On two particular N-state generalisations of the quantum Ising model

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featuring ongoing work with Remy Adderton and Alex Henry

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Outline of this talk

1) superintegrable chiral Potts model

$\Longrightarrow$ coupled Temperley-Lieb algebra

$\Longrightarrow$ pictorial representation

2) Baxter-Fendley $Z(N)$ model

$\Longrightarrow$ free parafermions

$\Longrightarrow$ exceptional points

$Z(N)$ spin chains

Building blocks are the $N \times N$ ('shift' and 'clock') matrices

$$\begin{aligned} (\tau)_{\ell m} &= \delta_{\ell, m+1} \pmod N \\ \sigma &= \mathrm{diag}\left(1, \omega, \omega^2, \ldots, \omega^{N-1}\right) \end{aligned}$$

with $\omega = \mathrm{e}^{2\pi i/N}$. For $N = 3$,

$$\tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}.$$

With 1 the identity, they satisfy

$$\tau^N = \sigma^N = 1, \qquad \tau^\dagger = \tau^{N-1}, \qquad \sigma^\dagger = \sigma^{N-1},$$

$$\sigma\tau = \omega\tau\sigma.$$

Some well studied Yang-Baxter integrable N-state quantum spin chains are of the form

$$H = -\sum_{j=1}^{L} \sum_{n=1}^{N-1} a_n \left(\lambda \tau_j^n + \sigma_j^n \sigma_{j+1}^{N-n} \right)$$

$$\tau_j = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \tau \otimes 1 \otimes \cdots \otimes 1$$
$$\sigma_j = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma \otimes 1 \otimes \cdots \otimes 1$$

where 1, τ and σ are $N \times N$ matrices, τ and σ occur in position j.

special cases

- N-state quantum Potts model

$$\boxed{a_n = 1} \tag{1}$$

- Fateev-Zamolodchikov $Z(N)$ model

$$\boxed{a_n = \frac{1}{\sin(\pi n/N)}} \tag{2}$$

- N-state superintegrable chiral Potts model

$$\boxed{a_n = \frac{2}{1-\omega^{-n}}} \tag{3}$$

Each model reduces to the quantum Ising model for $N = 2$.

Models (1) and (2) are equivalent for $N = 3$.

Model (3) still something of an enigma..

Potts and Temperley-Lieb

Recall the N-state quantum Potts representation of the TL algebra

$$
\begin{aligned}
e_{2j-1} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \tau_j^n & j = 1, \ldots, L \\
e_{2j} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left(\sigma_j \sigma_{j+1}^\dagger\right)^n & j = 1, \ldots, L-1
\end{aligned}
$$

with

$$
\begin{aligned}
e_j^2 &= \sqrt{N}\, e_j \\
e_j e_{j\pm1} e_j &= e_j \\
e_j e_i &= e_i e_j \qquad |i-j| > 1
\end{aligned}
$$

Potts model hamiltonian

$$
H_{\mathrm{P}} = -\sum e_j ,
$$

Superintegrable chiral Potts (SICP) chain

$$H_{\mathrm{CP}} = -\sum_{j=1}^{L}\sum_{n=1}^{N-1}\left(\lambda\,\alpha_n\,\tau_j^n + \overline{\alpha}_n\,\left(\sigma_j\sigma_{j+1}^{\dagger}\right)^n\right)$$

$$\alpha_n = \frac{\mathrm{e}^{\mathrm{i}(2n-N)\phi/N}}{\sin(\pi n/N)}, \qquad \overline{\alpha}_n = \frac{\mathrm{e}^{\mathrm{i}(2n-N)\overline{\phi}/N}}{\sin(\pi n/N)}$$

- The chiral Potts model has an R-matrix when

$$\lambda \cos\phi = \cos\overline{\phi}.$$

- The special values $\phi = \overline{\phi} = \frac{\pi}{2}$ define the superintegrable case.
- H_{SICP} admits an infinite set of commuting conserved charges.
- H_{SICP} only solved for periodic bc's.
  (N-state free parafermions only solved for open bc's)

H_{SICP} can be written in terms of a coupled TL algebra!

