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Particle-like Interactions of Two-Dimensional Solitary Waves in Continuous Media

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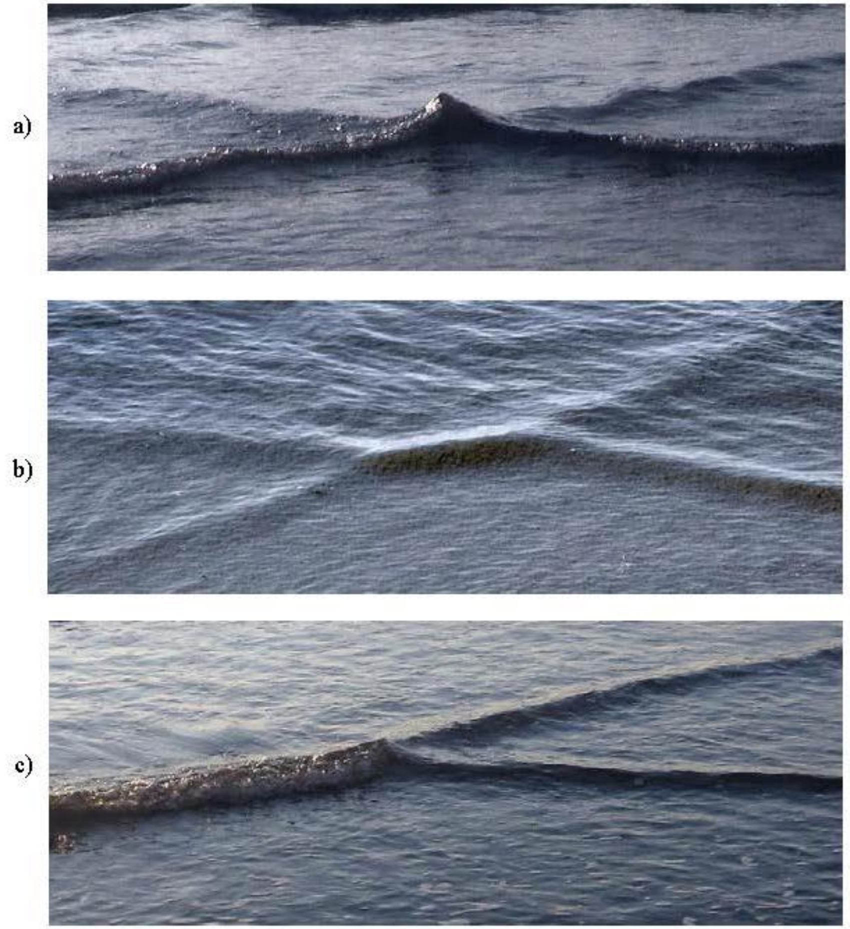
Observable two-dimensional wave patterns

One of the first photos of plane soliton interaction on the surface of shallow water from the book by *Ablowitz & Segur* (1981).

Various types of soliton interaction



Different types of solitary wave interactions on shallow water: X-type, H-type, and Y-type as per *Ablowitz & Baldwin* (*2012*).

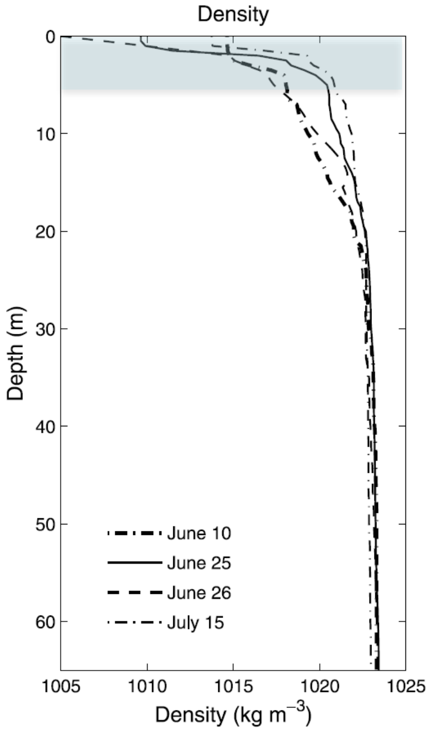
Solitary waves on a water surface

Cross waves
(from the open sources in the Internet).

Solitary waves on a water surface

Rectangular solitary waves
(from the open sources in the Internet).

