Exploring higher order dispersion: Families of exact soliton solutions

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Solitons

- Pulses unchanging in propagation even in presence of dispersion and nonlinearity
- Optical solitons widespread in optical systems:
  - Ultrafast pulse generation
  - Supercontinuum generation
  - Frequency Combs

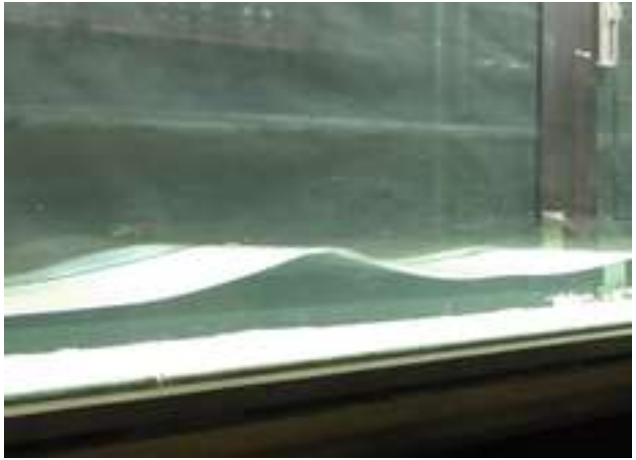


Image: Redor, I. (2019), Phys. Rev. Lett. 122, 214502

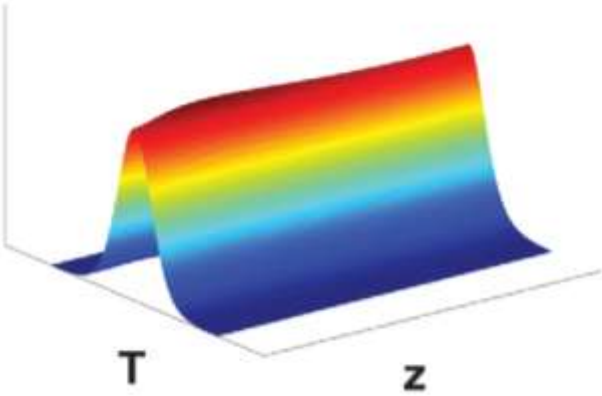
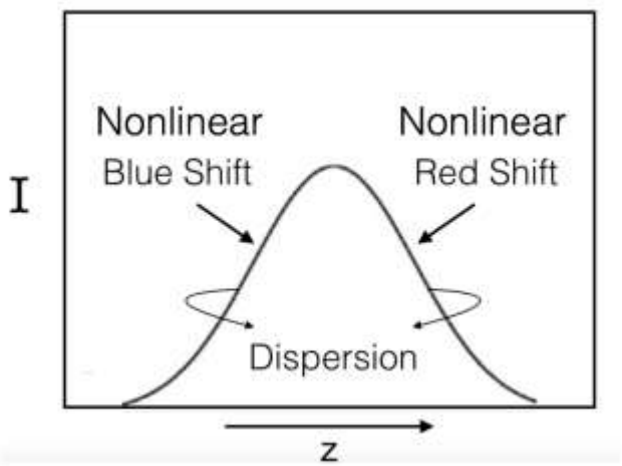


Image: Marko et al.(2013) Disturbance of soliton pulse propagation from higher-order dispersive waveguides

Soliton Formation

– Kerr Nonlinearity: Leading edge – red shifted, Trailing edge – blue shifted
– Dispersion: Phase velocity of light depends on frequency
– Anomalous Dispersion: Red light moves slower than blue light (occurs when 2nd order dispersion is negative)



Dispersion

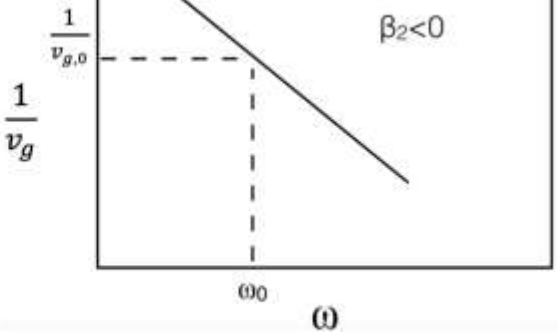
– Modal propagation constant β: $$\beta = \beta_0 + \frac{d\beta}{d\omega}(\omega - \omega_0) + \frac{1}{2}\frac{d^2\beta}{d\omega^2}(\omega - \omega_0)^2 \ldots$$

