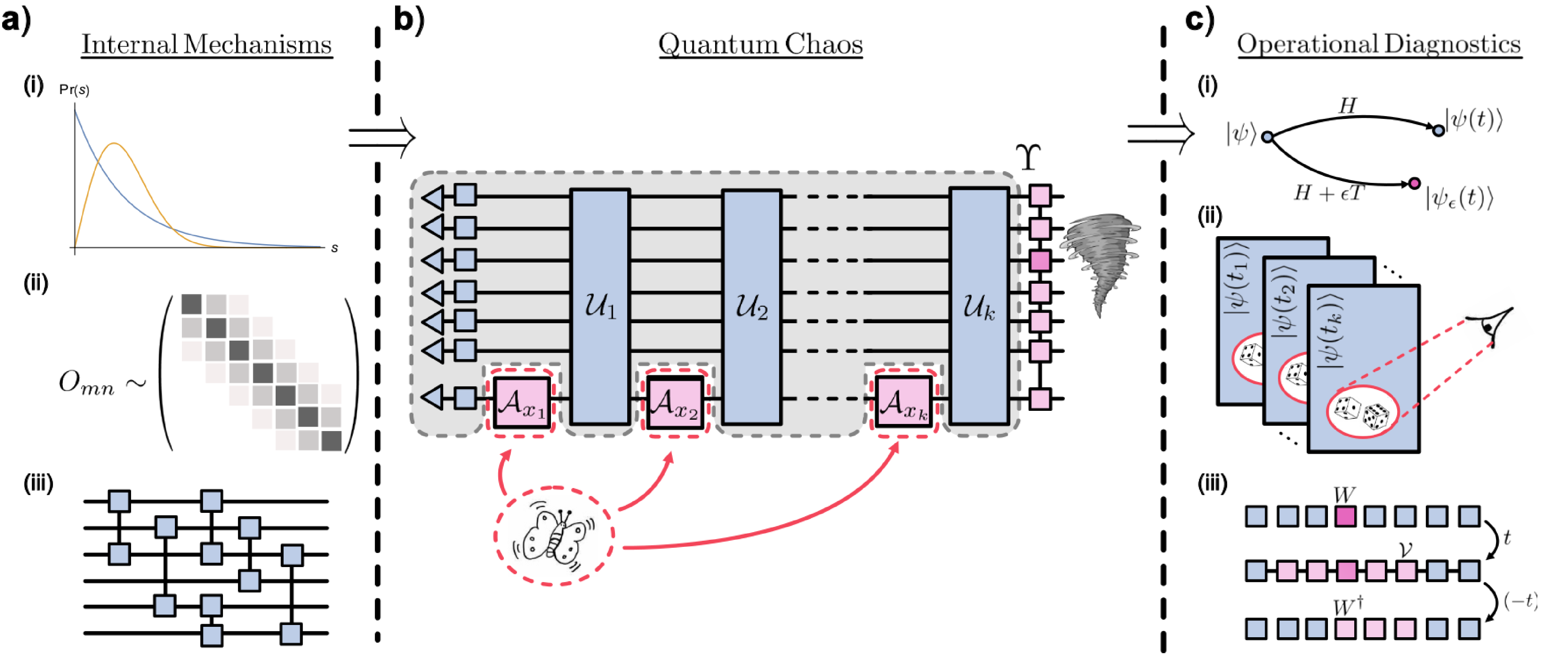
Pollock et al., PRL (2018)
Pollock et al., PRA (2018).