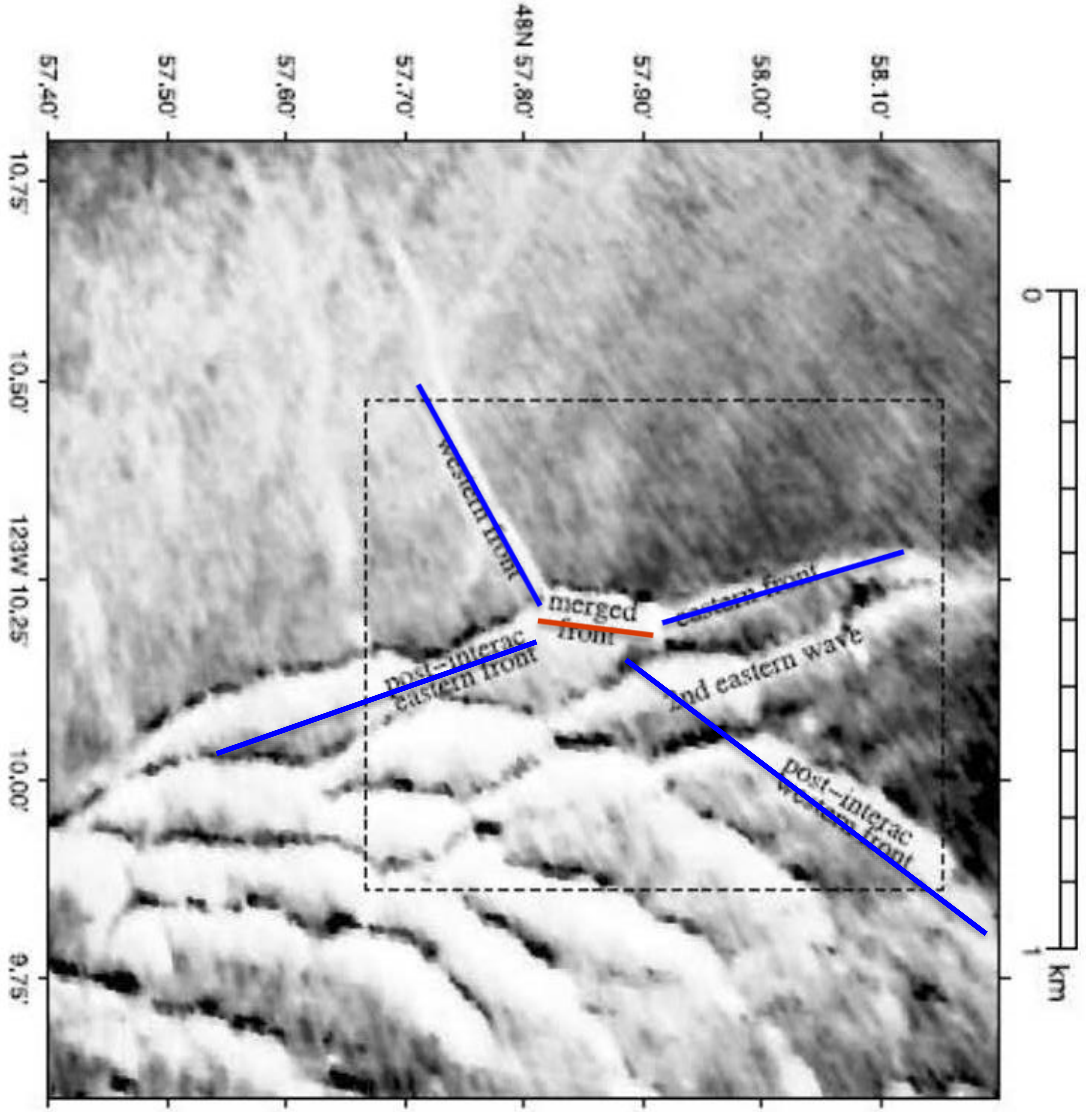
Internal solitary waves



Observations of internal wave interactions in the Strait of Georgia, British Columbia (west coast of North America).

The deep-water case; water depth is 150 m, the pycnocline at the depth $h \sim 5$–7 m (*Wang & Pawlowicz*, 2012).

Internal solitary waves



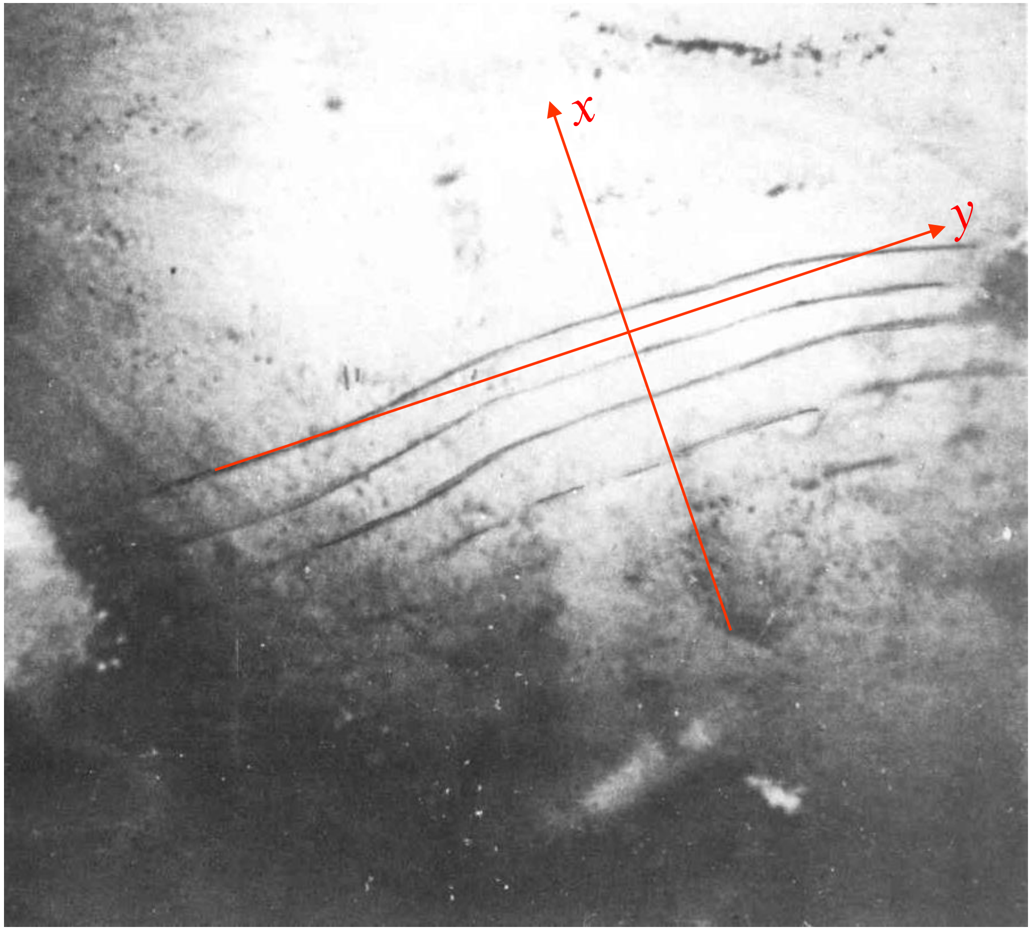
Internal solitary waves in deep water after image processing (*Wang & Pawlowicz*, 2012).

The Kadomtsev-Petviashvili equation

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t}+c\frac{\partial \eta}{\partial x}+\alpha\eta\frac{\partial \eta}{\partial x}+\beta\frac{\partial^3 \eta}{\partial x^3}\right)=-\frac{c}{2}\frac{\partial^2 \eta}{\partial y^2}}$$

- Derived in 1970 by *Kadomtsev & Petviashvili* for the description of weakly nonlinear waves with dispersion propagating along the x-axis and experiencing a weak diffraction in the lateral direction.
- In 1974 Druma found Lax's representation of the KP equation in terms of the LA-pair of operators.
- The same year Zakharov & Shabat (1974) developed a "dressing method" of solution of the KP equation. Later it was shown that this equation is <u>completely integrable</u> for $\beta > 0$ (the ocean wave case) and "<u>weakly integrable</u>" for $\beta < 0$.