– Anomalous dispersion: high frequencies faster than low frequencies

– Inverse group velocity monotonically decreasing with frequency

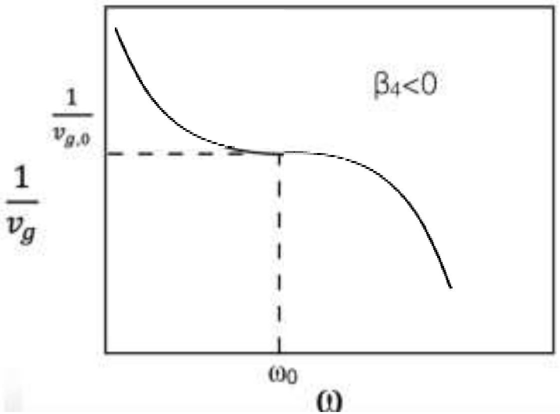
$$\beta_1 \equiv \frac{d\beta}{d\omega} = \frac{1}{v_g}$$

$$\beta_2 \equiv \frac{d^2\beta}{d\omega^2} = \frac{d\frac{1}{v_g}}{d\omega}$$



For even order dispersion

$$\beta_4 \equiv \frac{d^4\beta}{d\omega^4} = \frac{d^3\frac{1}{v_g}}{d\omega^3}$$



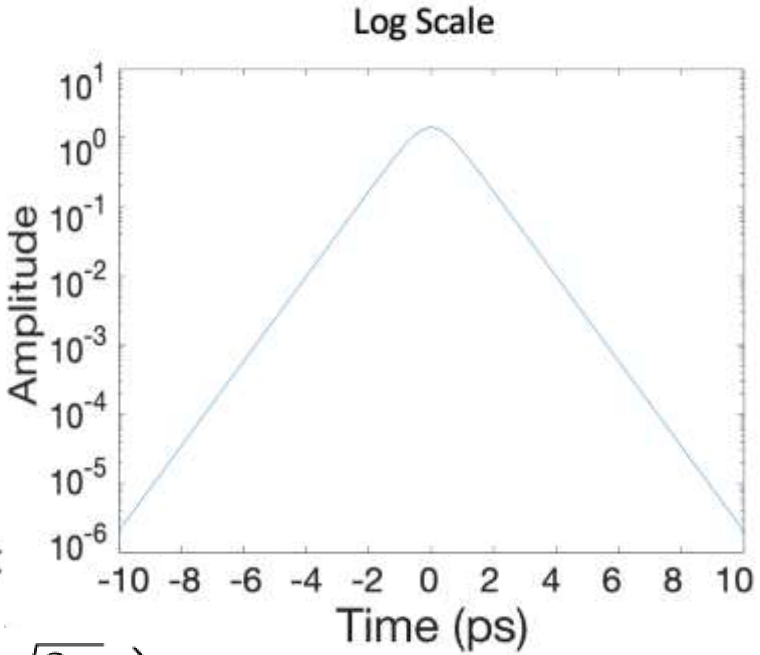
Conventional Soliton

– Nonlinear Schrödinger Equation:

$$i\frac{\partial\psi}{\partial z} - \frac{\beta_2}{2}\frac{\partial^2\psi}{\partial\tau^2} + \gamma|\psi|^2\psi = 0$$