Quantum Chaos = Volume-Law spatiotemporal entanglement



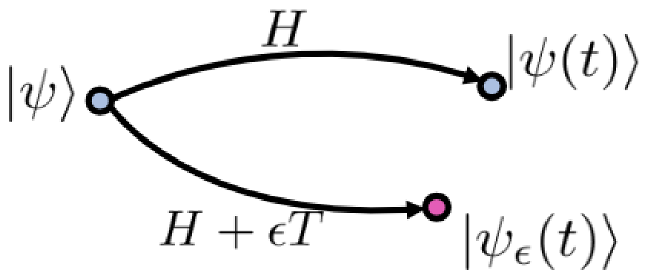
$$\zeta(\Upsilon) := \sup_{\bar{R}, V, \langle x|y\rangle = 0} \left(|\langle \Upsilon_{R|x} | V_{\bar{R}} | \Upsilon_{R|y} \rangle|^2 \right)$$

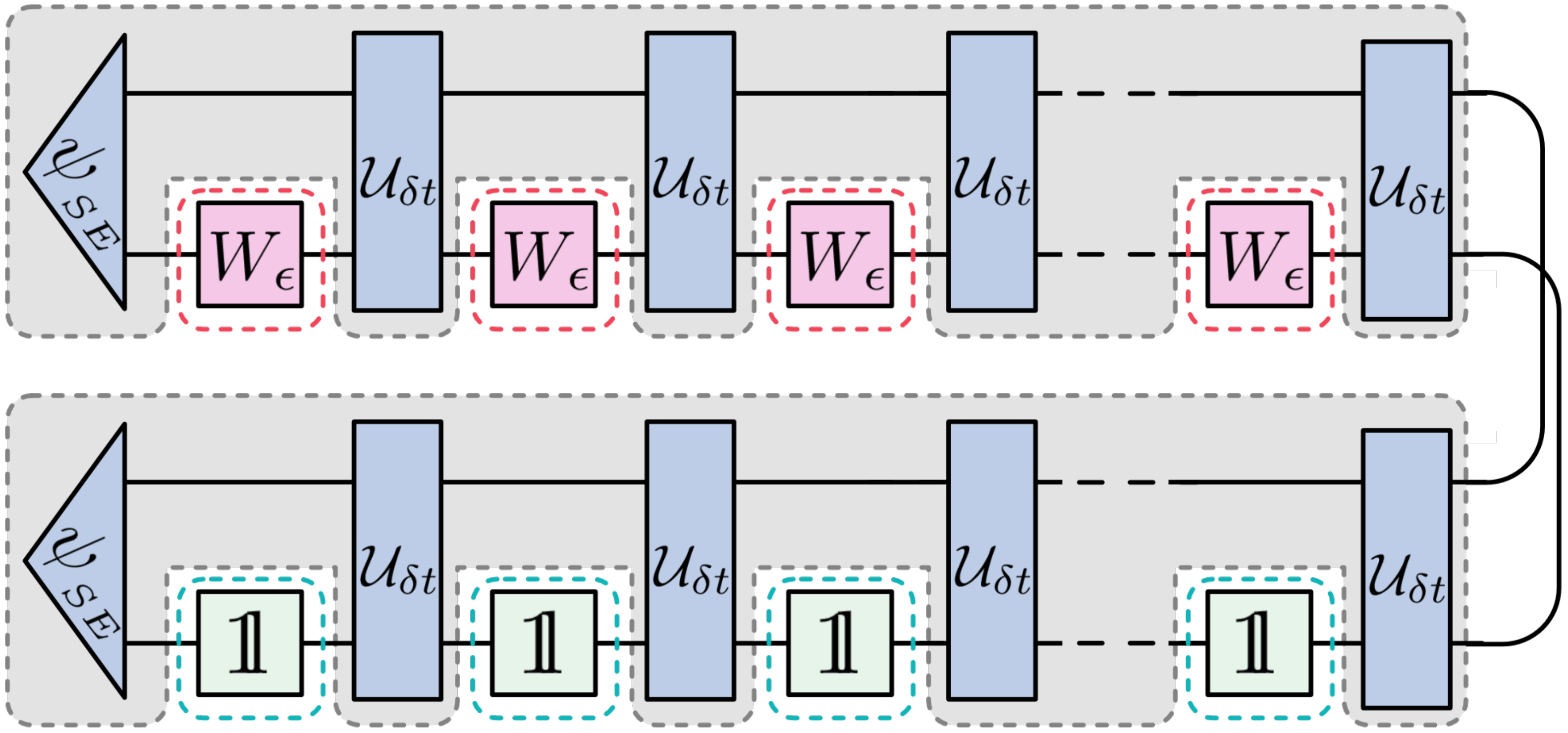
Quantum Chaos and its Causes and Effects