[$N = 3$ case, J Fjelstad and T Månsson, JPA 45, 155208 (2012)]

For general N there are $N-1$ generators $e_j^{(k)}$ which satisfy

$$
\begin{aligned}
\left(e_j^{(k)}\right)^2 &= Q\, e_j^{(k)} & \\
e_j^{(k)} e_{j\pm1}^{(\ell)} e_j^{(k)} &= e_j^{(k)} & \\
e_i^{(k)} e_j^{(\ell)} &= e_j^{(\ell)} e_i^{(k)} & \quad |i-j| > 1 \\
e_j^{(k)} e_j^{(\ell)} = e_j^{(\ell)} e_j^{(k)} &= 0 & \quad k \neq \ell
\end{aligned}
$$

with $Q = \sqrt{N}$.

For $N = 2$ this reduces to the single TL generator e_j.

For $N = 3$ we label the generators by $e_j = e_j^{(1)}$ and $f_j = e_j^{(2)}$.

In general we can write

$$e_{2j-1}^{(k)} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left(\omega^k \tau_j\right)^n \qquad j = 1, \ldots, L$$

$$e_{2j}^{(k)} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left(\omega^k \sigma_j \sigma_{j+1}^\dagger\right)^n \qquad j = 1, \ldots, L-1$$

for $k = 1, \ldots, N-1$. Here $\omega = e^{2\pi \mathrm{i}/N}$.

Then, for periodic bc's

$$\boxed{H_{\mathrm{SICP}} = \frac{2}{\sqrt{N}} \sum_{j=1}^{L} \sum_{k=1}^{N-1} (N-k) \left(\lambda e_{2j-1}^{(k)} + e_{2j}^{(k)}\right) - (\lambda+1)(N-1)L}$$

And for open bc's

$$
\begin{aligned}
H_{\mathrm{SICP}} &= -(N-1)(L(\lambda+1)-1) \\
&\quad + \frac{2}{\sqrt{N}} \sum_{j=1}^{L} \sum_{k=1}^{N-1} \lambda (N-k) e_{2j-1}^{(k)} \\
&\quad + \frac{2}{\sqrt{N}} \sum_{j=1}^{L-1} \sum_{k=1}^{N-1} (N-k) e_{2j}^{(k)}
\end{aligned}
$$

The generators $e_j^{(k)}$ also satisfy additional cubic relations.

For the $N = 3$ case

$$\begin{aligned} f_j e_{j\pm 1} e_j &= \pm \mathrm{i} \left(\omega^{\mp 1} e_{j\pm 1} e_j - f_{j\pm 1} e_j \right) + \omega^{\pm 1} e_j \\ &= \pm \mathrm{i} \left(\omega^{\mp 1} f_j e_{j\pm 1} - f_j f_{j\pm 1} \right) + \omega^{\pm 1} f_j \\ e_j e_{j\pm 1} f_j &= \mp \mathrm{i} \left(\omega^{\pm 1} e_{j\pm 1} f_j - f_{j\pm 1} f_j \right) + \omega^{\mp 1} f_j \\ &= \mp \mathrm{i} \left(\omega^{\pm 1} e_j e_{j\pm 1} - e_j f_{j\pm 1} \right) + \omega^{\mp 1} e_j \\ f_j f_{j\pm 1} e_j &= \omega^{\pm 1} f_j e_{j\pm 1} e_j \\ e_j f_{j\pm 1} f_j &= \omega^{\mp 1} e_j e_{j\pm 1} f_j \end{aligned}$$

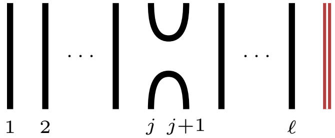
For $N = 4$ with $e_j = e_j^{(1)}$, $f_j = e_j^{(2)}$ and $g_j = e_j^{(3)}$, a typical cubic relation is of the type

$$f_1 e_2 e_1 = \frac{1}{2}(1 - \mathrm{i}) e_2 e_1 - \frac{1}{2}(1 + \mathrm{i}) g_2 e_1 - \mathrm{i} f_2 e_1 + \mathrm{i}\, e_1 \,.$$

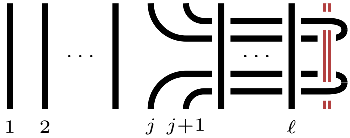
Pictorial representation

We give a pictorial representation of the generators. For $N = 3$:

$$e_j = e_j^{(1)} =$$



$$f_j = e_j^{(2)} =$$



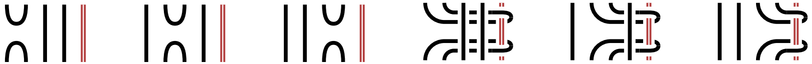
The key feature of the pictorial representation is a pole around which loops can become entangled. Here we choose the position of the pole to be at one end of the chain. In the associated loop diagrams, closed (contractible) loops have weight Q, with $Q = \sqrt{3}$. The weight of closed (non-contractable) loops encircling the red line is zero.