Quasi-plane internal solitary waves



Satellite image of internal solitons in the Andaman Sea (*Osborne & Burch*, 1980).

In the frame moving with the linear speed c the KP equations reads:

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t}+\alpha\eta\frac{\partial \eta}{\partial x}+\beta\frac{\partial^3 \eta}{\partial x^3}\right)=-\frac{c}{2}\frac{\partial^2 \eta}{\partial y^2}}$$

Soliton solutions to this equation can be easily obtained through the Hirota transform (*Satsuma*, 1976):

$$\eta(x,y,t)=12\frac{\beta}{\alpha}\frac{d^2}{dx^2}\ln F(x,y,t)$$

Equation for the function $F(x,y,t)$ reads:

$$\boxed{F\left(F_{tx}+\beta F_{xxxx}+\frac{c}{2}F_{yy}\right)=F_t F_x+\frac{c}{2}\left(F_y\right)^2+\beta\left[4F_x F_{xxx}-3\left(F_{xx}\right)^2\right].}$$

The one-soliton solution to the KP equation in terms of function $F(x,y,t)$ is: $\boxed{F(x,y,t) = 1 + e^{\omega t - kx - ly}.}$

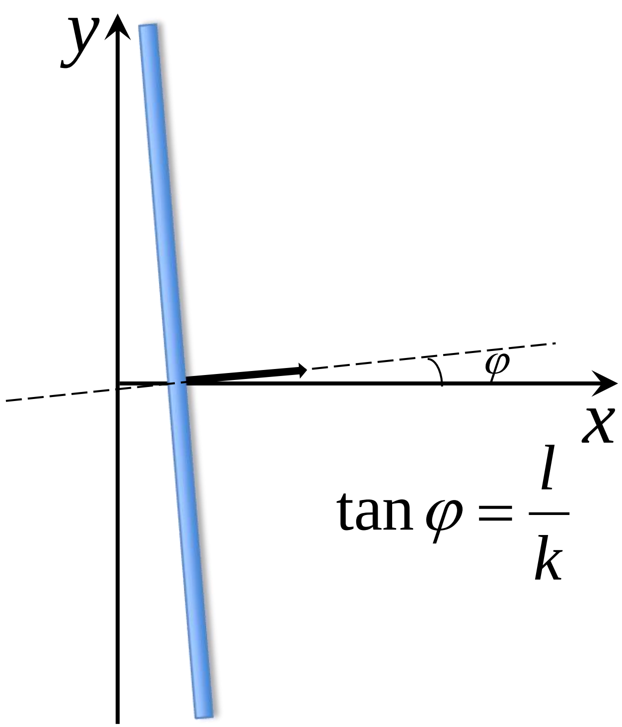
Soliton solutions of the KP2 equation

In the original variables, we obtain:

$$F(x, y, t) = 1 + e^{\omega t - kx - ly};$$

$$\eta(x,y,t) = 12\frac{\beta}{\alpha}\frac{d^2}{dx^2}\ln F(x,y,t)$$

$$= A_s \operatorname{sech}^2 \frac{x + y\tan\varphi - V_s t}{\Delta}.$$



$$\tan\varphi = \frac{l}{k}$$

$$A_s = \frac{3\beta}{\alpha}k^2; \quad \Delta = \frac{2}{k};$$

$$\omega(k,l) = k\left(\frac{\alpha A_s}{3} + \frac{c}{2}\tan^2\varphi\right);$$

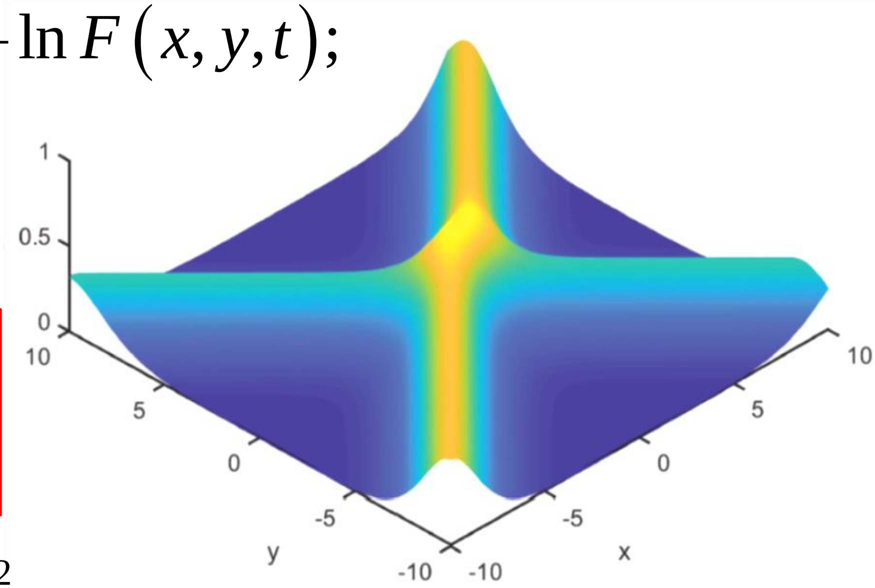
$$V_s = \frac{\alpha A_s}{3}\left(1 + \frac{3}{2}\frac{c}{\alpha A_s}\tan^2\varphi\right)\cos\varphi.$$

The two-soliton solution

The two-soliton solution to the KP equation

can be presented again through the Hirota transform:

$$\eta(x,y,t)=12\frac{\beta}{\alpha}\frac{d^2}{dx^2}\ln F(x,y,t);$$



$$\boxed{F(x,y,t)=1+e^{\omega_1 t-k_1 x-l_1 y}+e^{\omega_2 t-k_2 x-l_2 y}+e^{(\omega_1+\omega_2)t-(k_1+k_2)x-(l_1+l_2)y+\Phi}.}$$

$$\omega_i(k_i,l_i)=\beta k_i^3+\frac{c}{2}\frac{l_i^2}{k_i}; \qquad \boxed{\Phi=\ln B}.$$

$$B=\frac{6\beta(k_1-k_2)^2-c(\tan\varphi_1-\tan\varphi_2)^2}{6\beta(k_1+k_2)^2-c(\tan\varphi_1-\tan\varphi_2)^2}\geq 0.$$