Linear effect:
Anomalous 2nd Order
Dispersion

Nonlinear effect:
Positive Kerr Effect

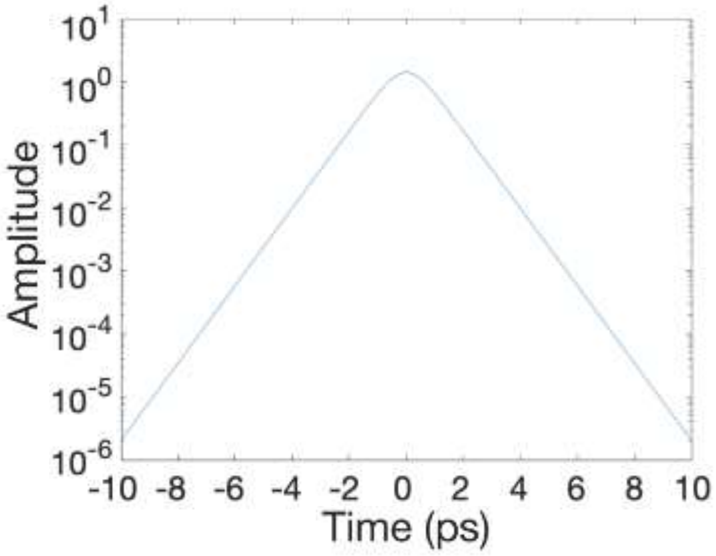


– Analytic Solution: $\psi(\tau, z) = \sqrt{\frac{2\mu}{\gamma}}\mathrm{sech}\left(\sqrt{\frac{2\mu}{\beta_2}}\tau\right)e^{i\mu z}$

Zakharov, V.E. and Shabat, A.B. (1972) Journal of Experimental and Theoretical Physics, 34, 62