SICP $N = 3$ example, generators e_j and f_j

The generators e_j are like the usual TL generators, with loops not encircling a line.

The generators f_j involve loops around the single red line.

Generators for the $L = 2$ site open chain:



e_1 e_2 e_3 f_1 f_2 f_3

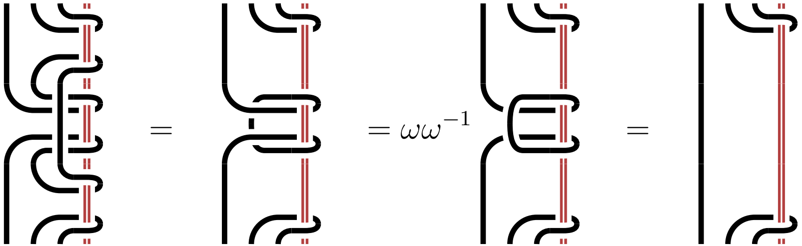
SICP $N = 3$ example, generators e_j and f_j

All algebraic relations can be proved via the diagrams.

We make use of the usual Kauffman-type relations.

The most interesting cubic relations are $f_1 f_2 f_1 = f_1$ and $f_2 f_1 f_2 = f_2$.

Proof of the relation $f_2 f_1 f_2 = f_2$



The knot can be resolved!

Key ingredients are crossing relations for loops encircling a red line.

Line crossing relations

 $=$ ω  ,  $=$ ω 

For this example with $N = 3$ the parameter is $\omega = \mathrm{e}^{\mathrm{i}2\pi/3}$.

This value can be derived topologically.

Based on the publication

Remy Adderton, MTB and **Paul Wedrich**,
J. Phys. A 53, 36LT01 (2020) (open access)

2) The Baxter-Fendley $Z(N)$ spin chain

A model that received **no attention** for a long time was found by Rodney Baxter in 1989.

For an L-site chain this model is defined as

$$\boxed{-H = \sum_{j=1}^{L} \tau_j + \lambda \sum_{j=1}^{L-1} \sigma_j^\dagger \sigma_{j+1}}$$

It reduces to the quantum Ising model for $N = 2$.

H is non-Hermitian!

The eigenvalues of H have a simple form!

$$\boxed{-E = \omega^{p_1}\epsilon_1 + \omega^{p_2}\epsilon_2 + \cdots + \omega^{p_L}\epsilon_L}$$

for any choice of $p_k = 0, \ldots, N-1$. Recall $\omega = \mathrm{e}^{2\pi i/N}$.

- cf free fermions ($N=2$) $\boxed{-E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_L}$
- Gives all N^L eigenvalues in the spectrum.
- The energy levels ϵ_k are known.
- Initially a numerical observation.
- The model originates as the hamiltonian limit of the τ_2 model, a variant of the chiral Potts model.

R J Baxter, Phys Lett A **140**, 155 (1989); J Stat Phys **57**, 1 (1989)

V V Bazhanov and Y G Stroganov, J Stat Phys **59**, 799 (1990)

R J Baxter, J Stat Phys **117**, 1 (2004)

- Fendley derived this result using a generalisation of the Jordan-Wigner transformation, namely the **Fradkin-Kadanoff transformation** to parafermionic operators originally introduced for the N-state clock models.

- Baxter (2014) and Au-Yang and Perk (2014,2016) applied Fendley's parafermionic approach to the more general τ_2 model with open boundaries.

P Fendley, J. Phys. A 47, 075001 (2014)

R J Baxter, J Phys A **47**, 315001 (2014)

H Au-Yang and J H H Perk, J Phys A **47**, 315002 (2014); arXiv:1606.06319

The hamiltonian is non-Hermitian, with complex energy eigenvalues for $N \geq 3$.

