Mapping the 3D structure of hadrons with lattice quantum chromodynamics

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THE UNIVERSITY *of* ADELAIDE

SPECIAL RESEARCH CENTRE FOR THE SUBATOMIC STRUCTURE OF MATTER

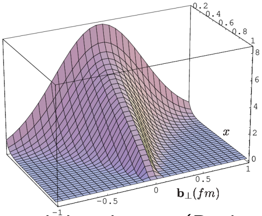
Outline of the problem

Generalised parton distributions are

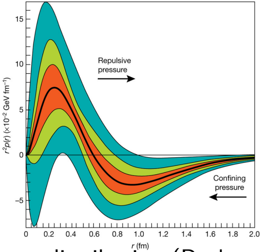
- extensions of PDFs
- related to elastic FFs and EMT

Contain a staggering amount of physical information:

- the spatial distributions of hadron constituents
- a solution to proton spin puzzle
- proton pressure distribution



Quark spatial distribution (Burkardt, 2002)



Proton pressure distribution (Burkert et al., 2018)

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Contain a staggering amount of physical information:

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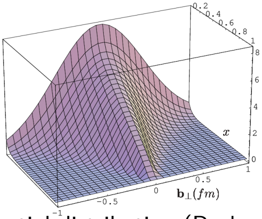
However...

- Difficult to measure experimentally
- and difficult to calculate on the lattice

In this talk: a new lattice method to calculate GPDs (Feynman-Hellmann), with strong parallels to experiment → electron-ion collider



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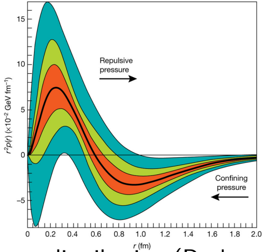
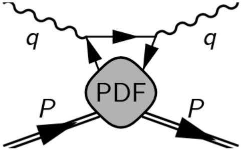


Proton pressure distribution (Burkert et al., 2018)

What are generalised parton distributions?

Compton amplitude



Parton distribution functions

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \overbrace{\langle P|\bar{\psi}_q(-\lambda n/2)\not{n}\psi_q(\lambda n/2)|P\rangle}^{\text{light-cone matrix elem}} = q(x)$$

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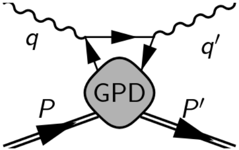

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$$+ E^q(x,\xi,t)\bar{u}(P')\frac{i\sigma^{\mu\nu}n_\mu(P'-P)_\nu}{2M}u(P).$$

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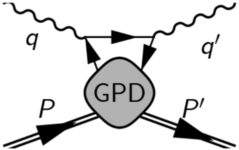

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- H^q and E^q helicity-conserving and -flipping GPDs
- $H^q(x,\xi,t) \xrightarrow{t\to 0} q(x)$
- $t = (P'-P)^2$ gives us access to spatial structure

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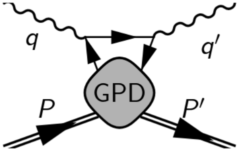
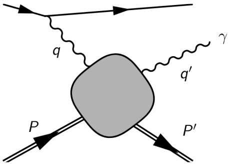


Deeply virtual Compton scattering



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Measuring GPDs

Very difficult to get GPDs from these measurements $\longrightarrow$ lattice QCD can be useful

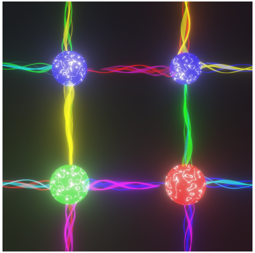
Lattice QCD

QCD path integral

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \mathcal{O} e^{iS_{\text{QCD}}}.$$

To evaluate this numerically:

1. discrete spacetime,
2. Wick rotatation $t \to -i\tau$, $e^{iS_{\text{QCD}}} \to e^{-S^E_{\text{QCD}}}$,
3. generate gauge configurations according to $e^{-S^E_{\text{QCD}}}$.