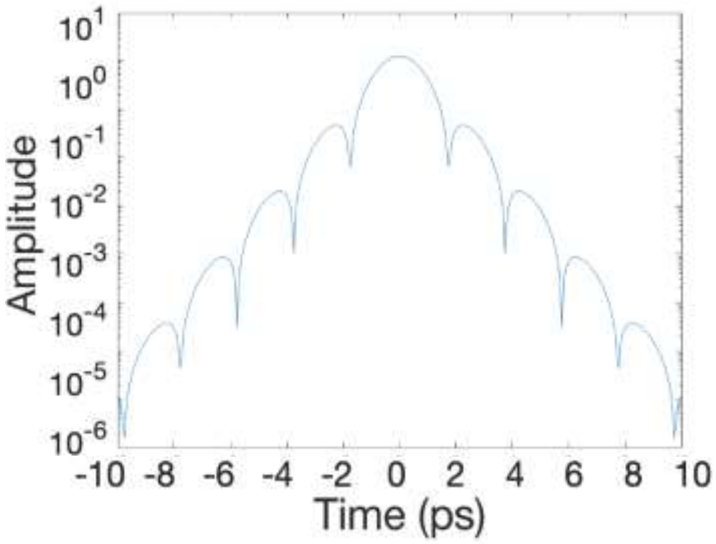
Conventional vs PQS

Conventional
Pure Exponential Decay



Log Scale

Pure Quartic Soliton
Exponential Decay with Oscillations

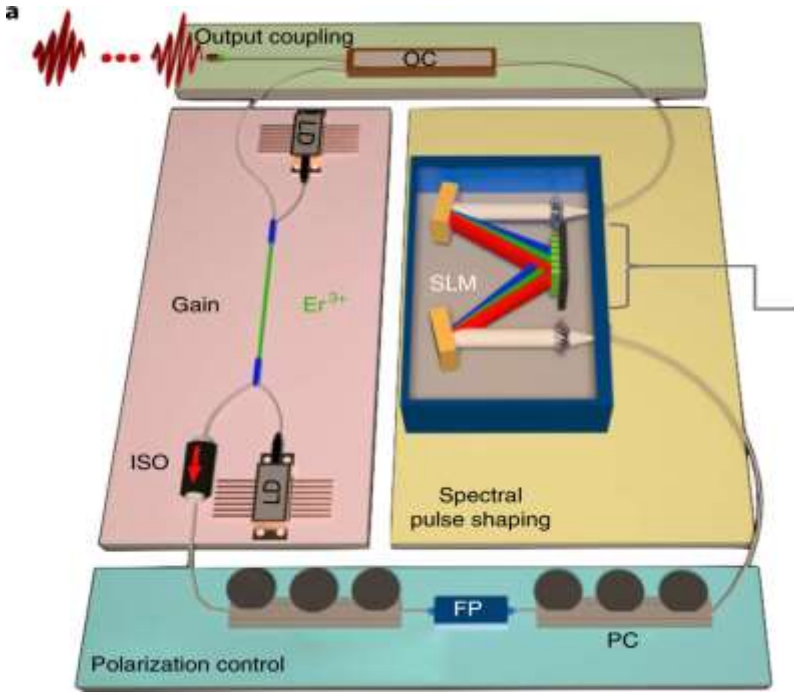


Log Scale

Akhmediev N, et al. (1994) Opt. Commun. 110 540–544
Blanco-Redondo, A. *et al.* (2016) *Nat. Commun.* **7**, 10427

Programmable Dispersion Solitons

- 2020 PQS laser constructed generate soliton pulses
- Spectral pulse shaper induces net cavity dispersion
- Programme **any** dispersion
- There exist infinite number of dispersion profiles with inverse group velocity decreasing



Runge, A.F.J. *et al. Nat. Photonics* **14**, 492–497 (2020).
Runge, A.F.J. *et al. Phys. Rev. Research* 3, 013166, (2021).
Lourdesamy, J.P. *et al. Nat. Physics* **18**, 59–66 (2022).

Programmable solitons at any dispersion order?

Do they exist?

How do we study them in a systematic way?

– Searching for exact soliton solutions at any dispersion order

– Characterising based on their properties

Analytic Solution: Single Term

- Search for analytic solution
- Conventional Soliton:

$$\psi = A\ \mathrm{sech}(\alpha\tau)$$

- Karlsson and Hook solution in 4th order:

$$\psi = A\ \mathrm{sech}^2(\alpha\tau)$$

- Predict analytic solutions of higher order of the form:

$$\psi = A\ \mathrm{sech}^{n/2}(\alpha\tau)$$

n = highest dispersion order

Karlsson M., Höök A., Optics Communications 104, 4–6, 303-307 (1994)

Families of Analytic Solutions

- We use a known property of the 2nd derivative of the hyperbolic secant: $$\frac{d^2\text{sech}^r(\tau)}{d\tau^2} = r^2\text{sech}^r(\tau) - r(r+1)\text{sech}^{r+2}(\tau).$$

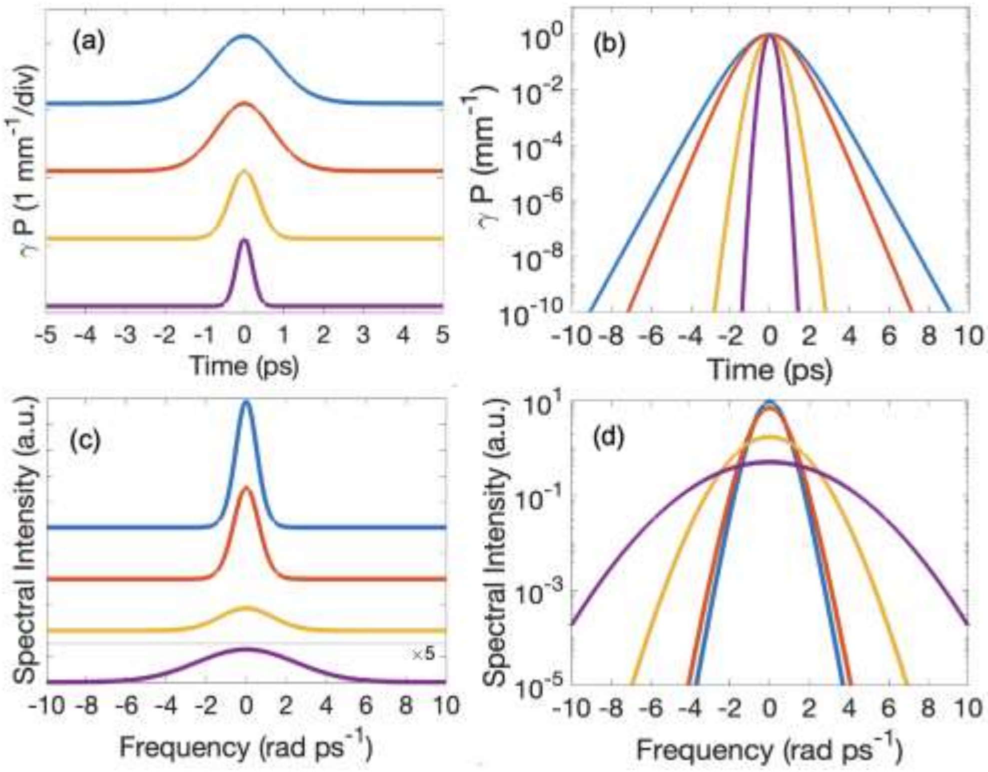
- We can match the hyperbolic secant terms at each power to create a set of equations