Two-soliton patterns

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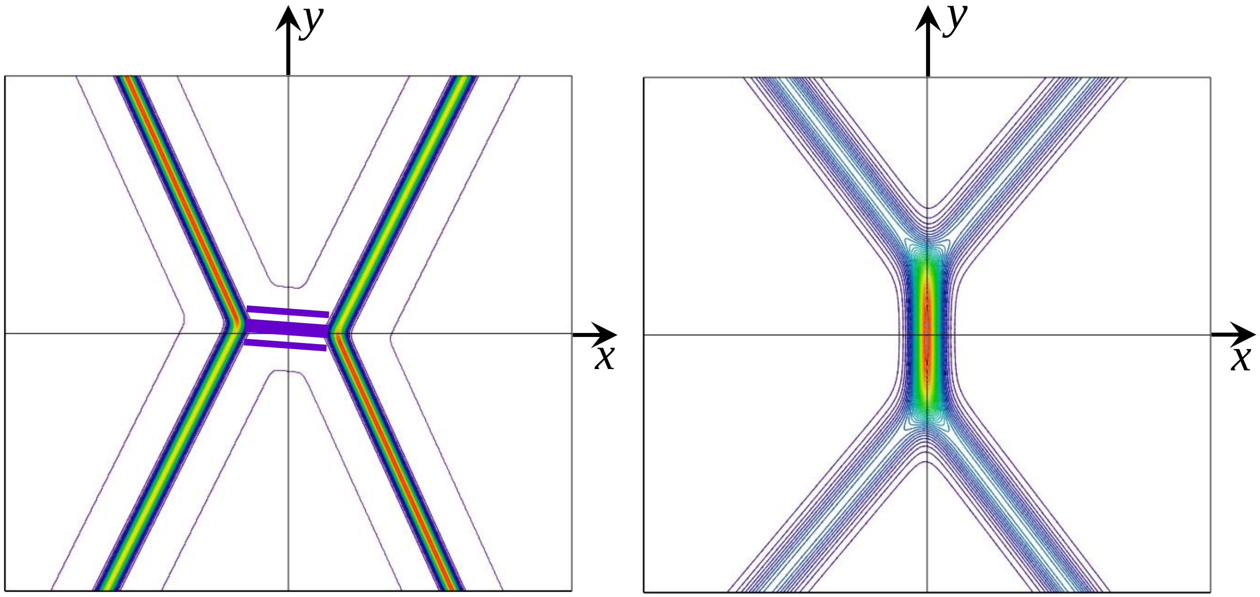


$$B = \frac{6\beta\left(k_1 - k_2\right)^2 - c\left(\tan\varphi_1 - \tan\varphi_2\right)^2}{6\beta\left(k_1 + k_2\right)^2 - c\left(\tan\varphi_1 - \tan\varphi_2\right)^2} \geq 0.$$

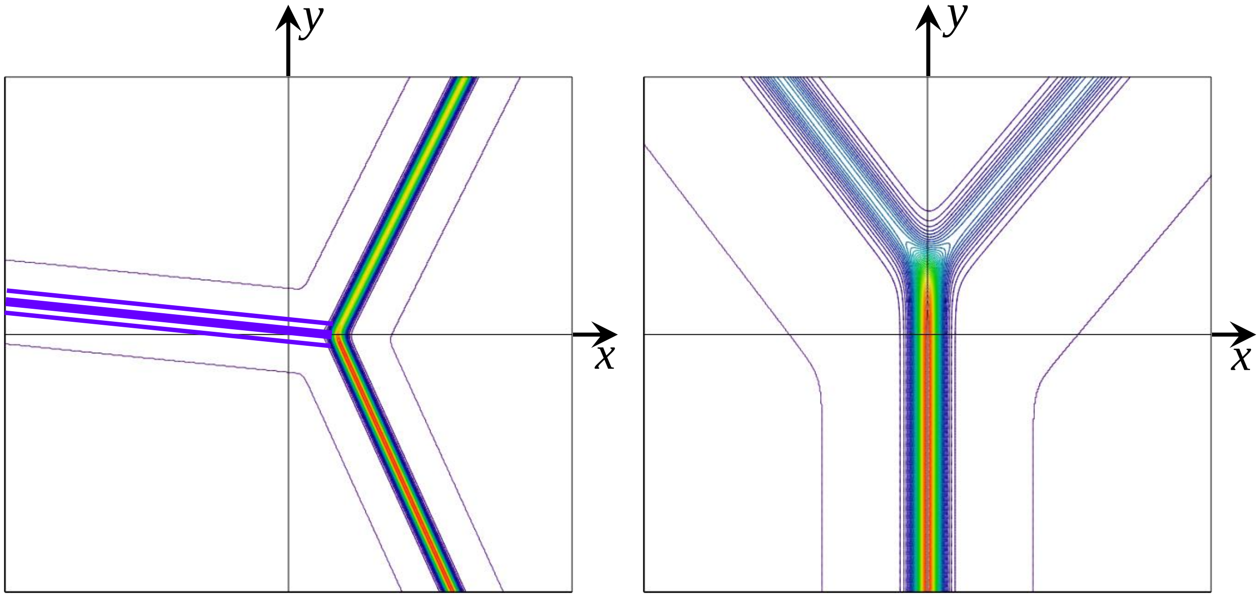
The length of the bridge between soliton fronts depends on the parameter B. It becomes infinite in two cases:

(i) when $B \to 0$, and (ii) $B \to \infty$,

provided that $\beta > 0$.