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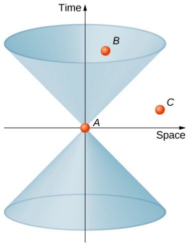

Wick rotated separation:

$$x^2 = (-i\tau)^2 - |\vec{x}|^2 = -\tau^2 - |\vec{x}|^2 < 0.$$

Separations are spacelike, but PDs require lightlike.
$\Rightarrow$ Can't calculate parton distributions on the lattice

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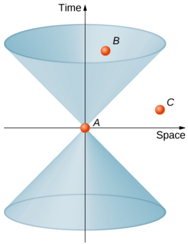
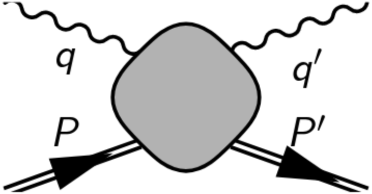


Related quantities we can calculate

1. Mellin moments (traditional)
2. Quasi- and pseudo-distributions (newer)
3. Scattering amplitude for unphysical kinematics (this talk)

The Compton amplitude from Feynman-Hellmann

Direct lattice calculation:



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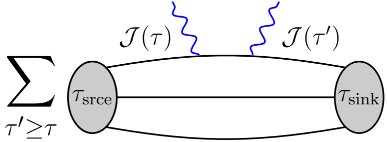

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- New inversion for each (τ, τ')
- Expensive!

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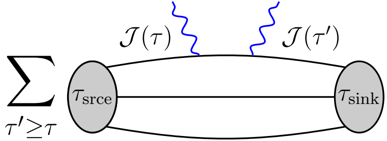

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Calculate 2-pt function, with quarks immersed in two magnetic fields:
$\vec{B}_1 = (0, 0, \lambda_1 \cos(\vec{q}_1 \cdot \vec{x}))$ and $\vec{B}_2 = (0, 0, \lambda_2 \cos(\vec{q}_2 \cdot \vec{x}))$.

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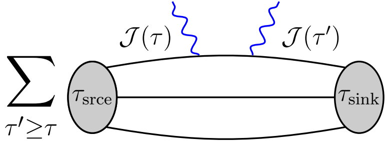

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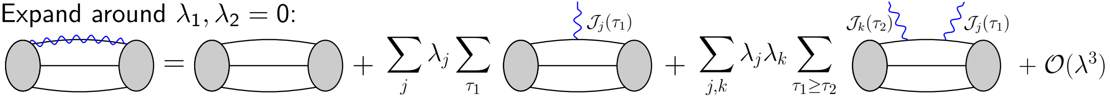
Expand around $\lambda_1, \lambda_2 = 0$:



Then $\lambda_1 \lambda_2$ term has $\vec{q}_1$ in $\vec{q}_2$ out $\Rightarrow$ isolate this to get OFCA.

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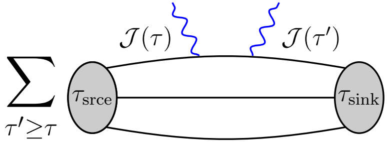

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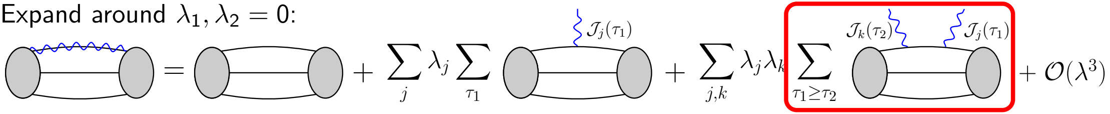

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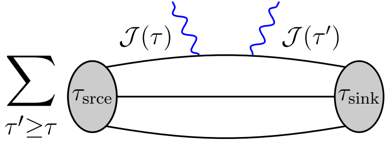

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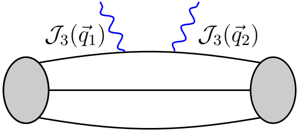
Expand around $\lambda_1, \lambda_2 = 0$:

$$\mathcal{G}_{(\lambda,\lambda)} + \mathcal{G}_{(-\lambda,-\lambda)} - \mathcal{G}_{(-\lambda,\lambda)} - \mathcal{G}_{(\lambda,-\lambda)} \simeq \lambda^2$$  $$+ \mathcal{O}(\lambda^4)$$

Then $\lambda_1 \lambda_2$ term has $\vec{q}_1$ in $\vec{q}_2$ out $\Rightarrow$ isolate this to get OFCA.