The Peres-Loschmidt Echo

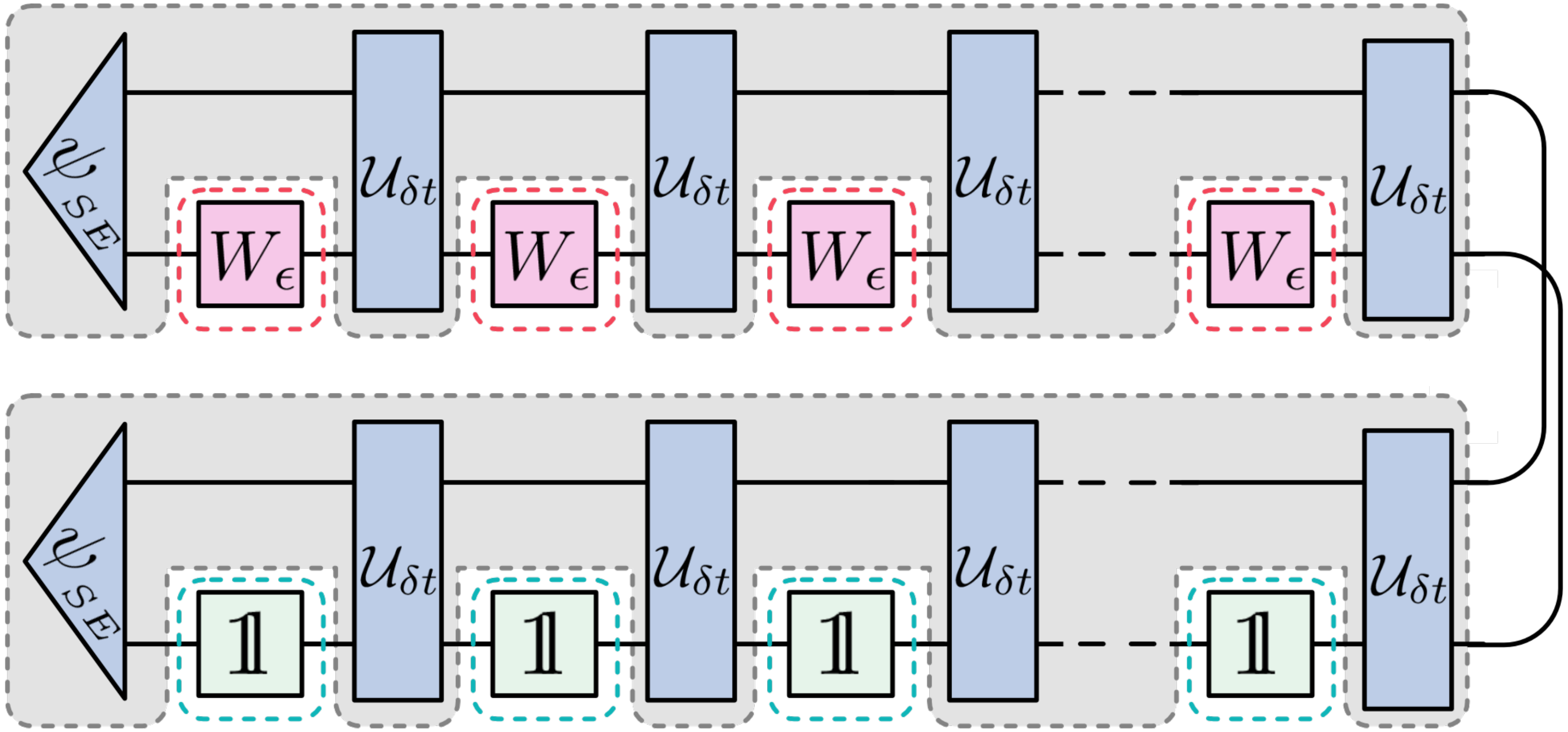
$$|\langle\psi_t|\psi_t^{\epsilon}\rangle|^2 = |\langle\psi|e^{iHt}e^{-it(H+\epsilon T)}|\psi\rangle|^2$$