Soliton triads

(*Newell & Redekopp*, 1977; *Miles*, 1977)



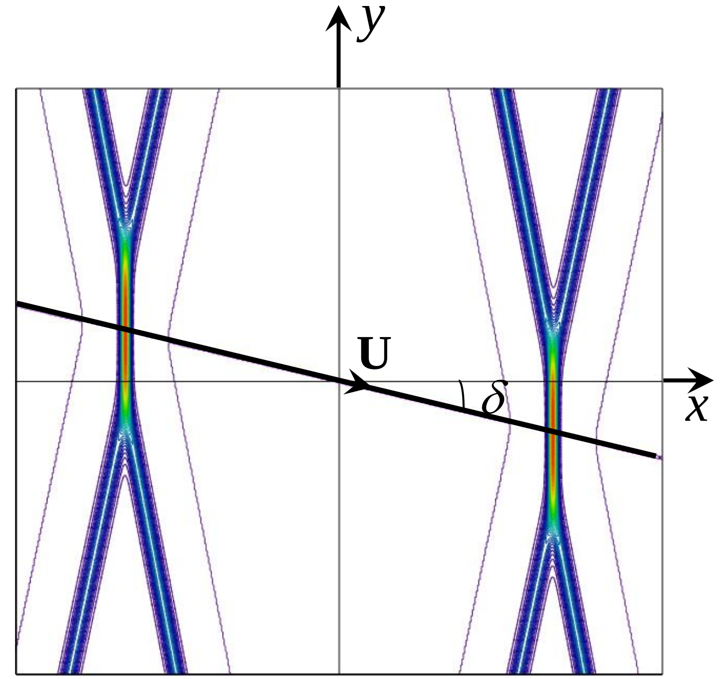
Examples of soliton patterns in two limiting cases:

(i) when $B \to 0$, and (ii) $B \to \infty$.

In the case of $B = 0$, the solution is very simple:

$$F(x, y, t) = 1 + e^{\omega_1 t - k_1 x - l_1 y} + e^{\omega_2 t - k_2 x - l_2 y}.$$

Where a pattern moves to?



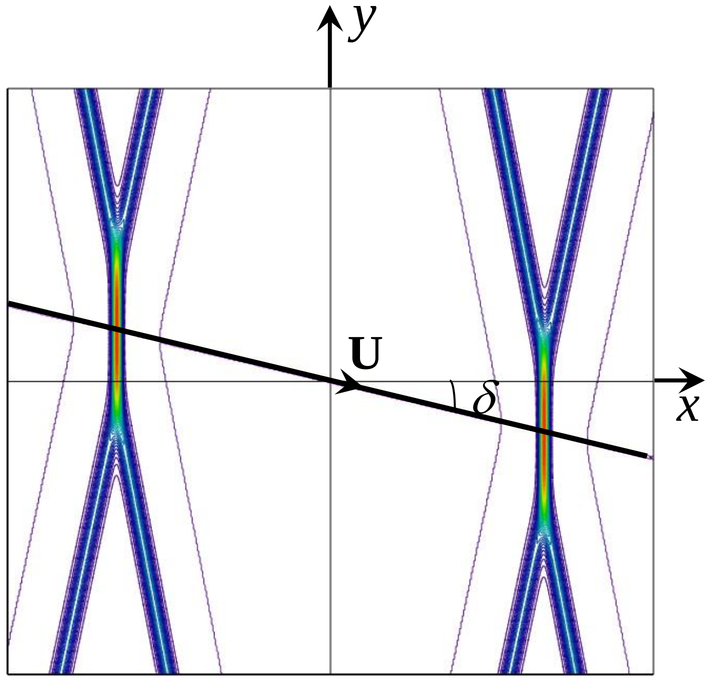
Let us make the orthogonal transformation:

$$\begin{cases} x = \xi\cos\delta - \eta\sin\delta, \\ y = \xi\sin\delta + \eta\cos\delta \end{cases}$$

and present the solution in the form:

$$F(x,y,t) = 1 + e^{-K_1(\xi - Ut) - L_1\eta} + e^{-K_2(\xi - Ut) - L_2\eta} + e^{-(K_1+K_2)(\xi - Ut) - (L_1+L_2)\eta + \Phi}.$$

Where a pattern moves to?



For the two-soliton solution this is always possible!

$$U = \omega_i / K_i \quad \text{where } i = 1, 2, \quad K_i = k_i\,(1 + \tan\delta \tan\varphi_i)\cos\delta,$$

$$\tan\delta = \frac{2\beta\left(k_2^2 - k_1^2\right) + c\left(\tan^2\varphi_2 - \tan^2\varphi_1\right)}{2\beta\left(k_1^2\tan\varphi_2 - k_2^2\tan\varphi_1\right) - c\tan\varphi_1\tan\varphi_2\left(\tan\varphi_2 - \tan\varphi_1\right)}.$$

Symmetric patterns

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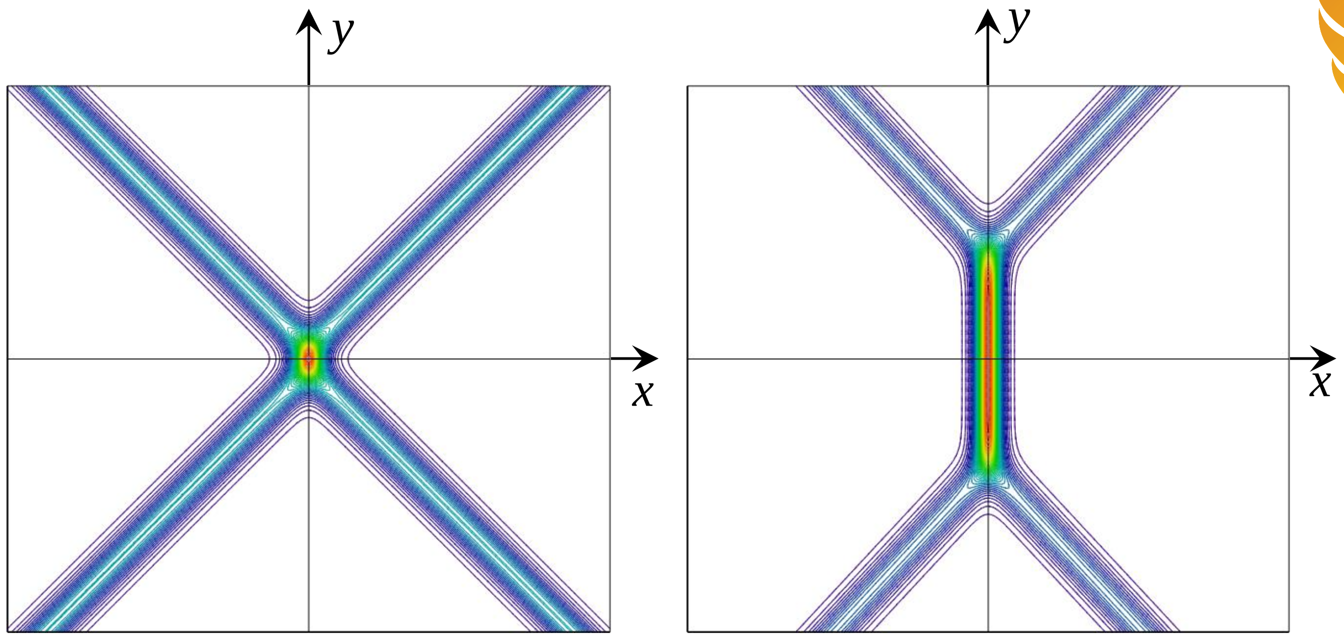