	Power of Hyperbolic Secant terms generated by terms in the general NLS				
Function	p				
2^{nd} derivative	p	$p+2$			
4^{th} derivative	p	$p+2$	$p+4$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
n^{th} derivative	p	$p+2$	$p+4$	$\cdots$	$p+n$
Nonlinear					$3p$

n = highest dispersion order
p = n/2

Results at different orders

– Solutions with same α and intensity, at highest dispersion orders: p=3 (blue), p=5 (red), p=15 (yellow), p=50 (purple)
– Time and spectral domains
– We can find the associated dispersion using the linear equations



Qiang, Y. L. et al. , J. Phys. A: Math. Theor. 55 385701. (2022)

Two Term Generalisation

– Simplest generalisation: Look for two term solutions

– Consider solution of the form: $u = A_1 \operatorname{sech}^{p-2}(\alpha\tau) + A_2 \operatorname{sech}^{p}(\alpha\tau)$

– We once again match hyperbolic secant powers

– Finding associated dispersion more challenging (use linear algebra)

	Power of Hyperbolic Secant terms generated by terms in the general NLS							
μ	$p-2$	p						
β_2	$p-2$	p	$p+2$					
β_4	$p-2$	p	$p+2$	$p+4$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	
β_n	$p-2$	p	$p+2$	$\cdots$	$p+n-6$	$p+n-4$	$p+n-2$	$p+n$
γ					$3p-6$	$3p-4$	$3p-2$	$3p$

Qiang, Y. L. et al. , J. Phys. A: Math. Theor. 55 385701. (2022)

Multi-term Generalisation

- Consider solutions with arbitrary number of terms:

$$u = A_1 \operatorname{sech}^{p-v}(\alpha\tau) + A_2 \operatorname{sech}^{p-v+2}(\alpha\tau) + \ldots + A_y \operatorname{sech}^{p}(\alpha\tau)$$

- Equivalent process as previous solutions
- Analytic properties known
- Also apply matrix method to find associated dispersion

Programmable solitons at any dispersion order?

– A superposition of hyperbolic secants is a very specific ansatz

– All these solutions all have straight tails.

– What about oscillating tails like the PQS?

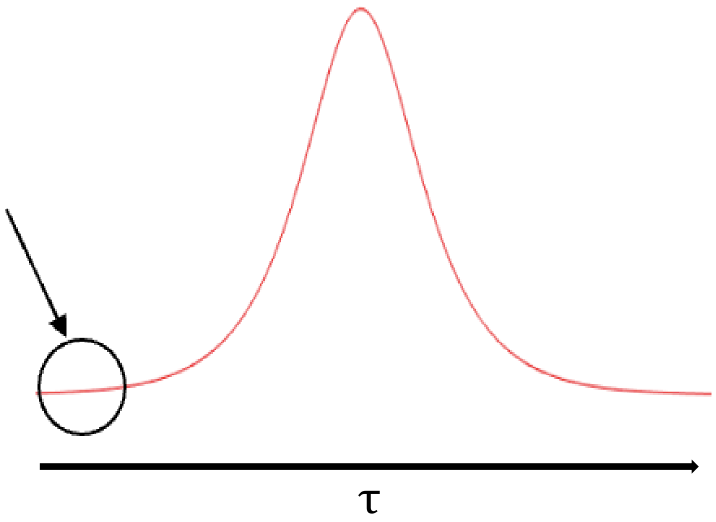
– How do we characterise these solutions?