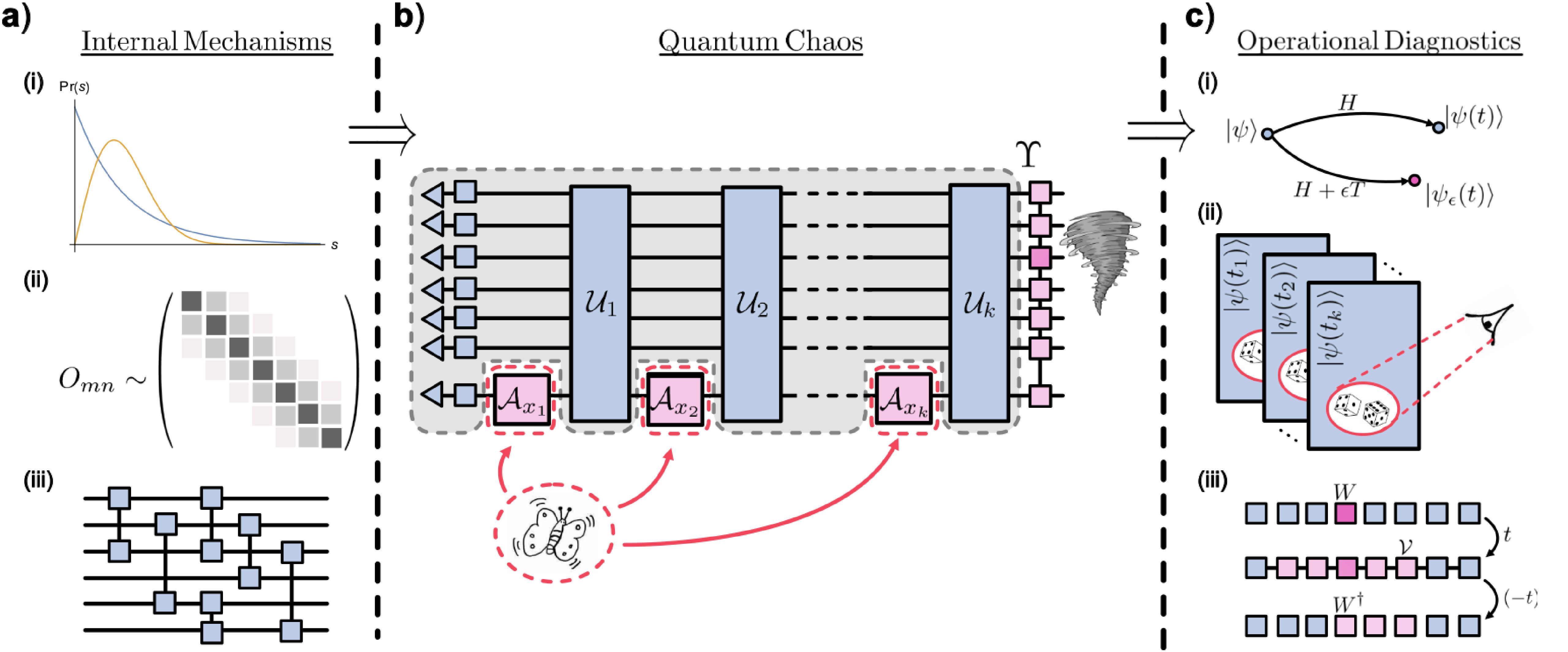


The Peres-Loschmidt Echo

Can show that for very chaotic systems (volume-entangled), this decays exponentially.