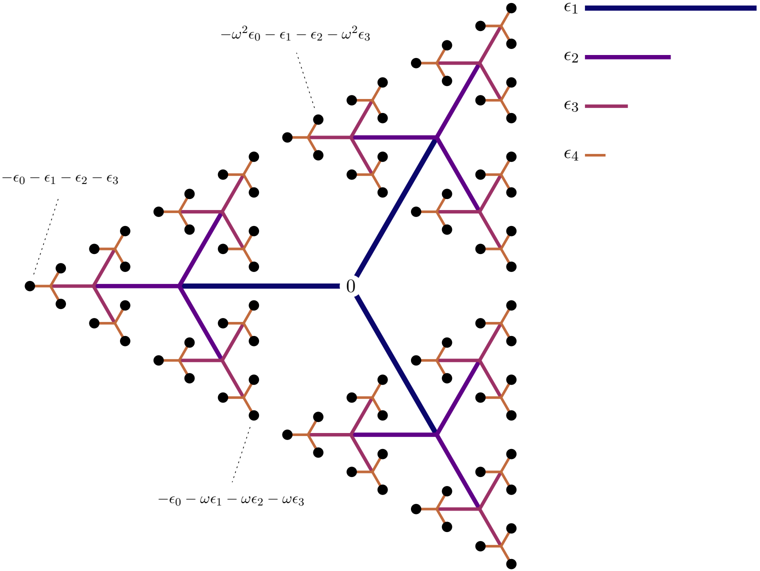
For any eigenvalue E, there are other eigenvalues $\omega E, \omega^2 E, \ldots$

This is the generalisation of the $E \leftrightarrow -E$ Ising symmetry (recall $\omega = -1$ for $N = 2$).

In general non-Hermitian hamiltonians describe the dynamics of physical systems that are not conservative.

The properties of the model are well worth exploring, being a rare example of an exactly solved non-Hermitian many-body system.

Free parafermion eigenspectrum ($N = 3$ $L = 4$)



The solution

F C Alcaraz, MTB and Z-Z Liu, J Phys A 50, 16LT03 (2017)

$$-H = \sum_{j=1}^{L} \tau_j + \lambda \sum_{j=1}^{L-1} \sigma_j \sigma_{j+1}^{\dagger}$$

$$-E = \sum_{j=1}^{L} \omega^{p_j} \epsilon_{k_j}, \quad p_j = 0, 1, \ldots, N-1, \quad \omega = \mathrm{e}^{2\pi i/N}$$

$$\begin{aligned} \epsilon_k &= \left(1 + \lambda^N + 2\lambda^{N/2} \cos k\right)^{1/N} \\ &= \left(1 + \lambda^{N/2}\right)^{2/N} \left(1 - \theta^2 \sin^2 \frac{k}{2}\right)^{1/N}, \qquad \theta^2 = \frac{4\lambda^{N/2}}{\left(1 + \lambda^{N/2}\right)^2} \end{aligned}$$

k_j satisfy

$$\boxed{\sin(L+1)k = -\lambda^{N/2} \sin Lk}$$

for $\lambda = 1$, $k_j = \frac{2j\pi}{2L+1}$, $j = 1, \ldots, L$ and $\epsilon_k = \left(2 \cos \frac{k}{2}\right)^{2/N}$.

What are exceptional points?

Exceptional points are spectral singularities in the parameter space of a system in which two or more eigenvalues, and their corresponding eigenvectors, simultaneously coalesce.

Such degeneracies are peculiar features of nonconservative systems that exchange energy with their surrounding environment.

EPs are level degeneracies induced by non-Hermiticity.

They exhibit exotic topological phenomena associated with the winding of eigenvalues and eigenvectors.

A vast and highly active topic!

Exceptional points

For real positive λ, the quasi-energies ϵ_j are always positive and distinct.

For **complex** λ, a pair of them may become equal at certain values of λ, which depend on N and L.

We call these *quasi-energy exceptional points*.

We call EPs in the energy spectrum *Hamiltonian exceptional points*.

Our point is that quasi-energy EPs give rise to Hamiltonian EPs.

Moreover, we can calculate them.