Linear Tail Solutions

– In the linear limit, we can look at the low amplitude tails
– Tail solutions take the form:

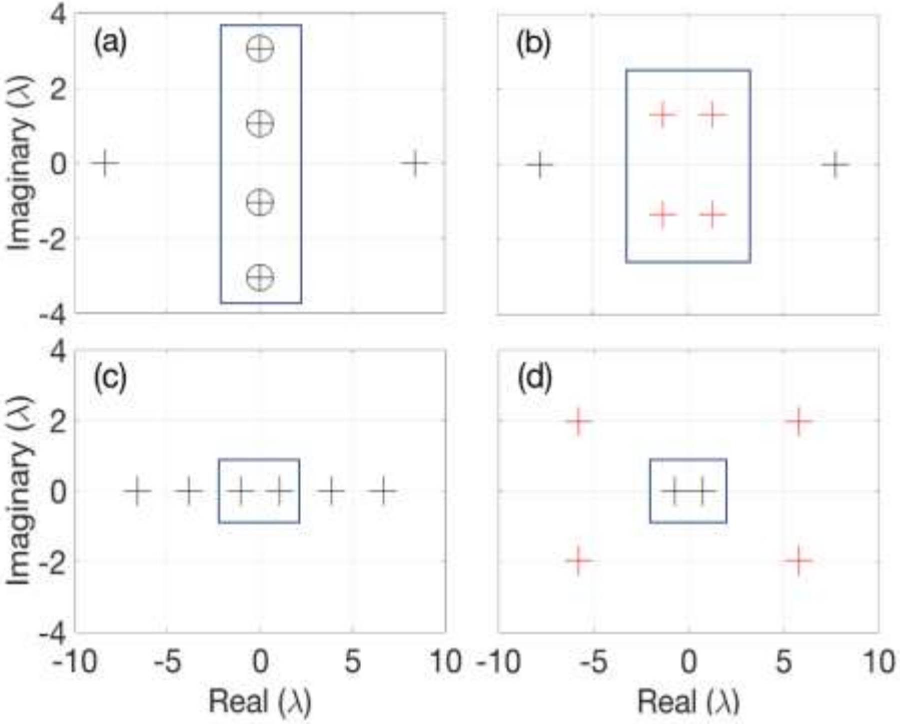
$$u = e^{\lambda\tau}$$

– λ real: Exponentially decay
– λ complex: Exponential decay with oscillations



Example: 6^{th} order case

- 6 roots for λ in $u = e^{\lambda\tau}$
- Smallest real root dominates
- Four solution types
- (a) No pulse-like solution
- (b) Exponential decay with oscillations
- (c) Exponential decay
- (d) Exponential decay