Lattice Compton amplitude

OFCA parameterised in terms of Compton form factors:

$$T_{\mu\nu} = \frac{1}{2\bar{P}\cdot\bar{q}}\left[-\left(h\cdot\bar{q}\mathcal{H}_1 + e\cdot\bar{q}\mathcal{E}_1\right)g_{\mu\nu} + \frac{1}{\bar{P}\cdot\bar{q}}\left(h\cdot\bar{q}\mathcal{H}_2 + e\cdot\bar{q}\mathcal{E}_2\right)\bar{P}_\mu\bar{P}_\nu + \mathcal{H}_3 h_{\{\mu}\bar{P}_{\nu\}}\right] + \ldots$$

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To get GPDs from our lattice data inverse problem

$$\overbrace{\mathcal{H}_1(\bar{\omega}, t)}^{\text{From lattice}} = 2\bar{\omega}^2 \int dx \frac{x\,\overbrace{H(x,t)}^{\text{GPD}}}{1-(x\bar{\omega})^2}$$

Hard to solve for $H(x,t)$!

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Hard to solve for $H(x,t)$!
But we can get the Mellin moments

$$\mathcal{H}_1(\bar{\omega},t) = 2\sum_{n\text{ even}}^{\infty}\bar{\omega}^n \int dx x^{n-1} H(x,t)$$

Moments defined as

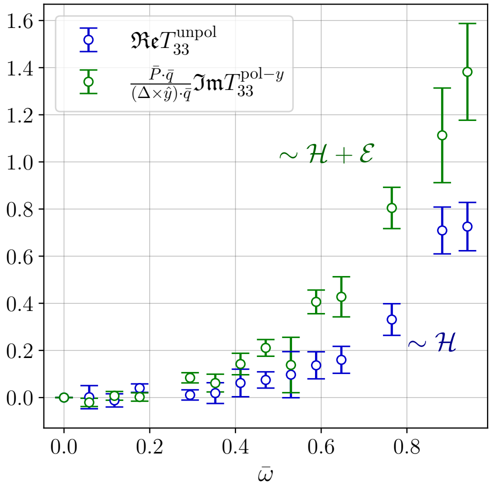
$$M_n(t) = \int dx x^{n-1} H(x,t)$$

- $n=1$ elastic FFs, $n=2$ gravitational FFs, at $t=0$ reduce to $\langle x^{n-1}\rangle$
- Can be individually calculated on lattice ($n=3$ highest so far)
- What we do is fit to power series

$$f(\bar{\omega}) = M_2\bar{\omega}^2 + M_4\bar{\omega}^4 + M_6\bar{\omega}^6 + \dots$$

Lattice results

- Can calculate unpolarised Compton amplitude $(T_\uparrow + T_\downarrow)/2$ or polarised $(T_\uparrow - T_\downarrow)/2$.
- Each is a different linear combination of $\mathcal{H}_1$ and $\mathcal{E}_1$.



Isovector results for $t = -0.57\ \mathrm{GeV}^2$.

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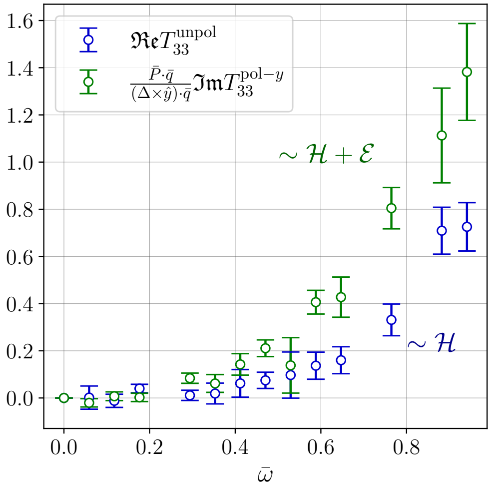

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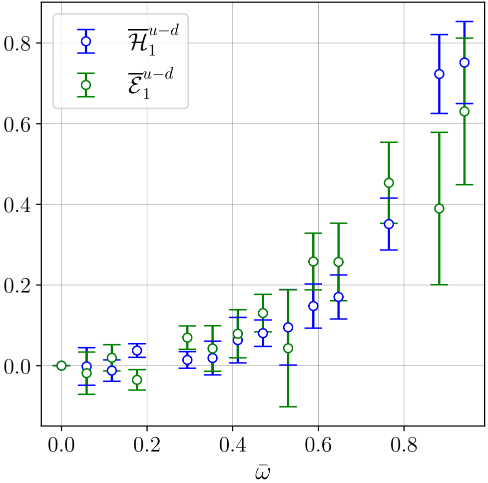