Quantum Chaos and its Causes and Effects



Mechanisms of Chaos

Random Dynamics are highly likely to generate a chaotic process.

- Exponentially so for Haar random unitary evolution.
- Polynomially so for t-designs.

$$\mathrm{tr}_R[\Upsilon_\mathrm{H}] :\approx \frac{\mathbb{1}}{d_S^{2k}}, \quad \text{for } d_B \ll d_R$$

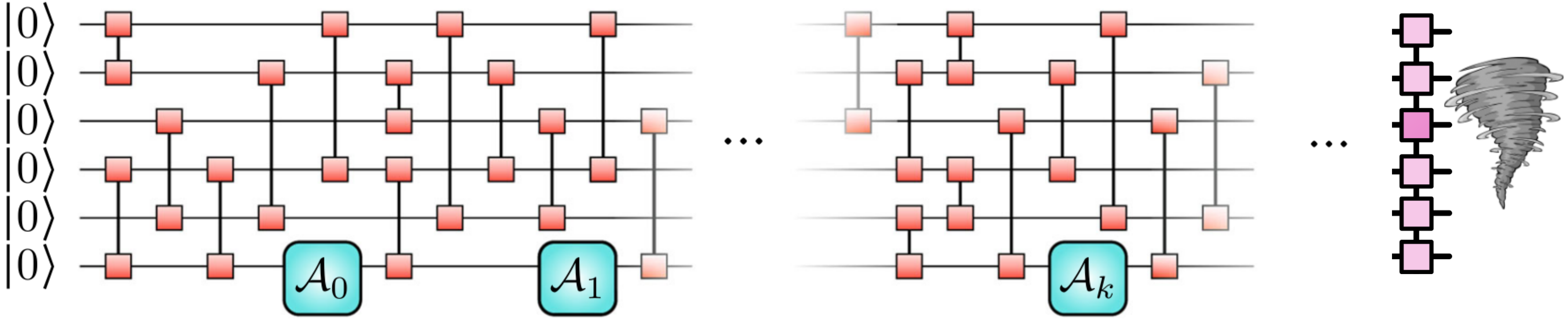


Fig. From: Figueroa-Romero et al., Nature Comm. Phys. (2021).