Symmetric patterns are very special and simple cases of soliton interaction. The patterns move along the x-axis with the parameters:

$$k_1 = k_2 = k > 0, \quad l_1 = -l_2 = l > 0, \ (\varphi_1 = -\varphi_2 = \varphi > 0).$$

$$\eta(\theta, y) = 24\frac{\beta}{\alpha}k^2 e^\theta \frac{\left(2 + e^\theta \cosh ly\right) Be^\theta + \cosh ly}{\left(1 + 2e^\theta \cosh ly + Be^{2\theta}\right)^2}, \quad \theta = \omega t - kx.$$

Symmetric patterns

$$B = \frac{cl^2}{cl^2 - 6\beta k^4}; \quad U = \frac{\alpha A}{3} + \frac{c}{2}\tan^2\varphi$$

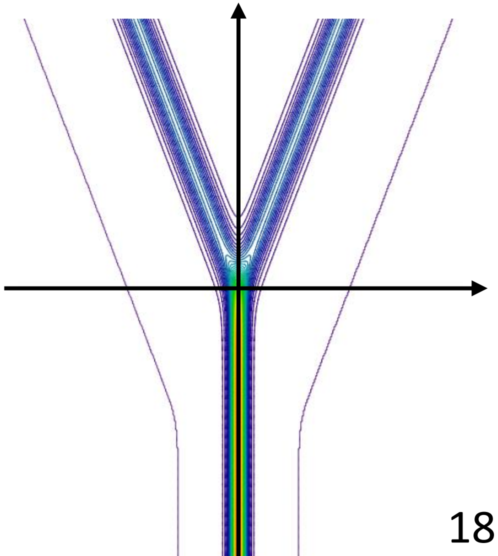
$$\eta_{\max} = 4A_s \frac{\sqrt{B}}{1+\sqrt{B}} \xrightarrow[B\to\infty]{} \boxed{\eta_{\max} = 4A_s}$$

Solution is nonsingular if

$$\boxed{l \geq l_{cr} \equiv k^2\sqrt{\frac{6\beta}{c}}, \quad \text{or} \quad \frac{2\alpha A}{c} \leq \tan^2\varphi \ll 1.}$$

This restriction is equivalent to $\boxed{2\alpha A/c \leq \varphi^2.}$

When $l = l_{cr}$, we obtain a symmetric triad.



Nonintegrable cases

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Quasi-one-dimensional Gardner equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t}+c\frac{\partial \eta}{\partial x}+\alpha\eta\frac{\partial \eta}{\partial x}+\alpha_1\eta^2\frac{\partial \eta}{\partial x}+\beta\frac{\partial^3 \eta}{\partial x^3}\right)=-\frac{c}{2}\frac{\partial^2 \eta}{\partial y^2}.$$

Quasi-one-dimensional Benjamin–Ono equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t}+c\frac{\partial \eta}{\partial x}+\alpha\eta\frac{\partial \eta}{\partial x}+\frac{\beta}{\pi}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty}\frac{\eta(x',t)\,dx'}{x-x'}\right)=-\frac{c}{2}\frac{\partial^2 \eta}{\partial y^2}.$$

Two-dimensional Boussinesq equations:

$$\frac{\partial \eta}{\partial t}=-(\nabla\phi\cdot\nabla\eta)+w\left[1+(\nabla\eta)^2\right],$$

$$\frac{\partial \phi}{\partial t}=-\eta-\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}w^2\left[1+(\nabla\eta)^2\right].$$

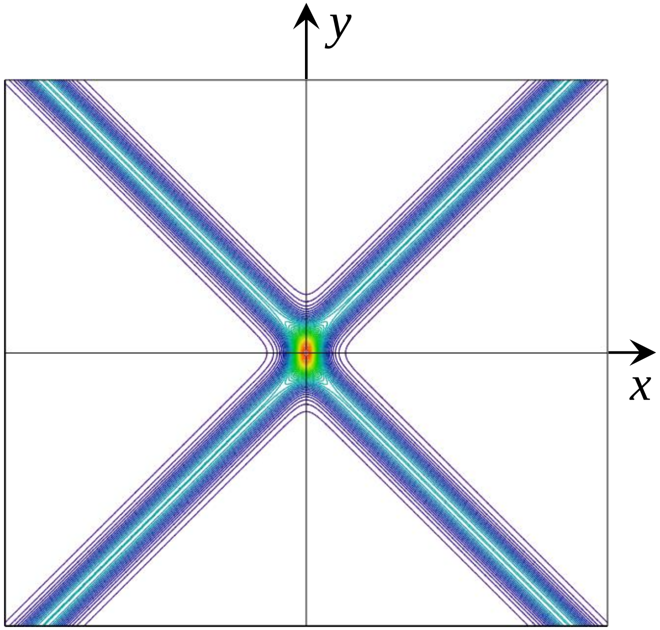
Asymptotic approach to the interacting soliton fronts