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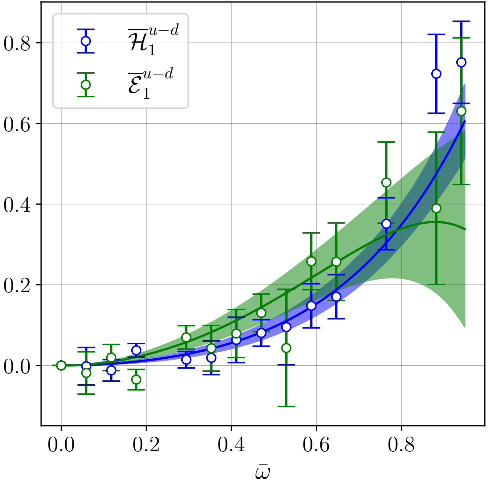

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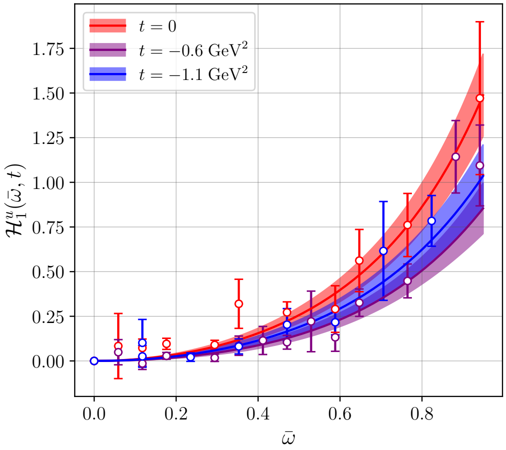
GPD moments

Fit to function

$$\mathcal{H}_1(\bar{\omega}, t, \bar{Q}^2) = 2 \sum_{n \text{ even}}^{N_{\text{max}}} \bar{\omega}^n M_n(t, \bar{Q}^2)$$

For $\bar{Q}^2 \gg \Lambda^2_{\text{QCD}}$, get GPD moments

$$M_n^{(H)}(t) = A_{n,0}(t), \quad M_n^{(E)}(t) = B_{n,0}(t)$$



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- Our leading even moment consistent with other lattice calculations
- First determination of $n \geq 4 \Rightarrow$ no other lattice results to compare
- Good first test—can we go further?

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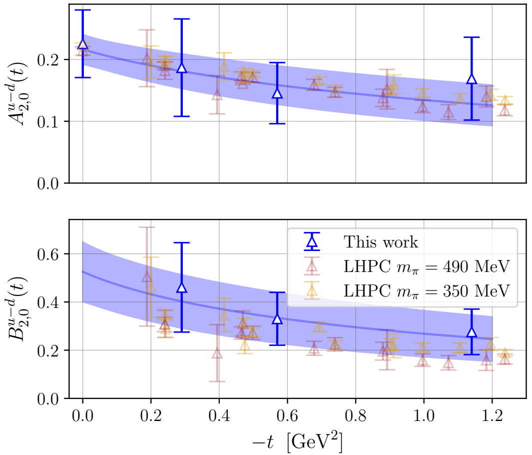

Inverse problem: accessing full GPDs

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Limited number of $\bar{\omega}$, large errors, lattice artefacts $\longrightarrow$ hard to solve!

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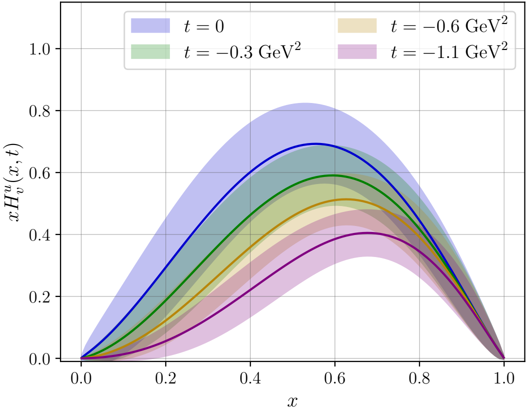
Widely used in expt. (e.g. proton pressure distribution)