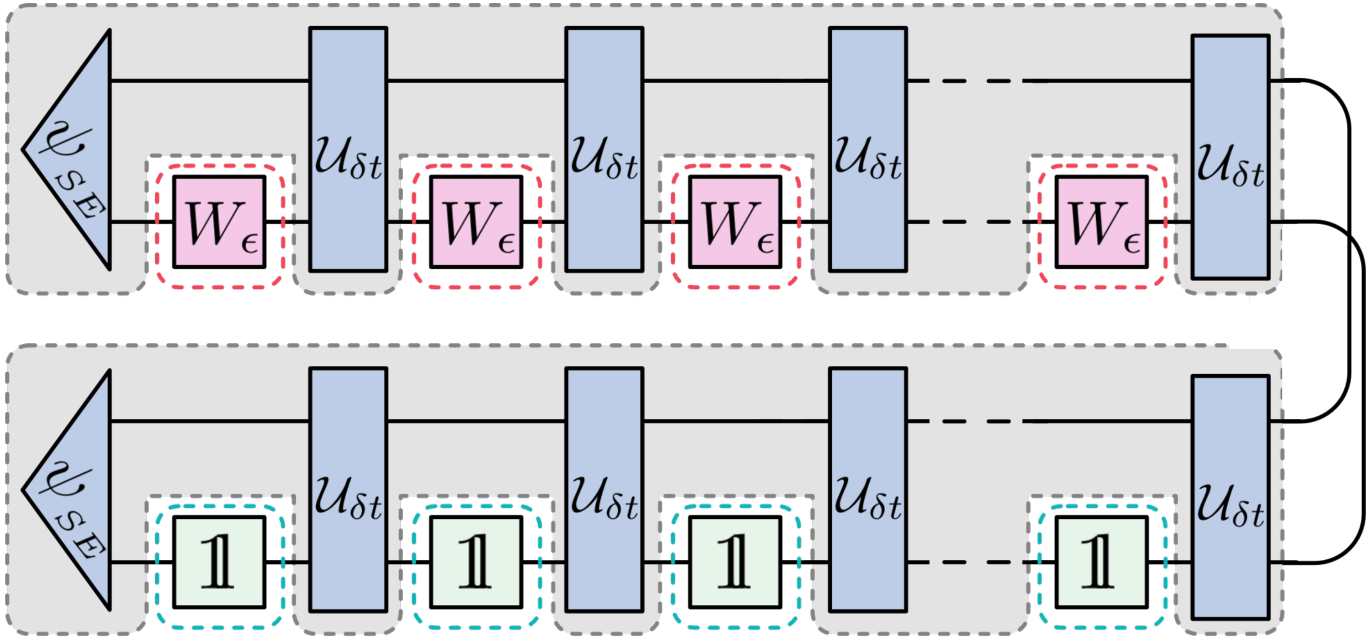
Summary and Applications

- We have constructed a quantum butterfly effect from a theory-independent principle of chaos.
- This allows us to identify chaos as the volume-law spatiotemporal entanglement property of a quantum process.
- Other operational diagnostics derive from this, and we show many mechanisms lead to it.
- Can systematically approach problems in many body physics.
- Other mechanisms of chaos: Random Matrix Theory?
- Emergence of a quantum Lyapunov exponent?

Thank you!

The Peres-Loschmidt Echo: decay

$$\zeta'(\Upsilon^{(\mathrm{H})}, \mathcal{X}_{LE}) \approx \frac{|\langle\vec{x}|\mathbb{1}/d_B|\vec{y}\rangle|^2}{\langle\vec{x}|\mathbb{1}/d_B|\vec{x}\rangle\langle\vec{y}|\mathbb{1}/d_B|\vec{y}\rangle}$$
$$= (1-\epsilon)^{2k} d_B^2 (1/d_B^2)$$
$$= (1-\epsilon)^{2k}$$
$$\approx \mathrm{e}^{-2k\epsilon}, \text{ for small } \epsilon,$$
$$\approx 0, \text{ for large } k.$$



Mechanisms of Chaos

Random Dynamics are highly likely to generate a chaotic process.

$$\mathbb{P}_{U_i \sim \mu}\left\{\tilde{\zeta}(\Upsilon, \mathcal{X}_{\mathrm{b}}) \geq \frac{\mathcal{J}_\mu(\delta)}{(d_{B\bar{R}} - 1)(1 - \sqrt{\mathcal{J}_\mu(\delta)})^2}\right\} \leq \mathcal{G}_\mu(\delta).$$

Haar evolution:

$$\mathcal{J}_{\mathrm{H}}(\delta) = d_{B\bar{R}}(\mathcal{B} + \delta) \approx d_{B\bar{R}}(\frac{1}{d_R} + \delta), \ \ and$$

$$\mathcal{G}_{\mathrm{H}}(\delta) = \exp[-\mathcal{C}\delta^2] \approx \exp[-\frac{(k+1)d_R}{8d_B}\delta^2],$$

ε-approximate t-deisgn:

$$\mathcal{J}_{\mu_{\epsilon-t}}(\delta) = d_{B\bar{R}}\delta, \ \ and$$

$$\mathcal{G}_{\mu_{\epsilon-t}}(\delta) = \frac{\mathcal{F}(d_B, d_R, m, t, \epsilon)}{\delta^m}$$