A quasi-energy EP will occur when

$$\sin(L+1)k = -\lambda^{N/2} \sin Lk$$

has a repeated root, meaning that both this equation and its derivative are satisfied.

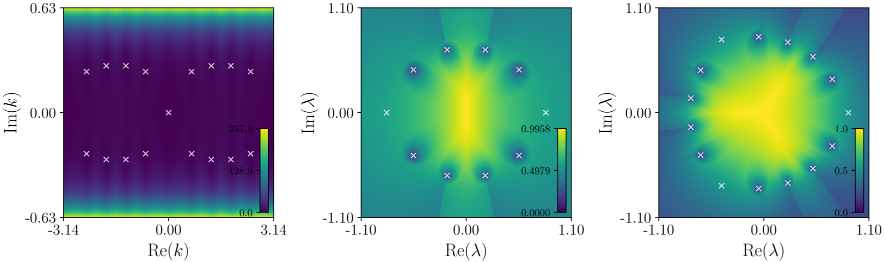
The EPs are pairs of values k_{EP} and λ_{EP} which satisfy these equations simultaneously.

In this way we obtain k_{EP} as the solution to

$$\sin(2L+1)k - (2L+1)\sin k = 0\,,$$

with the corresponding value λ_{EP} given by

$$\lambda = \left[\frac{-\sin(L+1)k}{\sin Lk}\right]^{2/N} .$$

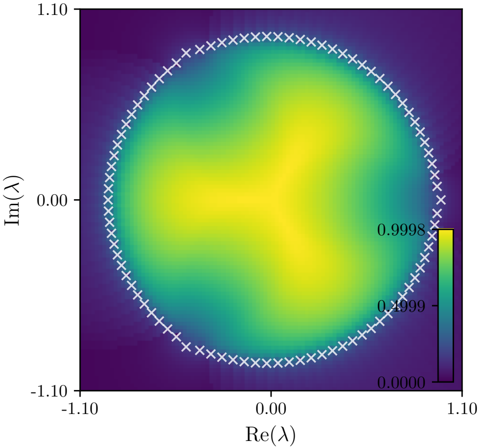


(left) k solutions for $L = 4$

(middle) difference between smallest and second-smallest quasi-energies for $N = 2$

(right) difference between smallest and second-smallest quasi-energies for $N = 3$

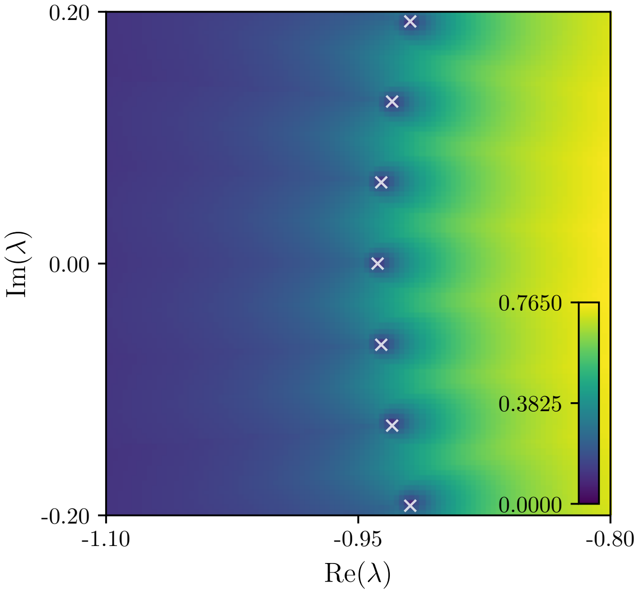
The corresponding values of λ_{EP} are also shown as crosses.



$N = 3 \quad L = 50$

Can apply large L expansion results for k to show that λ_{EP} satisfies

$$\lambda^N = \cos\left(\frac{2\pi j}{L}\right) \pm \mathrm{i} \sin\left(\frac{2\pi j}{L}\right) .$$



$N = 3 \quad L = 50$

Concluding remarks

- We have located the quasi-energy EPs in the complex plane.
- Numerical tests confirm they correspond to Hamiltonian EPs.
- And also confirm that the corresponding eigenvectors coalesce.
- There are other degeneracies in the energy eigenspectrum, but they are not EPs.
- Although in the complex plane, EPs can influence properties (such as correlations) along the real axis..