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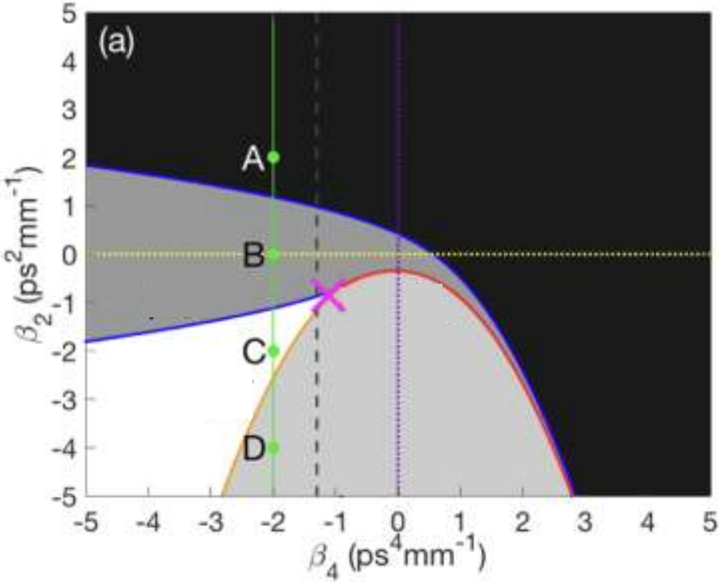
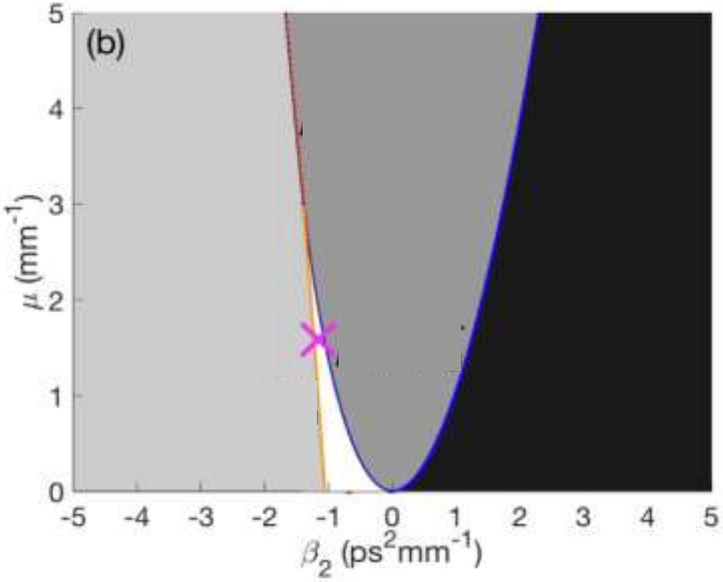
6^{th} order Parameter Space

- 3 dimensional parameter space (two cuts shown)
- All solution types represented





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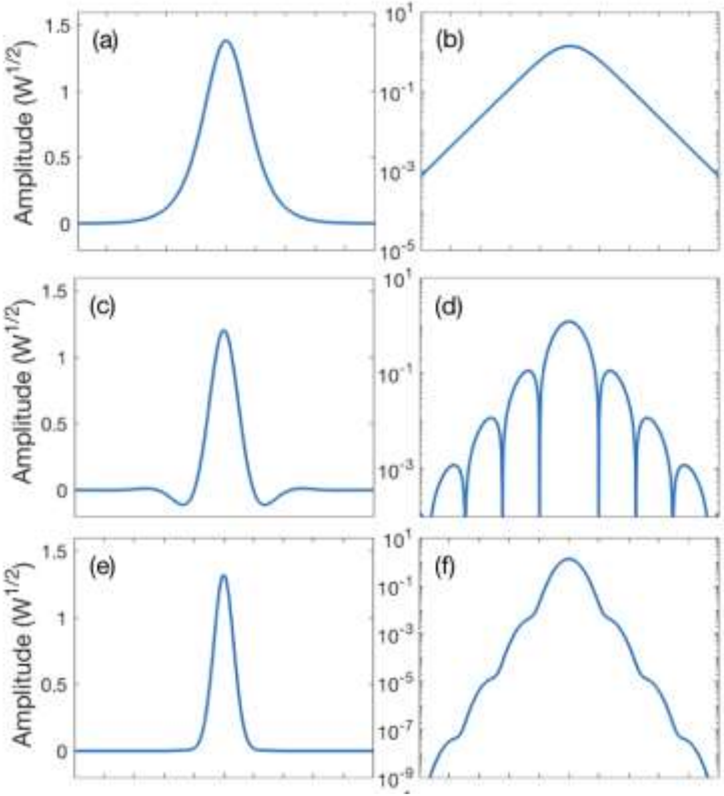
Stationary Pulse-Like Solutions

- Tail dominated by real roots
- Tail dominated by complex roots
- Solutions near a boundary between different types



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Conclusions

- Predict families of analytic solutions at high order dispersion
- Foundation for future numerical and experimental testing at higher dispersion orders
- Generation of associated dispersion terms allows us to program solutions experimentally
- Current numerical methods require accurate starting points
- Unique root structure to be further explored and understood for multi-term solutions