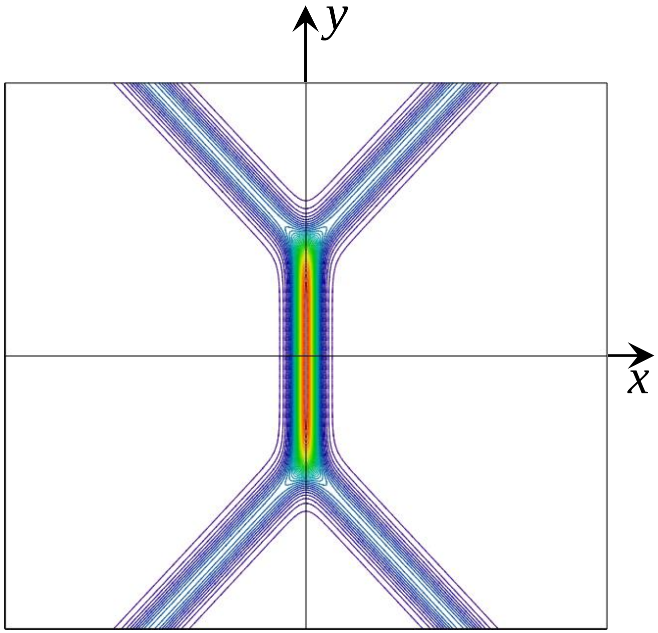
Let us consider for simplicity a symmetric soliton pattern

moving along the x-axis with the speed V_p

and consisting of two solitons with the parameters:

$$k_1 = k_2 = k > 0, \quad l_1 = -l_2 = l > 0, \quad (\varphi_1 = -\varphi_2 = \varphi > 0).$$





The Boussinesq-type equation

In the stationary case when $\eta(x, y, t) = \eta(\xi, y)$,

where $\xi = x - V_p t$, the KP2 equation reduces

to the Boussinesq-type equation:

$$\frac{\partial^2 \eta}{\partial y^2} - C_0^2 \frac{\partial^2 \eta}{\partial \xi^2} + \frac{\alpha}{c} \frac{\partial^2 \eta^2}{\partial \xi^2} + \frac{2\beta}{c} \frac{\partial^4 \eta}{\partial \xi^4} = 0,$$

where $\boxed{C_0^2 = 2V_p / c.}$

Then, we can think that y is the time variable τ, and soliton fronts in the ξ, y-plane represent space-time trajectories of one-dimensional solitons experiencing a head-on collision in the ξ, τ-plane.

The soliton solution

A soliton solution of the Boussinesq equation

on a constant pedestal p is:

$$\eta(\xi,\tau) = A\operatorname{sech}^2\left[\frac{\xi - W\tau}{\Delta}\right] + p,$$

$$\Delta = \sqrt{\frac{12\beta}{\alpha A}}, \quad W = \pm\sqrt{\frac{2}{c}\left(V_p - \alpha p - \frac{\alpha A}{3}\right)}.$$

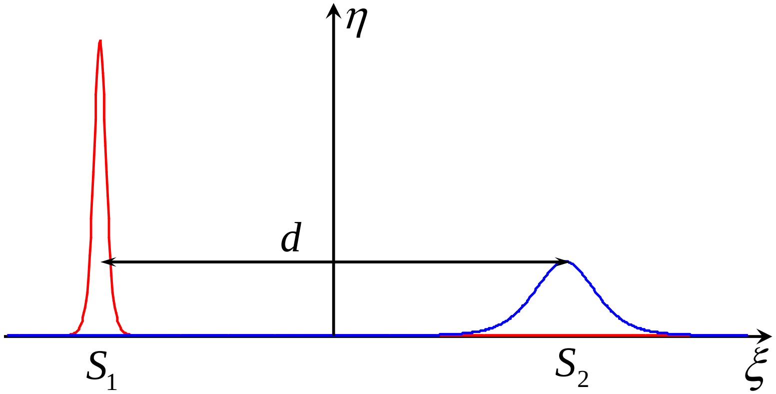
This solution is real if $V_p \geq \alpha(p + A/3)$.

This solution remains approximately correct even when the pedestal p is a gradually varying function of ξ with the characteristic length of variation $L >> \Delta$.

Particle-like interaction of solitons

One of the solitons can play a role

of a pedestal $p(\xi)$ for another soliton.



The tall and narrow soliton can be considered as a point particle in the external field created by the tail of another soliton.

Denoting the soliton position by $S(\tau)$, we obtain from the kinematic condition $\boxed{dS/d\tau = W.}$

Particle-like interaction of solitons

For two solitons moving in opposite directions,

we obtain the set of equation:

$$\frac{dS_1}{d\tau} = \sqrt{\frac{2}{c}\left(V_p - \alpha p_2 - \frac{\alpha A_1}{3}\right)},$$

$$\frac{dS_2}{d\tau} = -\sqrt{\frac{2}{c}\left(V_p - \alpha p_1 - \frac{\alpha A_2}{3}\right)}.$$

Substituting here expressions for V_p and p, we obtain:

$$\frac{dS_1}{d\tau} = \sqrt{\frac{3\beta l^2}{\alpha A_1}\left\{1 - \frac{2\alpha^2}{3\beta c l^2} A_1 A_2 \operatorname{sech}^2\left[\sqrt{\frac{\alpha A_2}{12\beta}}(S_2 - S_1)\right]\right\}},$$

$$\frac{dS_2}{d\tau} = -\sqrt{\frac{3\beta l^2}{\alpha A_2}\left\{1 - \frac{2\alpha^2}{3\beta c l^2} A_1 A_2 \operatorname{sech}^2\left[\sqrt{\frac{\alpha A_1}{12\beta}}(S_2 - S_1)\right]\right\}}.$$

Particle-like interaction of solitons

For solitons of equal amplitudes $A_1 = A_2$,

real solution exists if: $|l| \geq l_{cr} \equiv \alpha A\sqrt{2/3\beta c}$.

The set of equations can be readily solved provided that

the distance between the soliton centres is big, $|S_2 - S_1| >> \Delta$.