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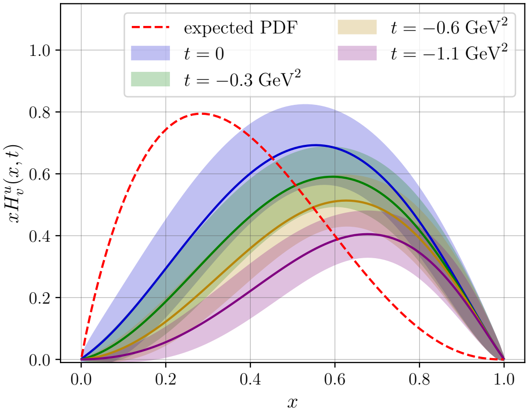

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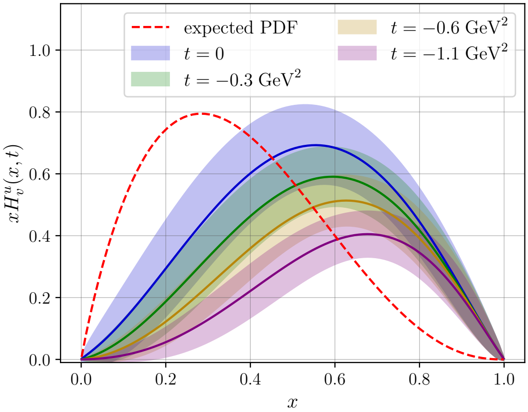
Q: What went wrong? The first moment was correct...



Inverse problem: accessing full GPDs

$$\overbrace{\mathcal{H}_1(\bar\omega, t)}^{\text{From lattice}} = 2\bar\omega^2 \int dx \frac{x\, \overbrace{H(x,t)}^{\text{GPD}}}{1-(x\bar\omega)^2}$$

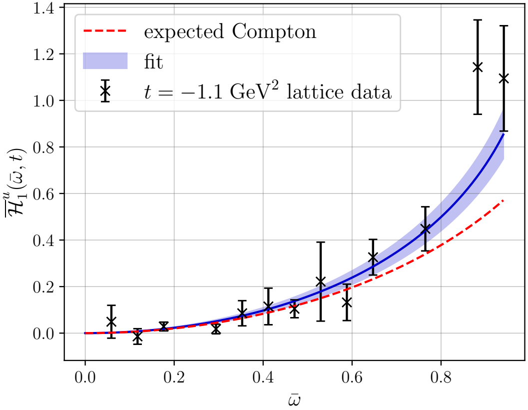
Limited number of $\bar\omega$, large errors, lattice artefacts $\longrightarrow$ hard to solve!

Model GPD:

$$H(x,t) = N x^{-\alpha-\alpha' t}(1-x)^\beta$$

Widely used in expt. (e.g. proton pressure distribution)

Q: What went wrong? The first moment was correct...
A: Higher ω vals not consistent with phenom $\longrightarrow \mathcal{O}(ap_\mu)$ artefacts

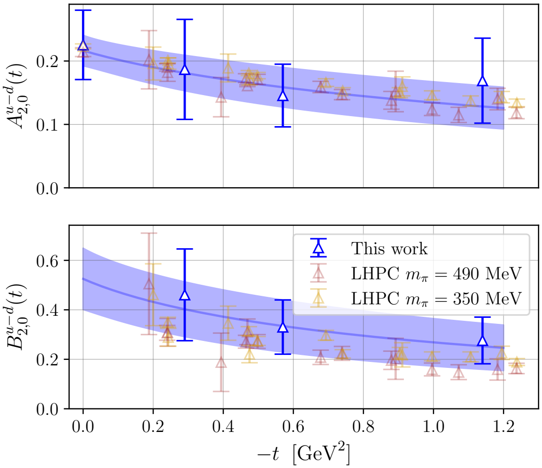


Conclusion

- New method to calculate OFCA $\rightarrow$ GPDs
- Allows strong parallels with experiment
- Calculation of leading moments of $\mathcal{H}_1$ and $\mathcal{E}_1$ demonstrates viability
- Need to understand lattice artefacts at large momentum

Outlook

- Control lattice artefacts $\longrightarrow$ investigation of GPD models and inversion
- Non-leading twist terms
- Subtraction function