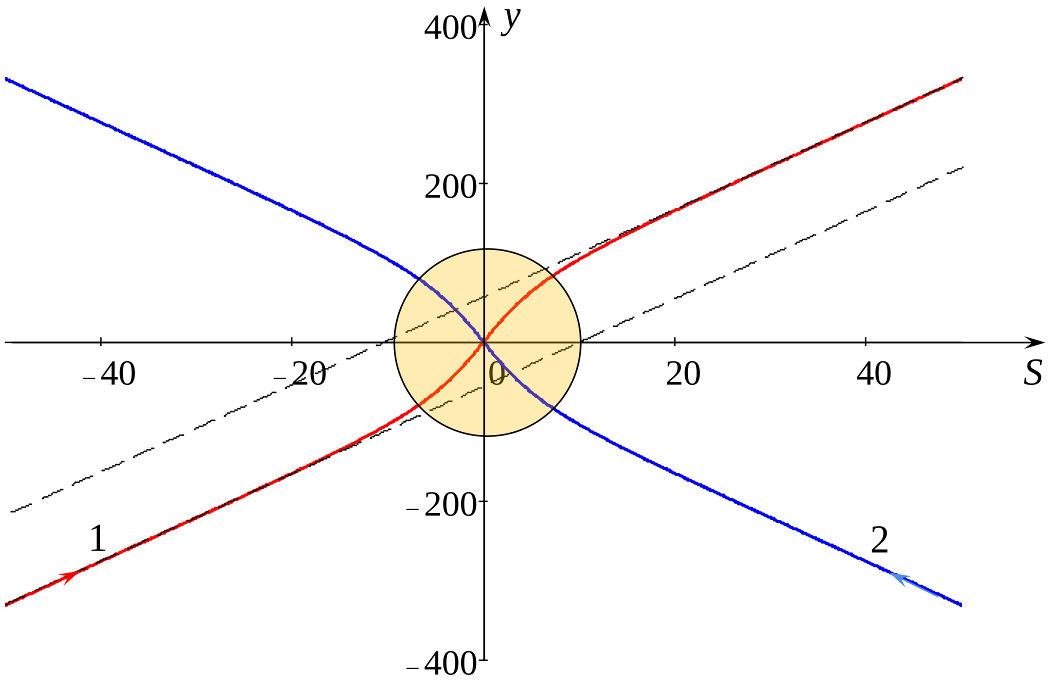
Then, we obtain:

$$\tau_{1,2} = \frac{1}{l}\,\text{Arctanh}\,\frac{\tanh\left(2S_{1.2}/\Delta\right)}{\sqrt{1-\dfrac{8\alpha A}{cl^2\Delta^2}\,\text{sech}^2\left(2S_{1,2}/\Delta\right)}},$$

or, bearing in mind that $\tau \equiv y$, we finally derive:

$$y_{1,2} = \frac{1}{l}\,\text{Arctanh}\,\frac{\tanh\left(2S_{1.2}/\Delta\right)}{\sqrt{1-\dfrac{8\alpha A}{cl^2\Delta^2}\,\text{sech}^2\left(2S_{1,2}/\Delta\right)}}.$$

Particle-like interaction of solitons



Soliton trajectories (configuration of soliton fronts). The phase shift of trajectories caused by soliton interaction is clearly seen.

The yellow circle shows the domain where the asymptotic theory is formally inapplicable because the distance between the solitons is not big here.

Particle-like interaction of solitons

As follows from the solution, asymptotically

when $y \to \infty$, we obtain:

$$S_1(y) \approx \sqrt{\frac{3\beta l^2}{\alpha A}}\, y + \sqrt{\frac{3\beta}{4\alpha A}} \ln\left(1 - \frac{2\alpha^2 A^2}{3\beta c l^2}\right).$$

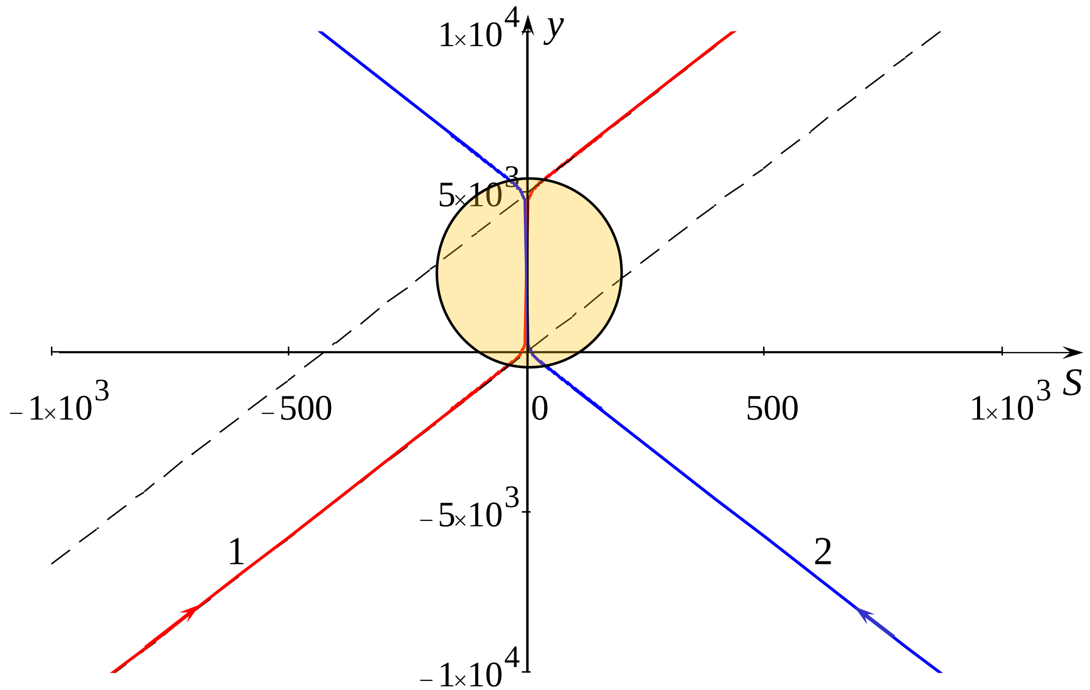
Then, the phase shift is:

$$\Phi_y \equiv 2\left[y_1(S) - y_{10}(S)\right] = -\frac{1}{l} \ln\left(1 - \frac{2\alpha^2 A^2}{3\beta c l^2}\right).$$

This exactly equals to the phase shift which follows from the analytical solution of the KP2 equation!

When $|l| \to l_{cr} \equiv \alpha A \sqrt{2/3\beta c}$, we obtain the triad solution.

Particle-like interaction of solitons



Soliton trajectories (configuration of soliton fronts)

with the big phase shift.

Selected publications

- Ostrovsky L.A., Stepanyants Y.A. Kinematics of interacting solitons in two-dimensional space. Russ. J. Earth Sci., 2020, v. 20, ES4007.

- Stepanyants Y.A. The asymptotic approach to the description of two-dimensional symmetric soliton patterns. Symmetry, 2020, v. 12, 1586.