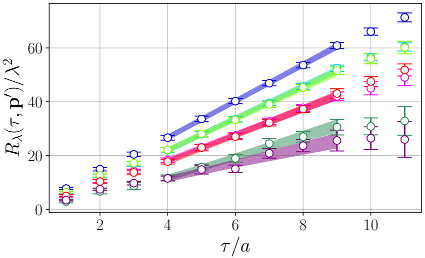


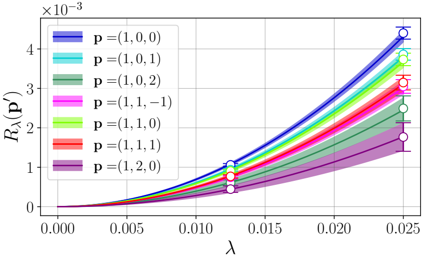
Lattice details

This calculation

Set	t [GeV^2]	$\bar{Q}^2$ [GeV^2]	N_{meas}
	0	4.86	10000
#1	-0.29	4.79	1000
#2	-0.57	4.86	1000
#3	-1.14	4.86	1000

- Lattice size: $L^3 \times T = 48^3 \times 96$
- Unphysical pion mass $m_\pi = 420$ MeV.
- For each set, calculate two couplings: $\lambda = 0.0125, 0.025$
- $\beta = 5.65$, $\kappa_l, \kappa_s = 12205$





Recall:

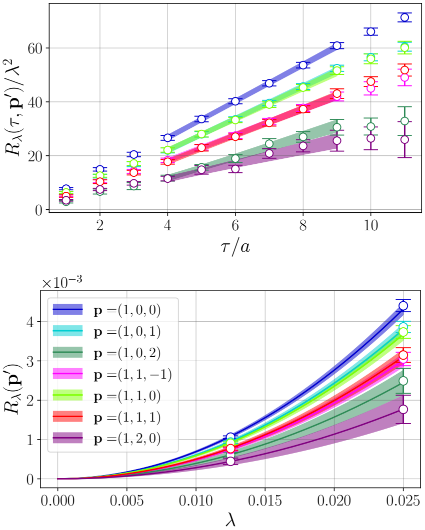
$$R_\lambda \equiv \frac{\mathcal{G}_{(\lambda,\lambda)} + \mathcal{G}_{(-\lambda,-\lambda)} - \mathcal{G}_{(-\lambda,\lambda)} - \mathcal{G}_{(\lambda,-\lambda)}}{\mathcal{G}_0}$$

$$\simeq 2\lambda^2 \tau \frac{T_{33}}{E_N(\vec{p})}$$

Fits

1. $f(\tau) = a\tau + b$
2. $g(\lambda) = c\lambda^2$

Final result is $T_{33}(\bar{\omega})$.



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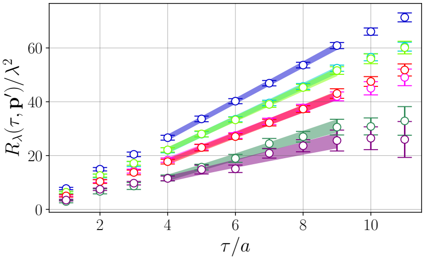
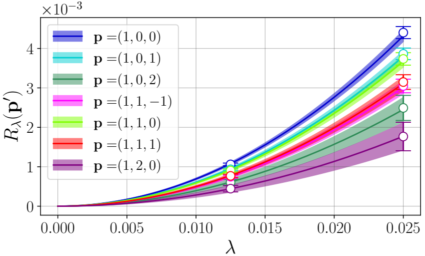
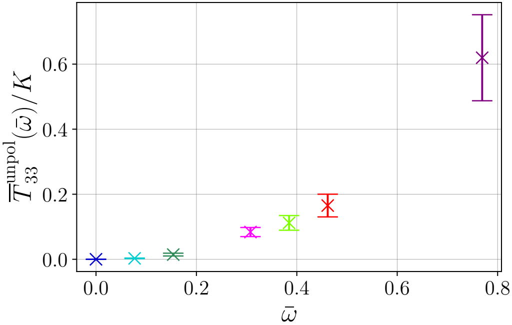
By varying the sink momentum, $\vec{p}$, we vary scaling variable:

$$\bar{\omega} = \frac{4\vec{p} \cdot (\vec{q}_1 + \vec{q}_2)}{(\vec{q}_1 + \vec{q}_2)^2}$$

Feynman-Hellmann application







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